Optimal Algorithm for the Planar Two-Center Problem

SoCG 2024

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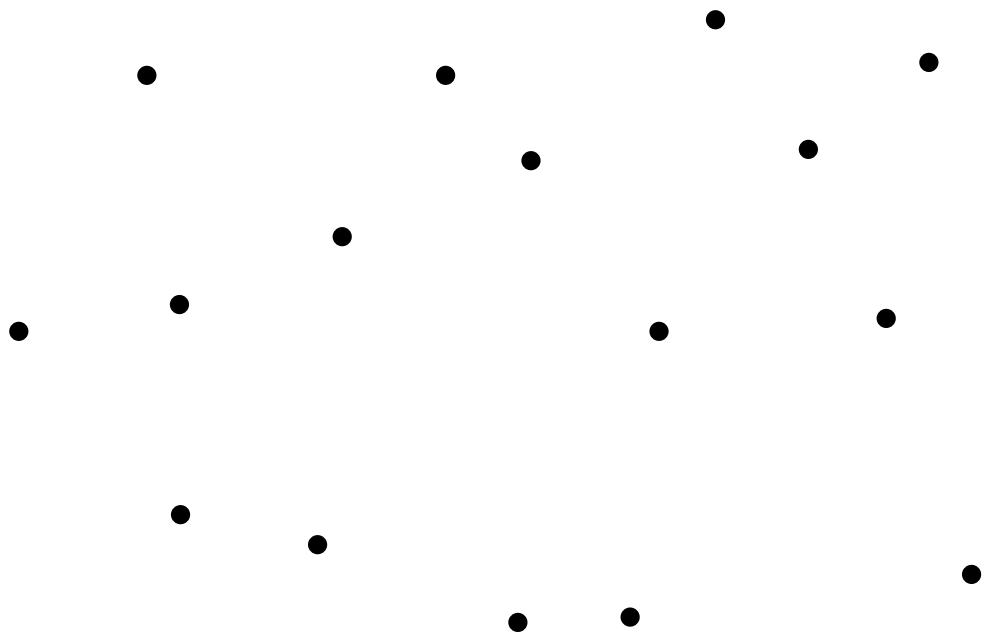
University of Utah

Jie Xue
Department of Computer Science
New York University Shanghai

Problem Definition

n points in $\mathbb{R}^2$



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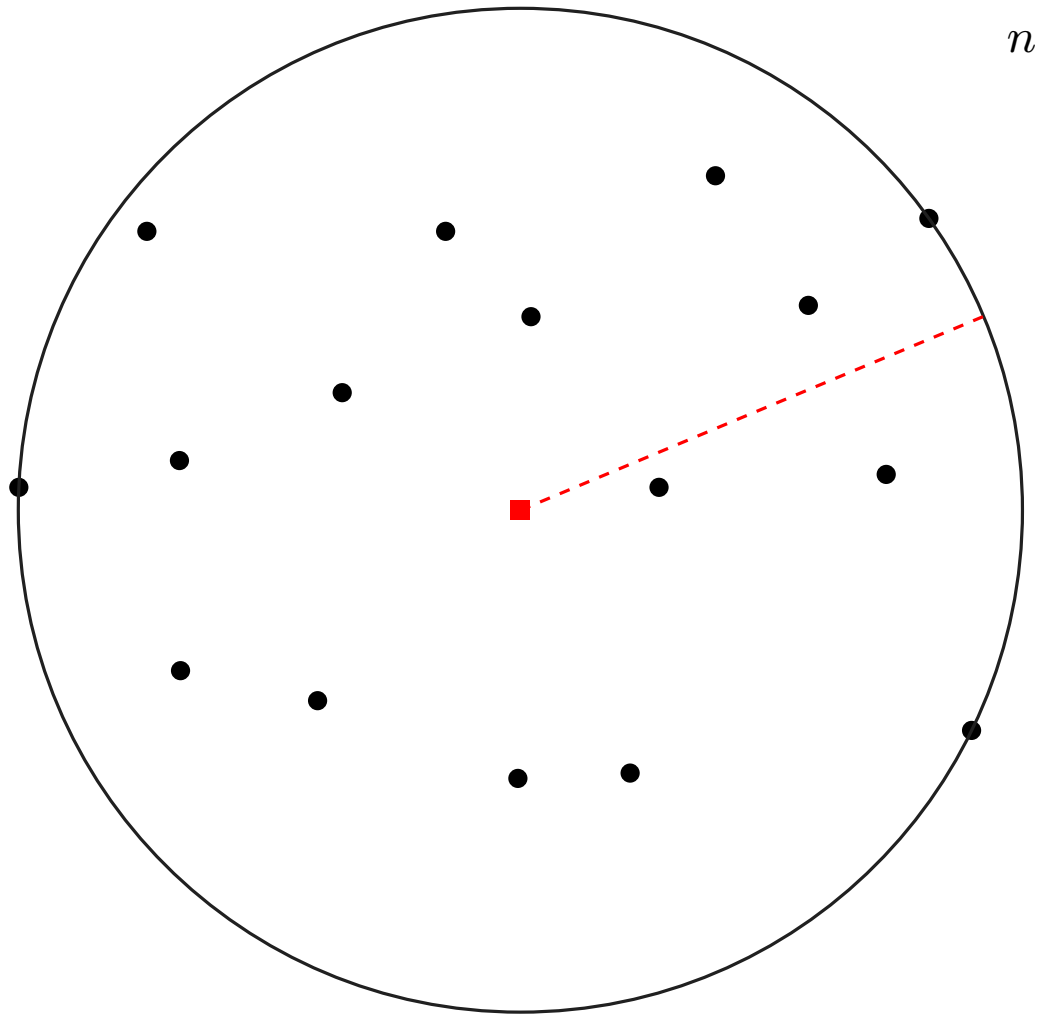

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1-center problem: It requires $\Theta(n)$ time to solve in $\mathbb{R}^d$.

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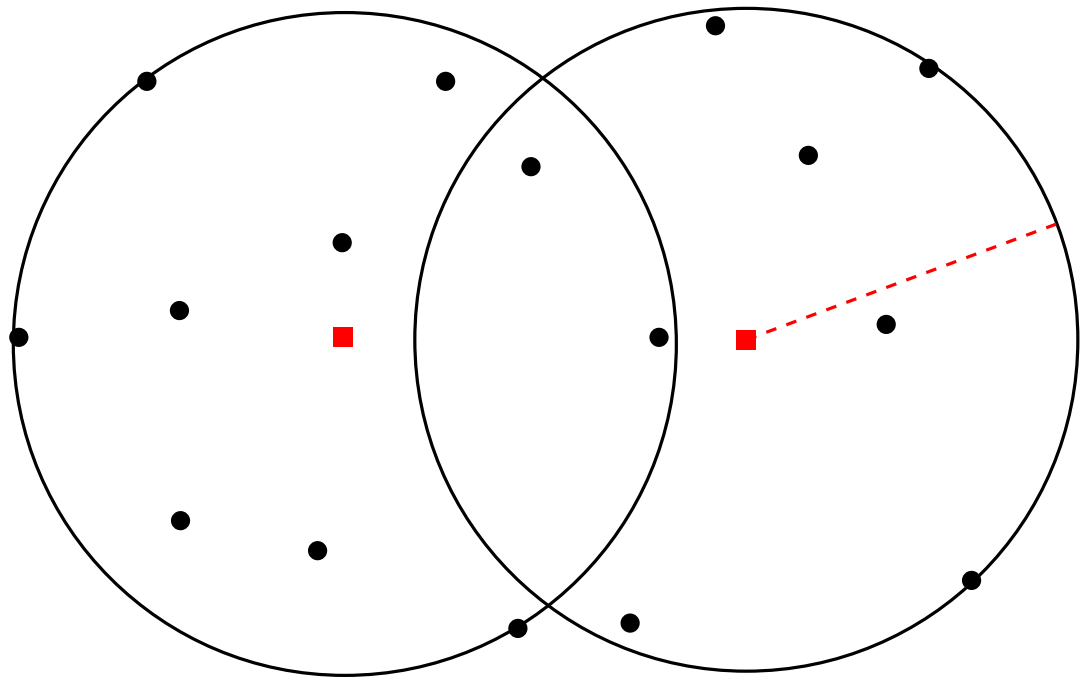

