Round Elimination for Vertex Coloring

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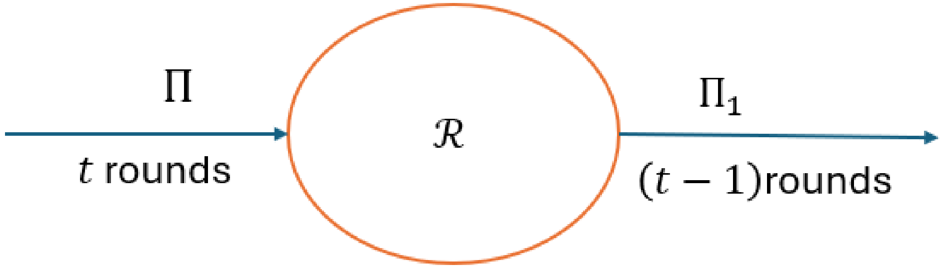
What is round elimination?

- A graph G with n nodes and maximum degree Δ with girth $\geq$ $2t + 2$ for some $t \geq 0$.
- A locally checkable problem Π on G that can be solved in t rounds.
- A speed up operation which will take Π and output another locally checkable problem Π_1 that can be solved in $(t-1)$rounds.



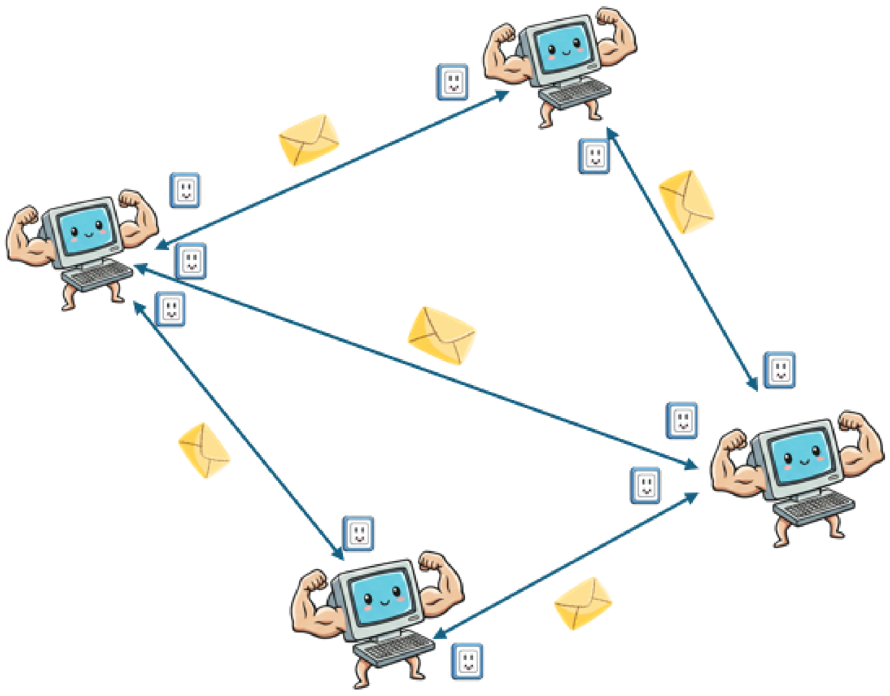
What is Π?

$$\Pi = (\Sigma_\Pi, \mathcal{N}_\Pi, \mathcal{E}_\Pi)$$

- ***Input:*** a graph $G \in \mathcal{G}_{n,\Delta}$ with input labels from $\mathcal{I}$.
  - $B(G)$:= set of all node-edge pairs (v, e), (equivalently half-edges), where $e \in E(G)$and e is incident to v.
  - $\Phi_G: B(G) \to 2^{\mathcal{I}}$
- Σ_Π is a finite subset of $\mathcal{O}$, where $\mathcal{O}$ is the possibly infinite set of output labels.
- $\mathcal{N}_\Pi$ (***set of node-constraints***) is set of multisets of size at most Δ with each element in the multiset from Σ_Π.
- $\mathcal{E}_\Pi$ (***set of edge-constraints***) is set of multisets of size 2 where both elements are taken from Σ_Π.
- An algorithm $\mathcal{A}$ solves Π on $\mathcal{G}_{n,\Delta}$ if, for any and any graph $G \in \mathcal{G}_{n,\Delta}$, $\mathcal{A}$ assigns an output $o_{(v,e)} \in \Sigma_\Pi$ to each pair $(v, e) \in B(G)$ such that it satisfies the node and edge constraints.