2-center problem: Finding two congruent disks D_1 and D_2 covering all points

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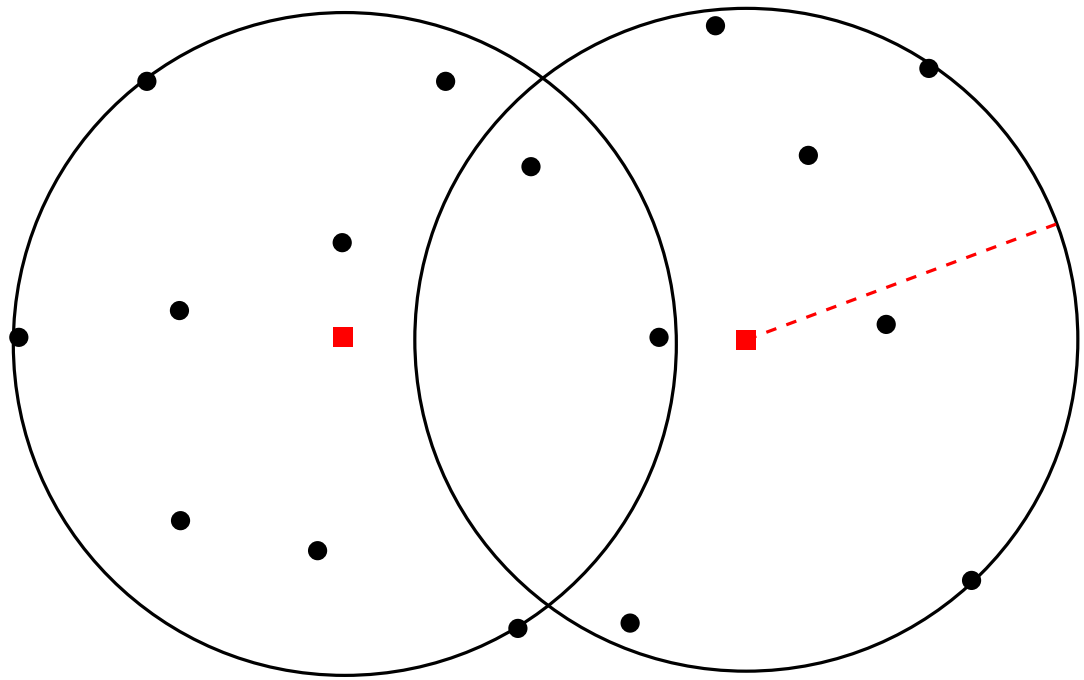

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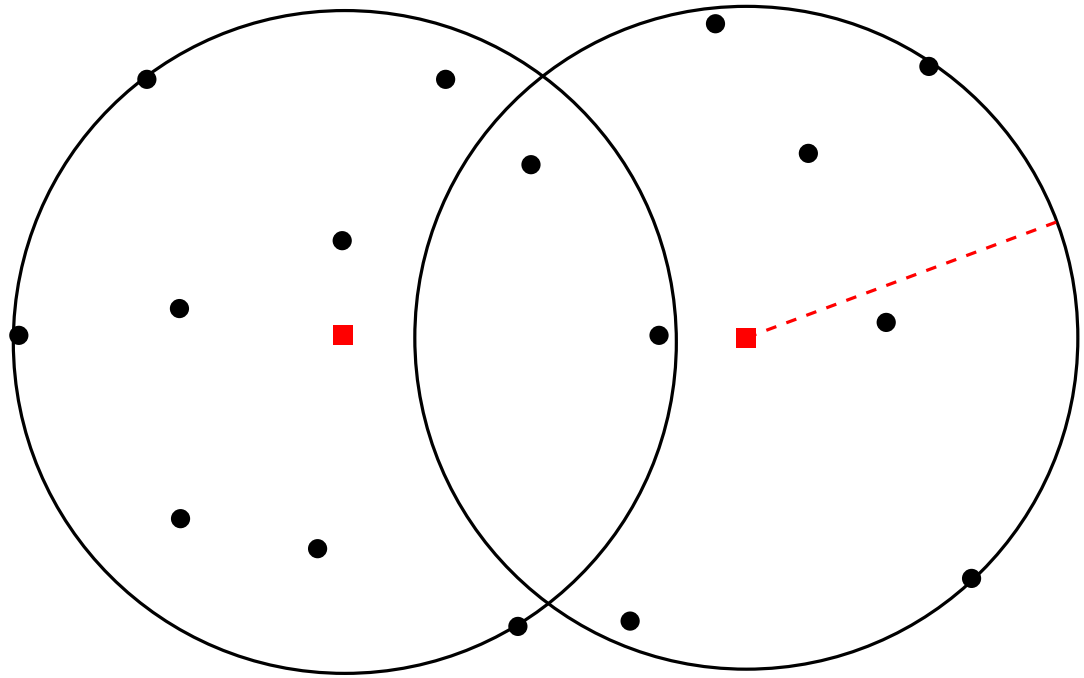

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Question: Is there a deterministic $O(n \log n)$ time algorithm?

Sketch of History

[Agarwal and Sharir 94] $O(n^2 \log^3 n)$ time algorithm

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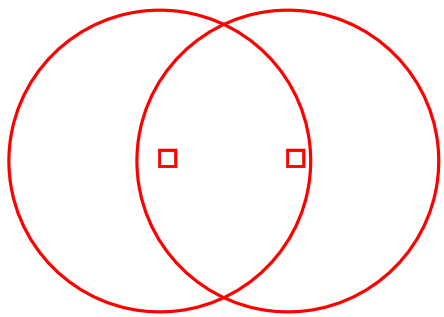
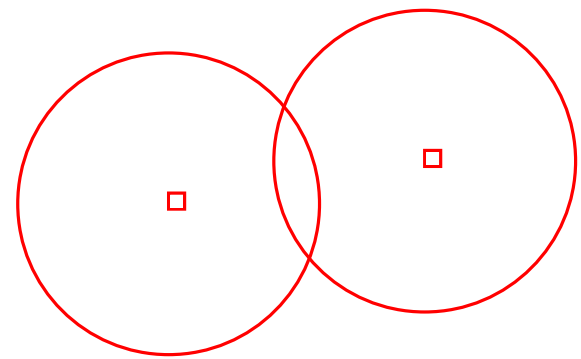
Nearby 2-center problem	Distant 2-center problem





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[Parametric Search]
Solving optimization using decision algorithms

Optimization: Find a min/max value r^*
Decision: Determine $r \geq r^*$ or not for given r

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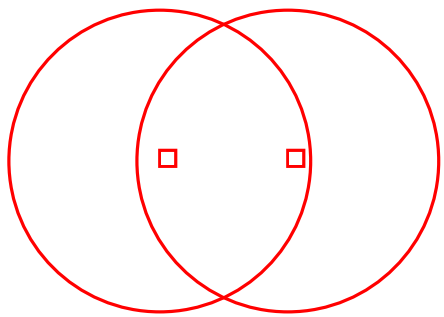

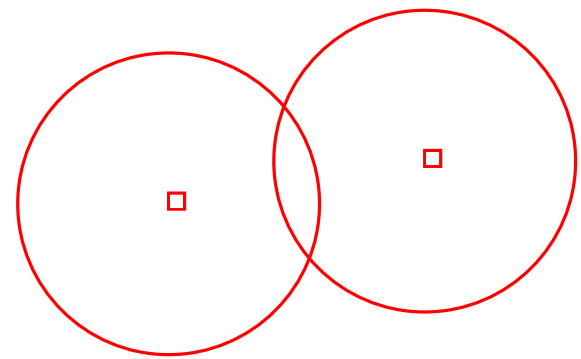

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Sequential Decision in $T_S(n)$ time
Paralallel Decision in $T_P(n)$ parallel step with $P(n)$ processors.

Then the optimal value r^* is returned in $O(T_S(n) \log P(n) + T_P(n)P(n))$ time.

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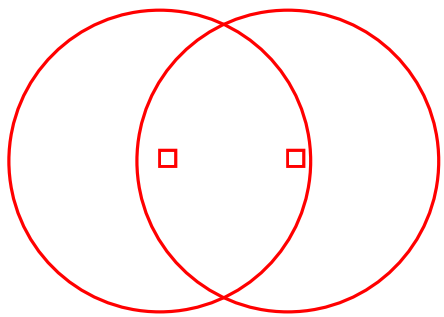

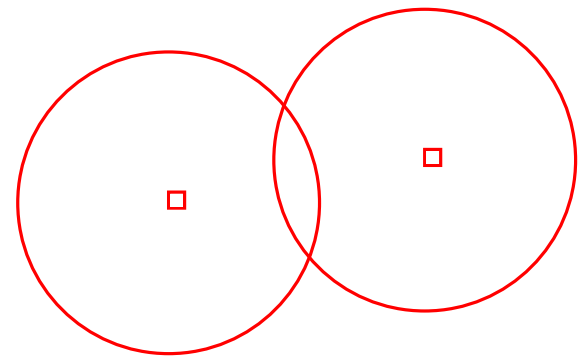

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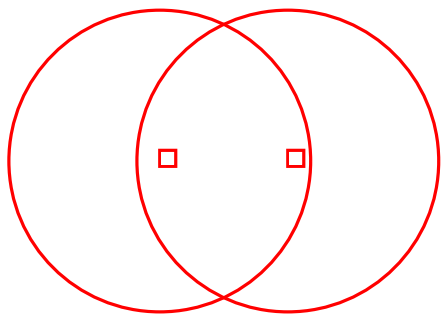

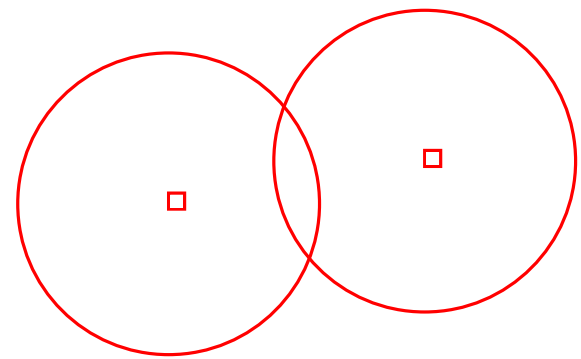

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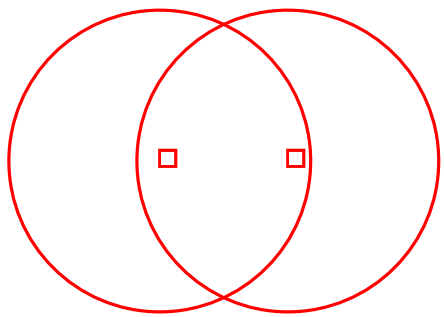	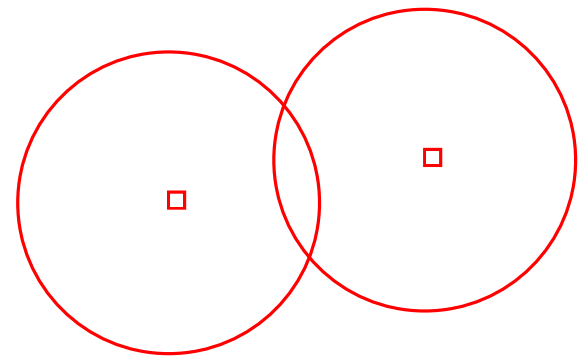
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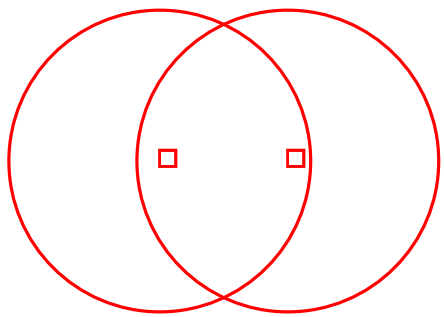	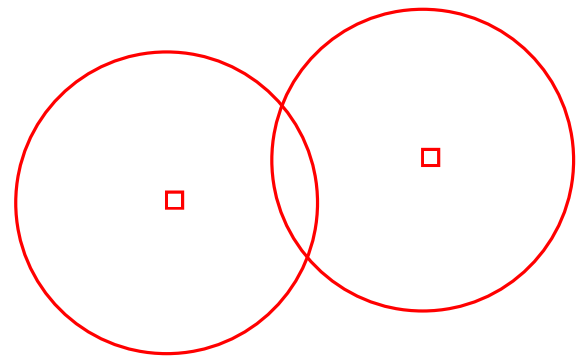
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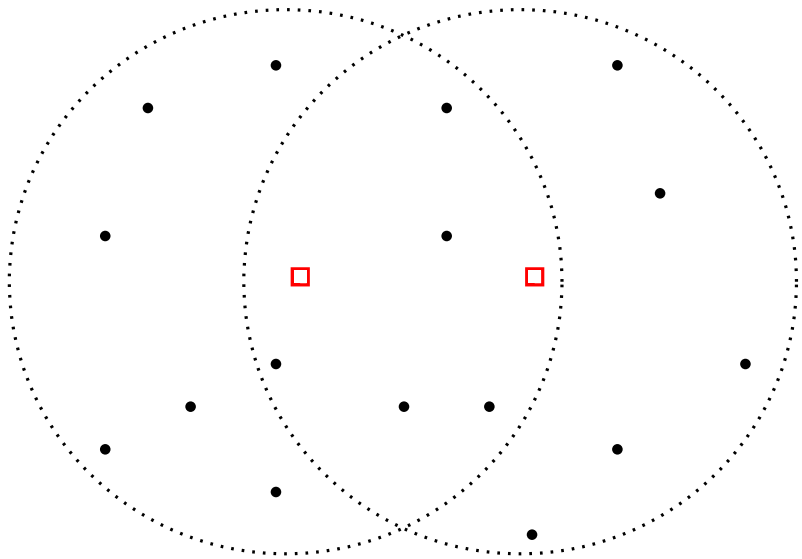
Decision: For a given r,decide whether $r^* \leq r$ or not (after $O(n \log n)$ preprocessing time)



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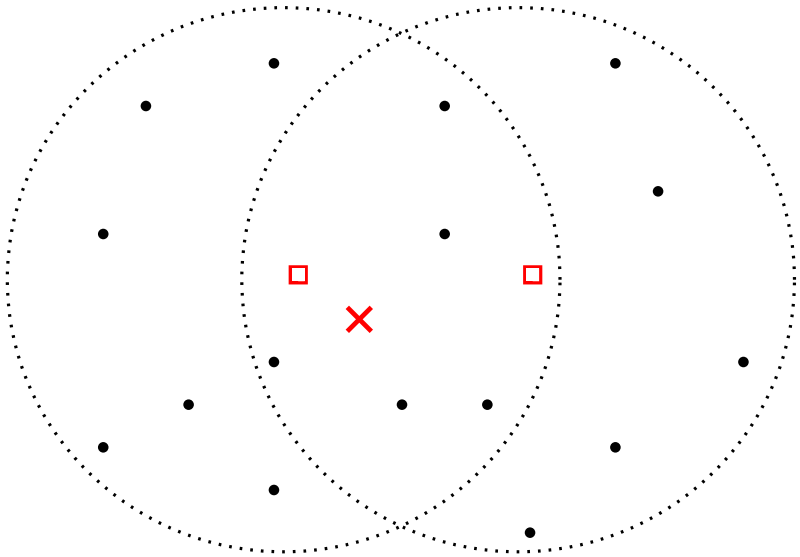