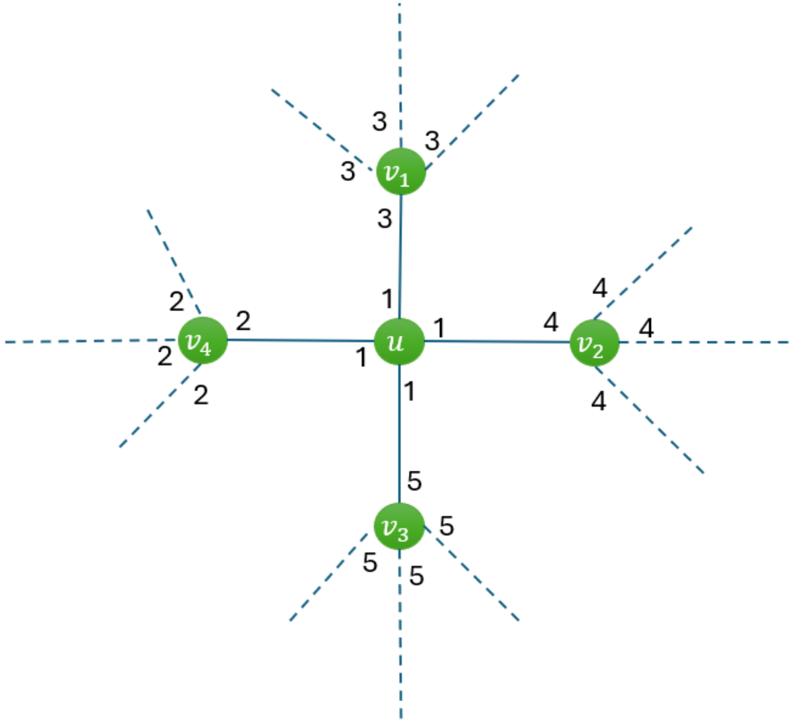
The Model

- Each node v is $\deg(v)$ many ports corresponding to all the incident edges.
- Arbitrarily large message size and computation power.
- Computation proceeds in synchronous rounds.
- In the beginning each node v is aware of the parameters n and Δ. And can see the input label assigned to $(v, e) \in B(G)$ for all incident edges e.
- When terminating, a node v assigns an output label from Σ_Π to each (v, e).



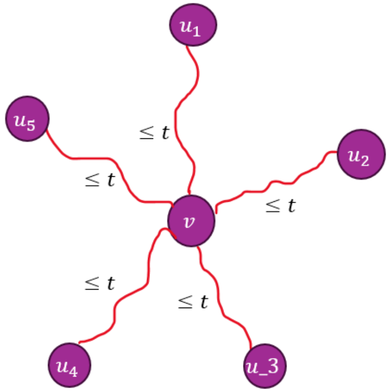
Example: 5-coloring on a 4-regular graph G

- $\Sigma_\Pi = \mathcal{O} = \{1, 2 \ldots 5\}$,
- (*Edge-constraint*) $\mathcal{E}_\Pi = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$,
- (*Node-constraint*)$\mathcal{N}_\Pi = \{\{1, 1, 1, 1\}, \{2, 2, 2, 2\}, \{3, 3, 3, 3\}, \{4, 4, 4, 4\}, \{5, 5, 5, 5\}\}$