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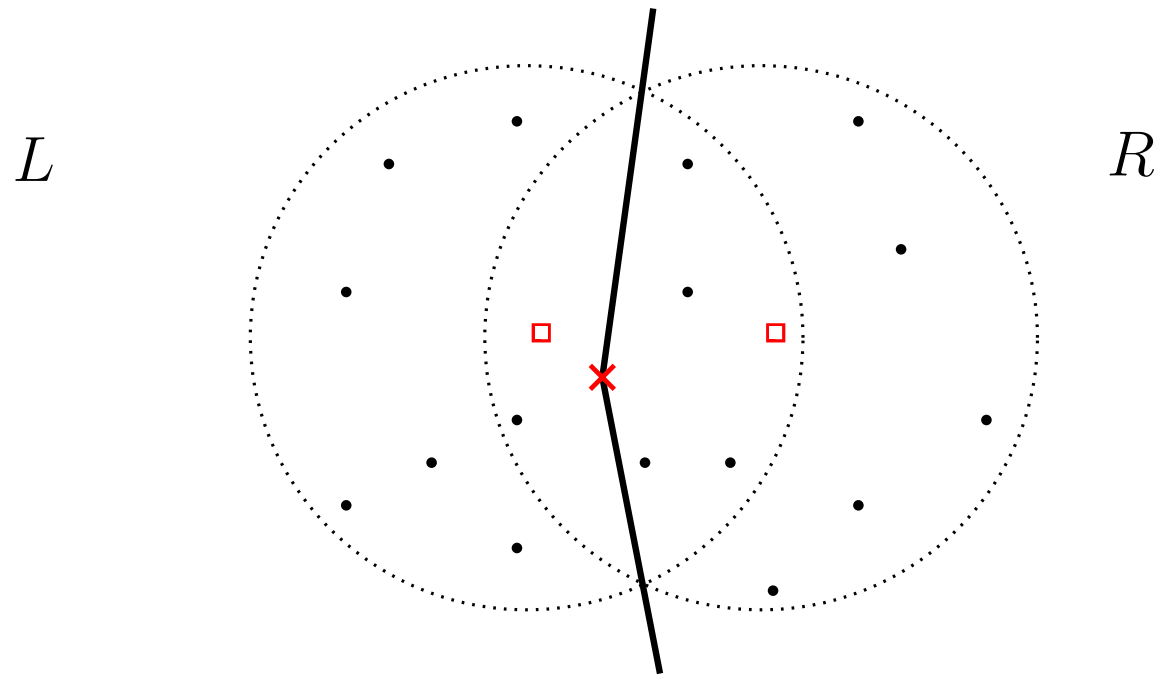

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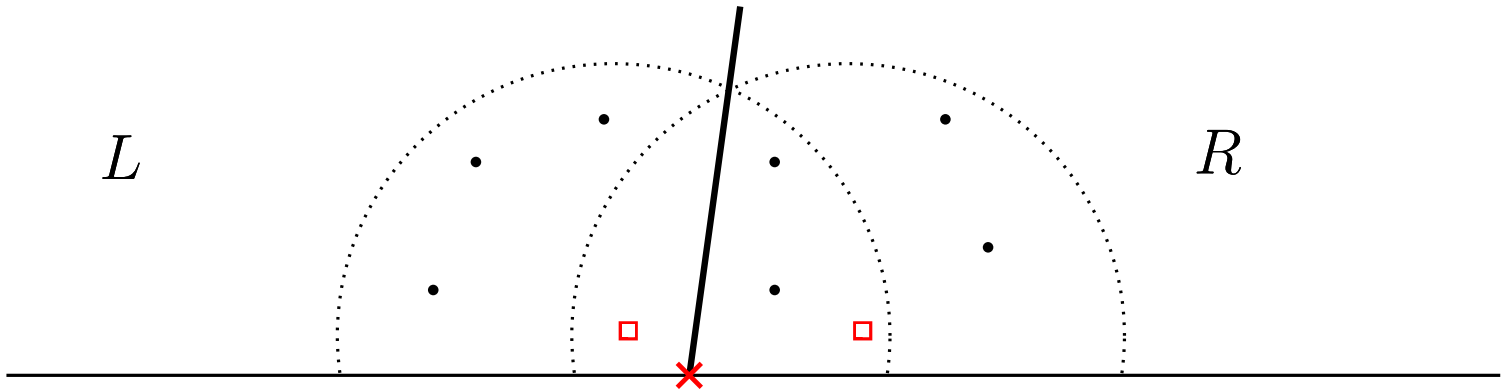

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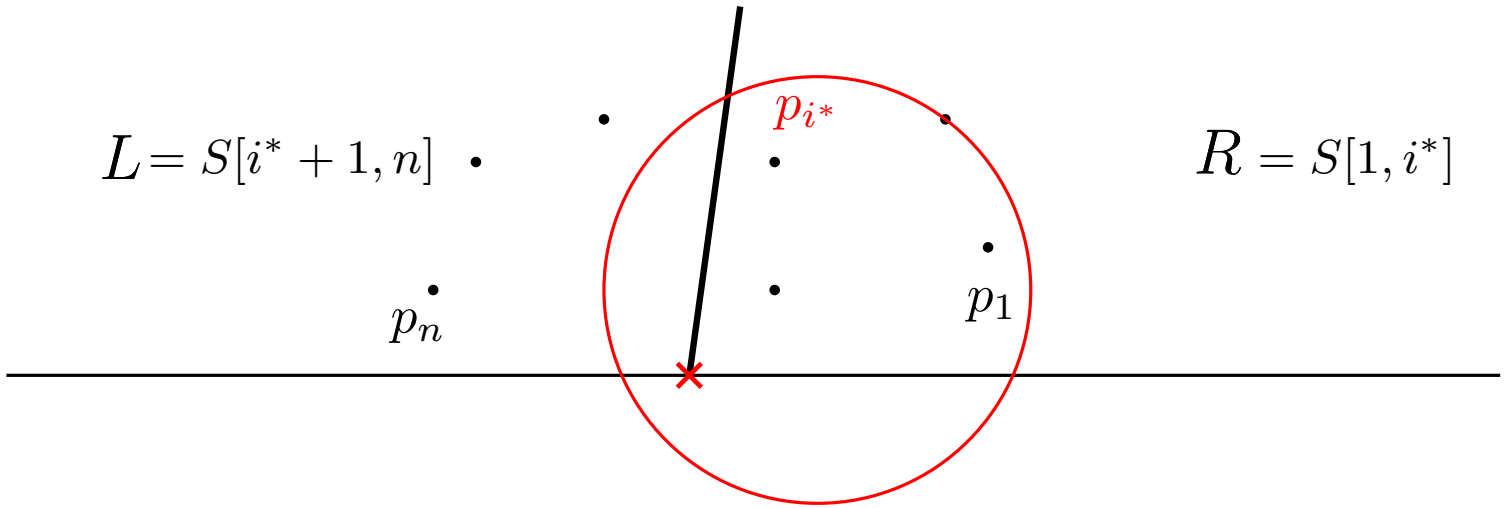

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Find the maximum i^* s.t. $S[1 : i^*]$ is covered by a disk of radius r

Check $S[i^* + 1 : n]$ is covered by a disk of radius r

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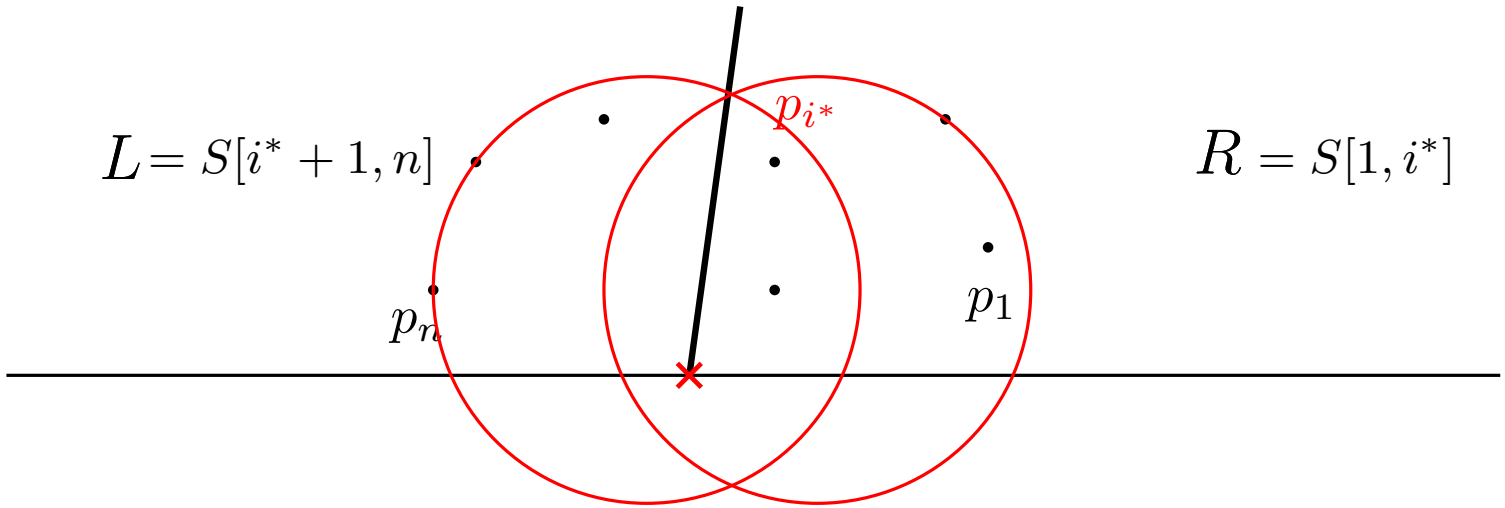

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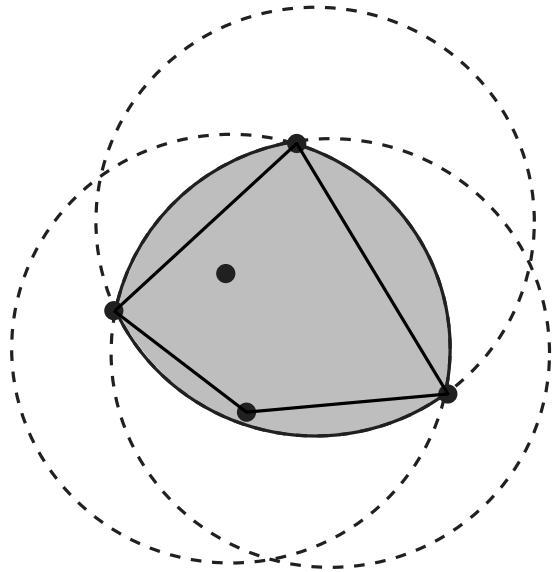

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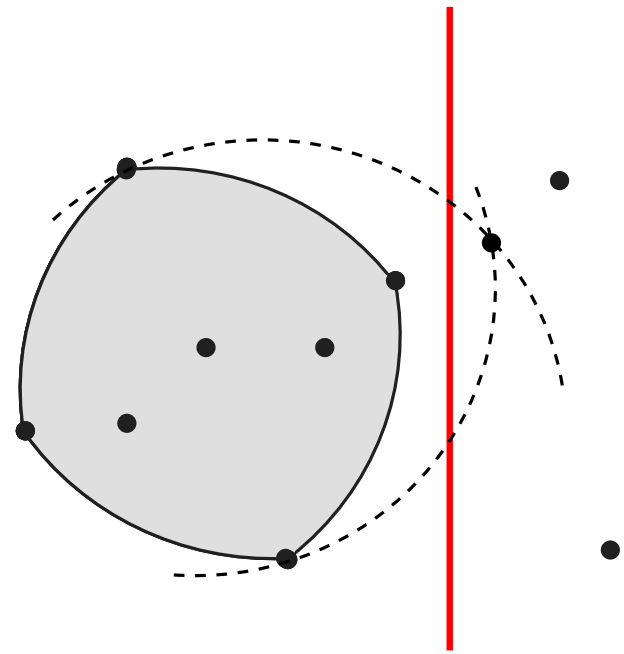

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- If points are sorted, $\mathrm{CH}(S)$ can be computed in $O(n)$ time.
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Initial: $S = \emptyset$

Insert the right most point to S

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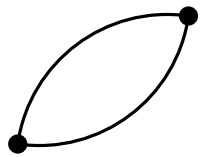

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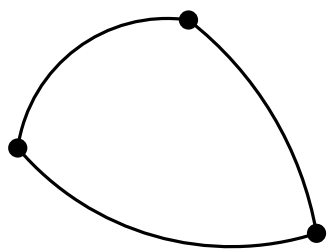

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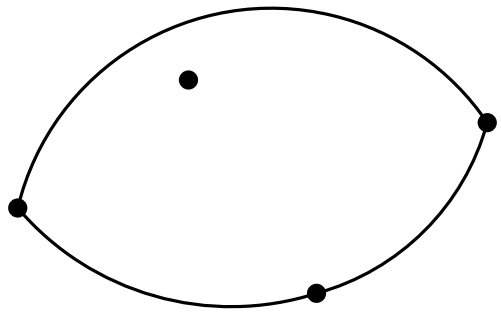

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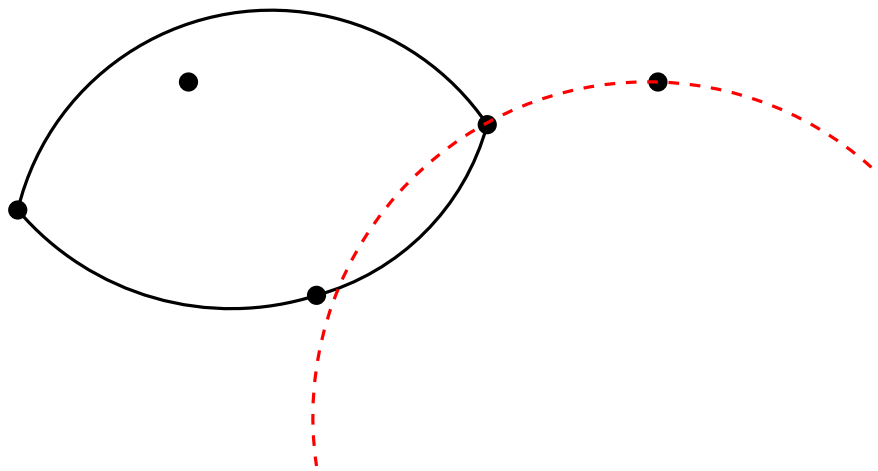

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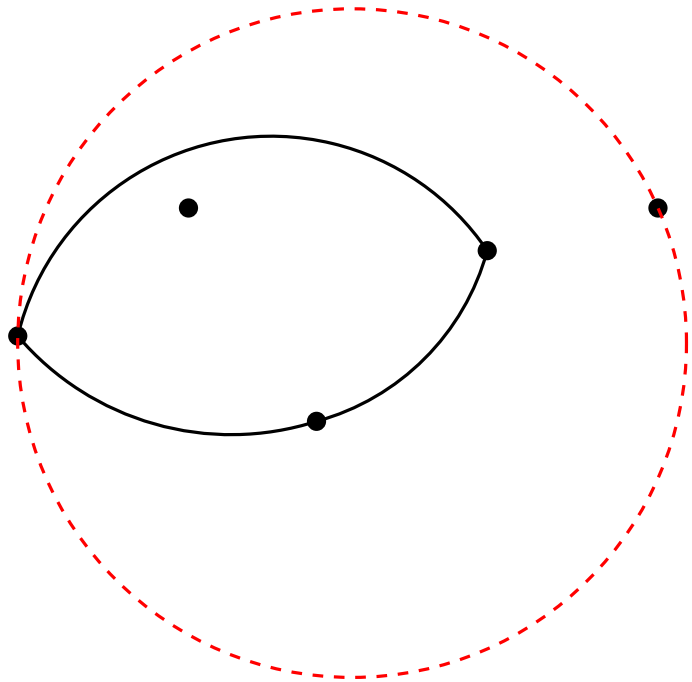