Neighborhoods and Extensions

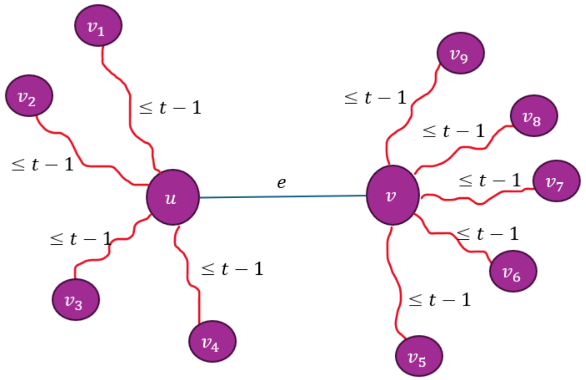
t-hop neighborhood of a node v



$$N^t(v)$$

$$Ext_e^t(v) = N^t(e) \setminus N^{t-1}(v)$$

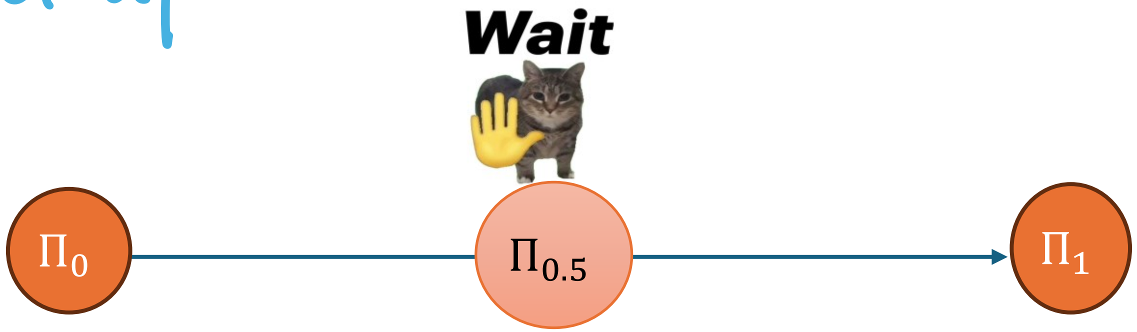
t-hop neighborhood of an edge $e = \{u, v\}$



$$N^t(e) = N^t(v) \cap N^t(u)$$

$$Ext_v^t(e) = N^t(v) \setminus N^t(e)$$

The speed up



- $\Pi_0 = \Pi$, and a node v uses its t-hop neighborhood $N^t(v)$ to compute an output for (v, e).

- $\Pi_{0.5}$is half-round faster than Π_0.

- In $\Pi_{0.5}$, output for (v, e) is computed using a neighborhood smaller than its t-hop neighborhood $N^t(v)$, but larger than its $(t-1)$-hop neighborhood $N^{t-1}(v)$. Which is precisely the t-hop neighborhood $N^t(e)$ of e.

- For each $o \in \Sigma_{\Pi_0}$ a node v determines whether $N^t(e)$ has an extension $Ext_v^t(e)$ in the graph class $\mathcal{G}_{n,\Delta}$ that v would assign output o to (v, e) in Π_0.
  ($N^t(e)$ has an extension $Ext_v^t(e)$ means $\exists$some $N^t(u)$ isomorphic to $N^t(e) \cup Ext_v^t(e)$)

$\Pi_{0.5} = (\Sigma_{\Pi_{0.5}}, \mathcal{N}_{\Pi_{0.5}}, \mathcal{E}_{\Pi_{0.5}})$

- $\Sigma_{\Pi_{0.5}} = 2^{\Sigma_{\Pi_0}} \setminus \{\{\}\}$ (output labels are sets)

- $\mathcal{E}_{\Pi_{0.5}}$ = the set of all multisets $\{Y, Z\}$ where $Y, Z \in \Sigma_{\Pi_{0.5}}$ such that for any $y \in Y$, $z \in Z$, such that the multiset $\{y, z\}$ is contained in $\mathcal{E}_{\Pi_0}$.(Universal)

- $\mathcal{N}_{\Pi_{0.5}}$ = the set of all multisets $\{Y_1, \dots, Y_i\}$, where $i \le \Delta$ and $Y_1, \dots, Y_i \in \Sigma_{\Pi_{0.5}}$ such that there exist elements $y_1 \in Y_1, \dots,\ y_i \in Y_i$ and the multiset $\{y_1, \dots, y_i\}$ is contained in $\mathcal{N}_{\Pi_0}$. (Existential)

How do we get Π_1 from $\Pi_{0.5}$?

- For each $o \in \Sigma_{\Pi_{0.5}}$ a node v determines whether $N^{t-1}(v)$ has an extension $Ext_e^t(v)$ in the graph class $\mathcal{G}_{n,\Delta}$ that v would assign output o to (v, e) in $\Pi_{0.5}$.
  ($N^{t-1}(v)$ has an extension $Ext_e^t(v)$ means $\exists$some $N^t(e)$ isomorphic to $N^{t-1}(v) \cup Ext_e^t(v)$)

$\Pi_1 = (\Sigma_{\Pi_1}, \mathcal{N}_{\Pi_1}, \mathcal{E}_{\Pi_1})$

- $\Sigma_{\Pi_1} = 2^{\Sigma_{\Pi_{0.5}}} \setminus \{\{\{\}\}\}$
- $\mathcal{E}_{\Pi_1}$ = the set of all multisets $\{Y, Z\}$ where $Y, Z \in \Sigma_{\Pi_1}$ such that there exists $y \in Y$, $z \in Z$, such that the multiset $\{y, z\}$ is contained in $\mathcal{E}_{\Pi_{0.5}}$.(Existential)
- $\mathcal{N}_{\Pi_1}$ = the set of all multisets $\{Y_1, \dots, Y_i\}$, where $i \le \Delta$ and $Y_1, \dots, Y_i \in \Sigma_{\Pi_1}$ such that for any $y_1 \in Y_1, \dots$, $y_i \in Y_i$ and the multiset $\{y_1, \dots, y_i\}$ is contained in $\mathcal{N}_{\Pi_{0.5}}$. (Universal)
- Π_1 is the dual of $\Pi_{0.5}$.