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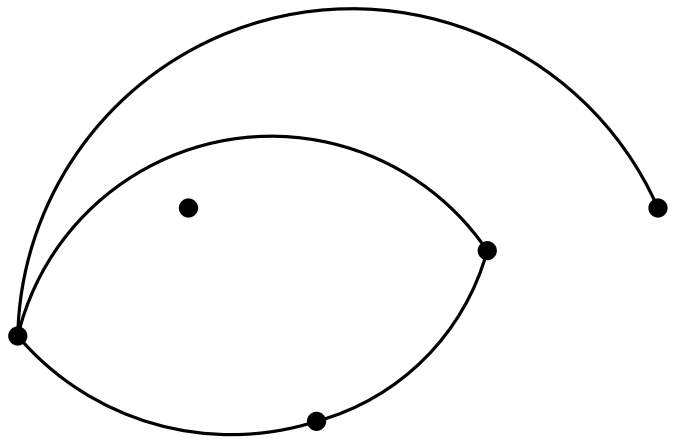

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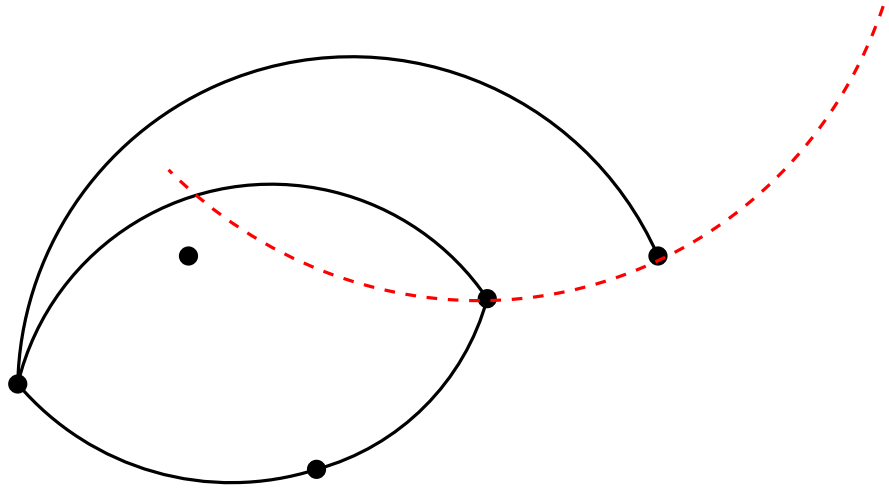

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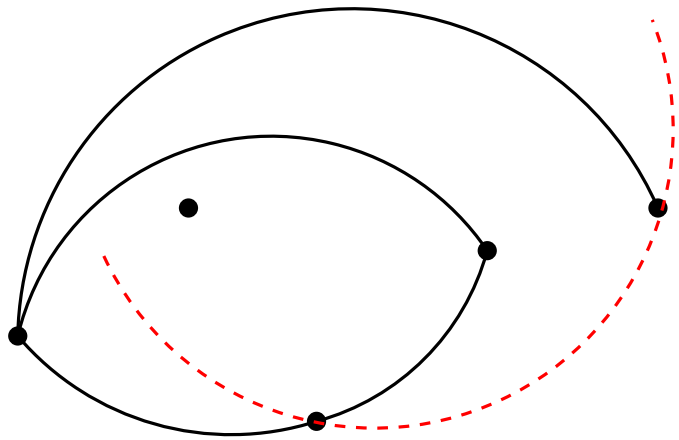

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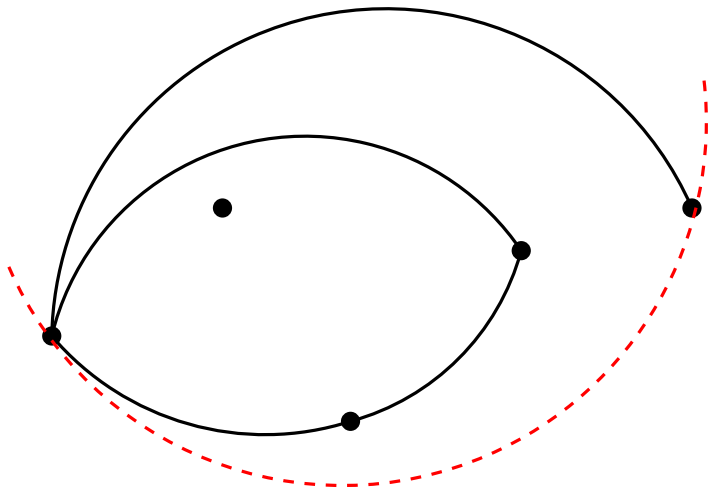

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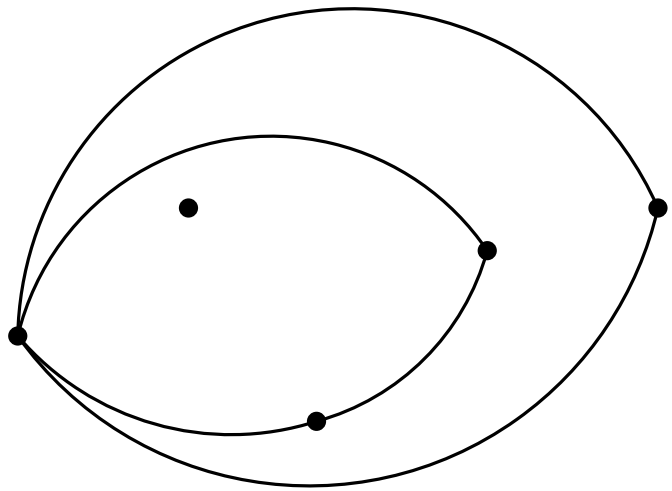

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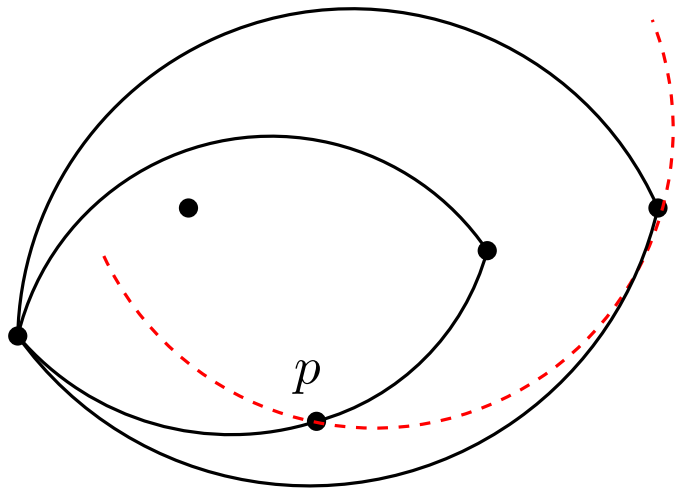


- p is a vertex of $\alpha_r(S)$
- p is lying inside $\alpha_r(S)$

  $\rightarrow$ p cannot be a vertex or boundary of α_r

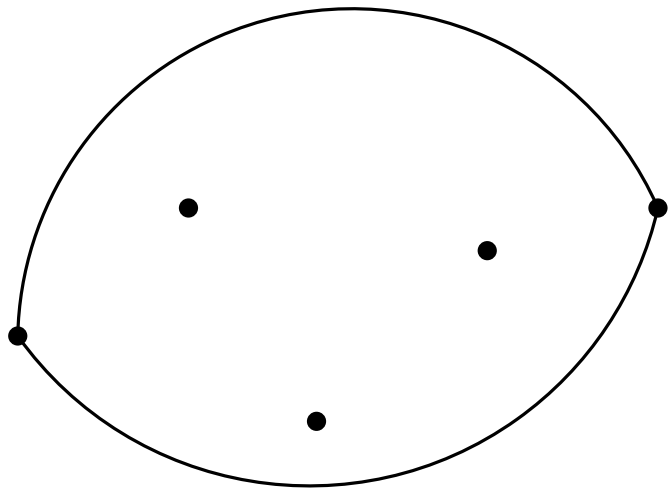
Inserting n vertices to S takes $O(n)$ time in total.

$\rightarrow$ amortized $O(1)$ update time

Dynamic r-Circular Hull: Maintaining $\alpha_r(S)$
Initial: $S = \emptyset$

Insert the right most point to S

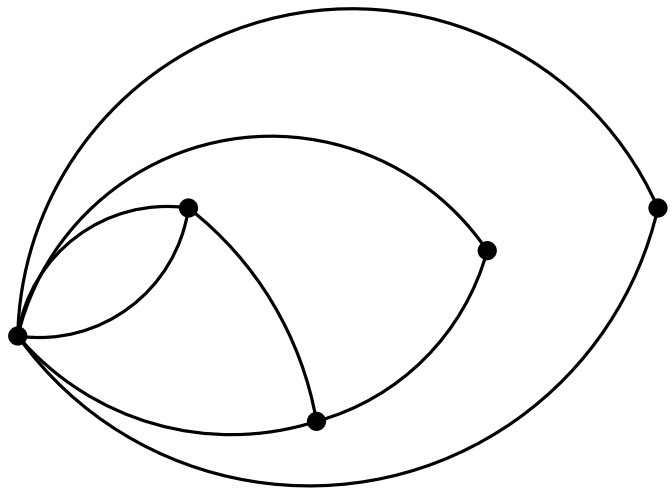
[Wang 20] Dynamic r-Circular Hull : Deletion



Dynamic r-Circular Hull: Maintaining $\alpha_r(S)$
Initial: S

Delete the right most point to S

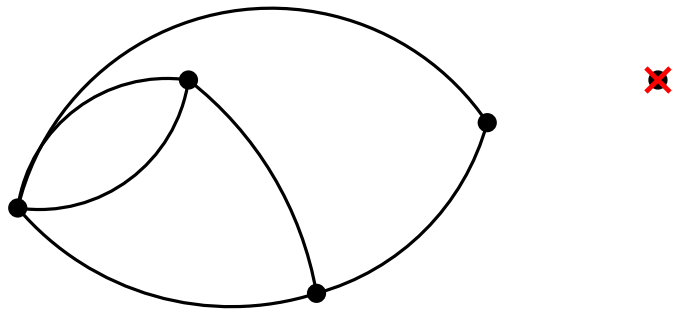
[Wang 20] Dynamic r-Circular Hull : Deletion



Dynamic r-Circular Hull: Maintaining $\alpha_r(S)$
Initial: S
We store $\alpha_r(S[1:1])$, $\alpha_r(S[1:2]), \ldots, \alpha_r(S[1:n])$

Delete the right most point to S

[Wang 20] Dynamic r-Circular Hull : Deletion



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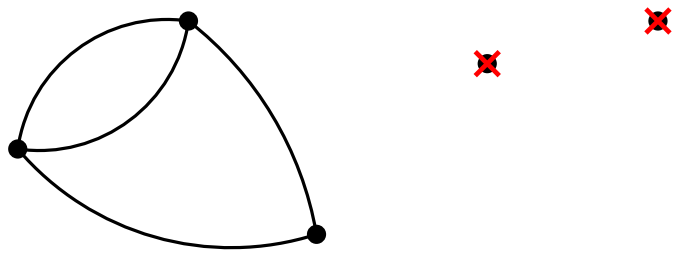
[Wang 20] Dynamic r-Circular Hull : Deletion



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[Wang 20] Dynamic r-Circular Hull : Deletion

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Delete the right most point to S

[Wang 20] Dynamic r-Circular Hull : Deletion

Initializing $O(n)$ time

Deleting a point takes $O(1)$ time.

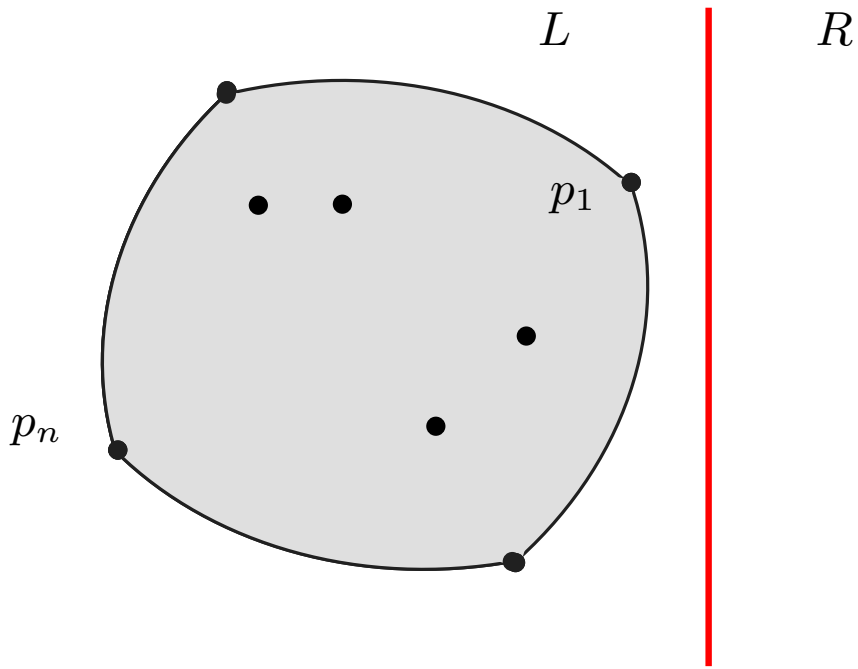
$\rightarrow$ amortized $O(1)$ update time

Dynamic r-Circular Hull: Maintaining $\alpha_r(S)$
Initial: S

We store $\alpha_r(S[1:1])$, $\alpha_r(S[1:2]), \ldots, \alpha_r(S[1:n])$

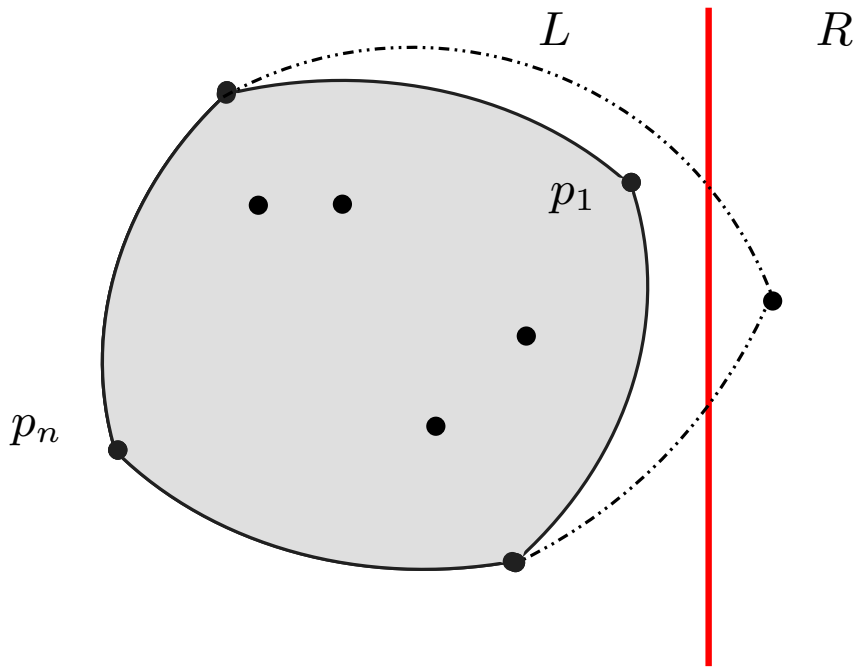
Delete the right most point to S

[Wang 20] Fully Dynamic r-Circular Hull



Dynamic r-Circular Hull: Maintaining $\alpha_r(L \cup R)$
Initial: $L = S$ and $R = \emptyset$
We store $\alpha_r(S[1:1])$, $\alpha_r(S[1:2]), \ldots, \alpha_r(S[1:n])$