What is known?

- It's already known that in LOCAL (without using unique IDs) , to reduce k colors from a m colored graph $G \in \mathcal{G}_{n,\Delta}$ in 1 round, we need $m \geq k(\Delta - k + 3)$. (*we keep the colors in* $[m-k]$ *and only recolor* $\{m-k+1, \ldots, m\}$) [Maus'21]
- If we repeat this 1 round color reduction, in we can reduce $(2k-1)$ colors (when $k \leq \frac{\Delta+4}{3}$) , otherwise less than $(2k-1)$ colors.

Can we do better if we increase the number of rounds? If yes, how much?

- A small example is when $k = 3, \Delta = 4$. We need $m \geq 10$ colors in the input.

Can we do it when $m \leq 9$.

Speed up for Vertex coloring

- Let $m = 9, \quad \Delta = 4 \; and \; k = 3$ be the input for Π_0.
- $\Sigma_{\Pi_0} = [6]$
- $\mathcal{E}_{\Pi_0} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$
- $\mathcal{N}_{\Pi_0} = \{\{1,1,1,1\},\{2,2,2,2\},\{3,3,3,3\},\{4,4,4,4\},\{5,5,5,5\},\{6,6,6,6\}\}$.

$$\boxed{\Pi_0 \rightarrow \Pi_{0.5} \rightarrow \Pi_1 \rightarrow \Pi_{1.5} \rightarrow \Pi_2}$$

- Π_0 can be solved in 2-rounds iff $\Pi_{1.5}$ can be solved in 0.5 rounds.

Speed up for Vertex coloring

The problem $\Pi_{0.5}$:

- $\Sigma_{\Pi_{0.5}} = 2^{[6]} \setminus \{\{\}\}$,
- $\mathcal{E}_{\Pi_{0.5}} = \left\{\{Y, Z\} \middle| Y, Z \in \Sigma_{\Pi_{0.5}}, Y \cap Z = \phi\right\}$ (Universal)
- $\mathcal{N}_{\Pi_{0.5}} = \left\{\{Y_1, Y_2, Y_3, Y_4\} \middle| Y_i \in \Sigma_{\Pi_{0.5}}, \ \cap_{i \in [4]} Y_i \neq \phi\right\}$ (Existential)

The problem Π_1:

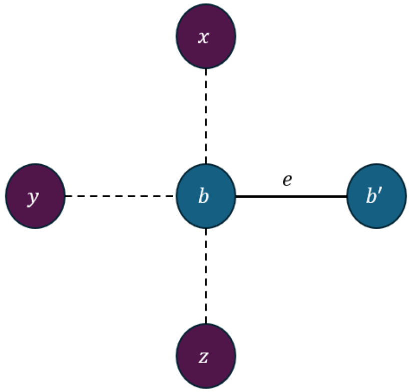
- $\Sigma_{\Pi_1} = 2^{2^{[6]}} \setminus \{\{\{\}\}\}$,
- $\mathcal{E}_{\Pi_1} = \left\{\{Y, Z\} \middle| Y, Z \in \Sigma_{\Pi_1}, \exists y \in Y, z \in Z \text{ such that } y \cap z = \phi\right\}$(Existential)
- $\mathcal{N}_{\Pi_1} = \left\{\{Y_1, Y_2, Y_3, Y_4\} \middle| Y_i \in \Sigma_{\Pi_1}, \text{and for any } y_i \in Y_i, \cap_{i \in [4]} y_i \neq \phi\right\}$ (Universal)

The problem $\Pi_{1.5}$:

- $\Sigma_{\Pi_{1.5}} = 2^{2^{2^{[6]}}} \setminus \{\{\{\{\}\}\}\}$,
- $\mathcal{E}_{\Pi_{1.5}} = \left\{\{Y, Z\} \middle| Y, Z \in \Sigma_{\Pi_{1.5}}, \text{for any } a \in Y, b \in Z, \exists y \in a, z \in b \text{ such that } y \cap z = \phi\right\}$(Universal)
- $\mathcal{N}_{\Pi_{1.5}} = \left\{\{Y_1, Y_2, Y_3, Y_4\} \middle| Y_i \in \Sigma_{\Pi_1}, \text{and } \exists a_i \in Y_i \text{ such that for any } y_i \in a_i, \cap_{i \in [4]} y_i \neq \phi\right\}$ (Existential)

The half round solution

- The output $o_{v,e}$ for an edge $e = \{u, v\}$is based on the information obtained from the $N^1(e)$.
- To satisfy the node constraints output must be computed considering all possible neighborhoods of v that contains u.



Solution: Let $\mathrm{A} = \{1,2,3,4,5,6\}$ and $\mathrm{B} = \{7,8,9\}$

- Nodes with color $a \in A$ outputs $\left\{\{\{a\}\}\right\}$ for any $e = \{a\ , x\}$.
- Nodes with color $b \in B$ outputs $\{\{A \setminus \{a\}\}\}$ for any $e = \{b, a\}$.
- 7 outputs: $\left\{\{\{1\}\}, \{\{2\}\}, \{\{3\},\{4\}\}, \{\{3\},\{5\}\}, \{\{3\},\{6\}\}, \{\{4\},\{5\}\}, \{\{4\},\{6\}\}, \{\{5\},\{6\}\}, \{\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}\right\}$ for both $e = \{7,8\}$ and $e = \{7,9\}$.
- 8 outputs: $\left\{\{\{3\}\}, \{\{4\}\}, \{\{1\},\{2\}\}, \{\{1\},\{5\}\}, \{\{1\},\{6\}\}, \{\{2\},\{5\}\}, \{\{2\},\{6\}\}, \{\{5\},\{6\}\}, \{\{1,2\},\{1,5\},\{1,6\},\{2,5\},\{2,6\},\{5,6\}\}\right\}$ for both $e = \{7,8\}$ and $e = \{8,9\}$.
- 9 outputs: $\left\{\{\{5\}\}, \{\{6\}\}, \{\{1\},\{2\}\}, \{\{1\},\{3\}\}, \{\{1\},\{4\}\}, \{\{2\},\{3\}\}, \{\{2\},\{4\}\}, \{\{3\},\{4\}\}, \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}\right\}$ for both $e = \{7,9\}$ and $e = \{8,9\}$.

Can we go lower?

Can we solve $\Pi_{1.5}$in 0.5 rounds when the input graph is 8 colored and $\Delta = 4$?

Yes!

Solution: $A = \{1,2,3,4,5\}$ and $B = \{6,7,8\}$

- Nodes with color $a \in A$ outputs $\left\{\{\{a\}\}\right\}$ for any $e = \{a\ , x\}$.
- Nodes with color $b \in B$ outputs $\{\{A \setminus \{a\}\}\}$ for any $e = \{b, a\}$.
- 6 outputs: $\left\{\{\{1\}\}, \{\{2\}\}, \{\{3\}, \{4\}\}, \{\{3\}, \{5\}\}, \{\{4\}, \{5\}\}, \{\{3,4\}, \{3,5\}, \{4,5\}\}\right\}$ for both $e = \{6,7\}$ and $e = \{6,8\}$.
- 7 outputs: $\left\{\{\{3\}\}, \{\{4\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{5\}\}, \{\{2\}, \{5\}\}, \{\{1,2\}, \{1,5\}, \{2,5\}\}\right\}$ for both $e = \{7,8\}$ and $e = \{7,6\}$.
- 8 outputs: $\{\{\{5\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{1\}, \{4\}\}, \{\{2\}, \{3\}\}, \{\{2\}, \{4\}\}, \{\{3\}, \{4\}\}, \{\{1,2\}, \{1,3\}, \{2,3\}\}$ $\{\{1,3\}, \{1,4\}, \{3,4\}\}, \{\{1,2\}, \{1,4\}, \{2,4\}\}, \{\{2,3\}, \{2,4\}, \{3,4\}\}, \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}\}$for both $e = \{8,6\}$ and $e = \{8,7\}$.

Summary and Future Directions

- Round elimination can be used to prove lower bounds for locally checkable problems.
- To reduce 3 colors in 2 rounds the input graph needs to be at least 8-colored when $\Delta = 4$.
- What is the lower bound on the number of input colors when want to reduce k colors for some $k \geq 0$ and some general Δ?

Any questions?