[Wang 20] Fully Dynamic r-Circular Hull



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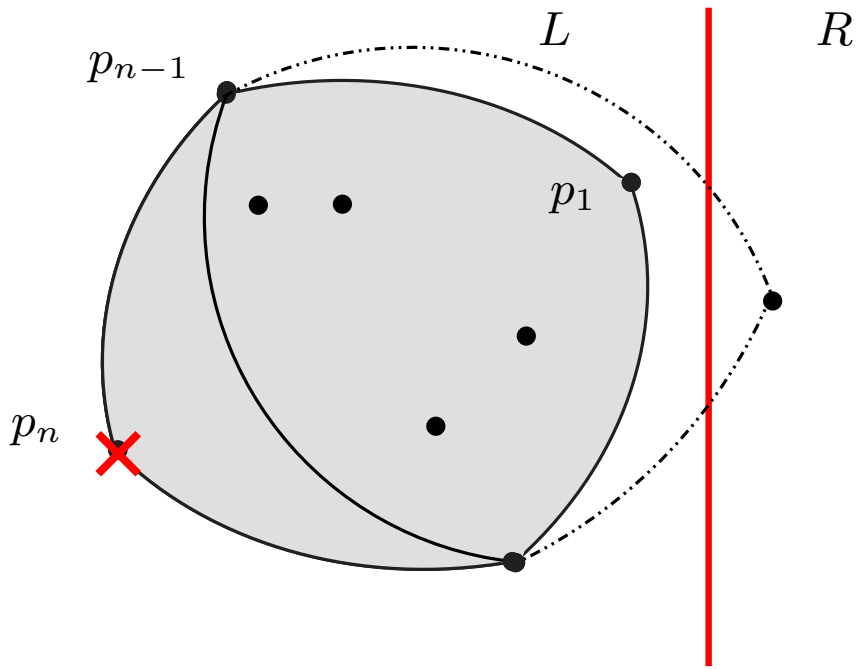
Initial: $L = S$ and $R = \emptyset$
We store $\alpha_r(S[1:1])$, $\alpha_r(S[1:2]), \ldots, \alpha_r(S[1:n])$

Insert the right most point q to R

[Wang 20] Fully Dynamic r-Circular Hull



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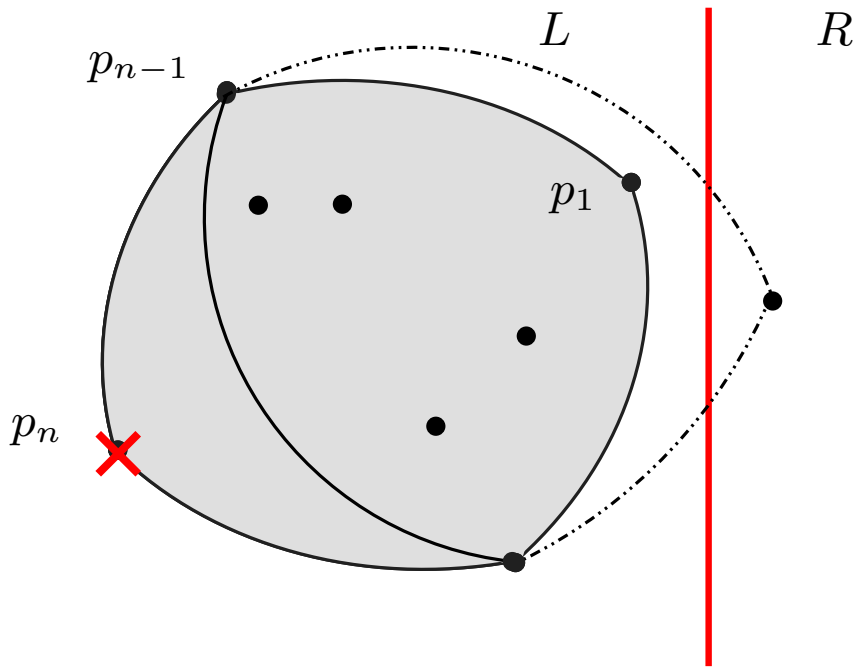
We store $\alpha_r(S[1:1])$, $\alpha_r(S[1:2]), \ldots, \alpha_r(S[1:n])$

Insert the right most point q to R
Delete the left most point p from L

[Wang 20] Fully Dynamic r-Circular Hull



Dynamic r-Circular Hull: Maintaining $\alpha_r(L \cup R)$
Initial: $L = S$ and $R = \emptyset$

We store $\alpha_r(S[1:1])$, $\alpha_r(S[1:2]), \ldots, \alpha_r(S[1:n])$

Initializing takes $O(n)$ time.

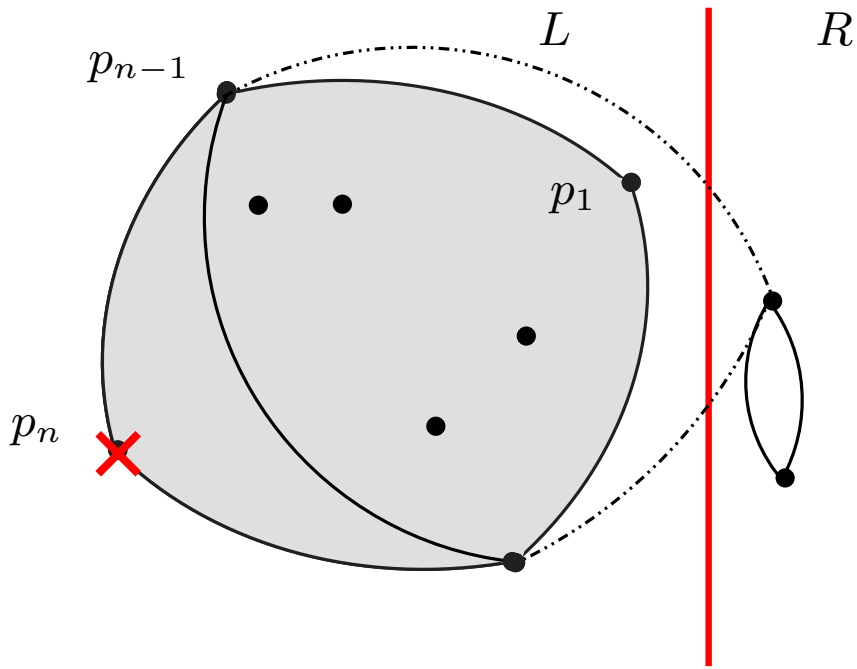
Insert the right most point q to R
Delete the left most point p from L

updating takes $O(1)$ amortized time.

Inserting m vertices to R and removing n vertices from L takes $O(n+m)$ time in total.

$\rightarrow$ updating takes $O(1)$ amortized time.

[Wang 20] Fully Dynamic r-Circular Hull



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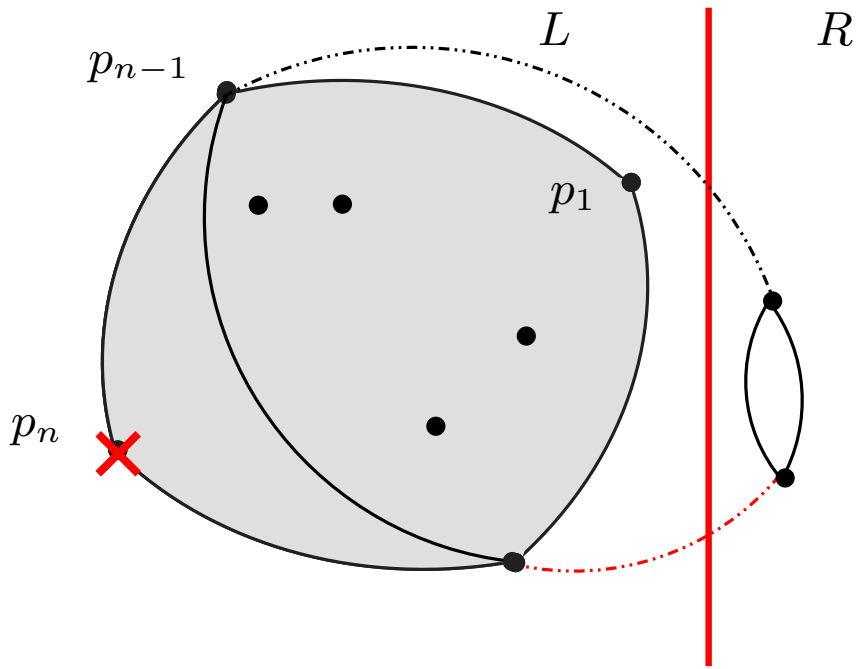

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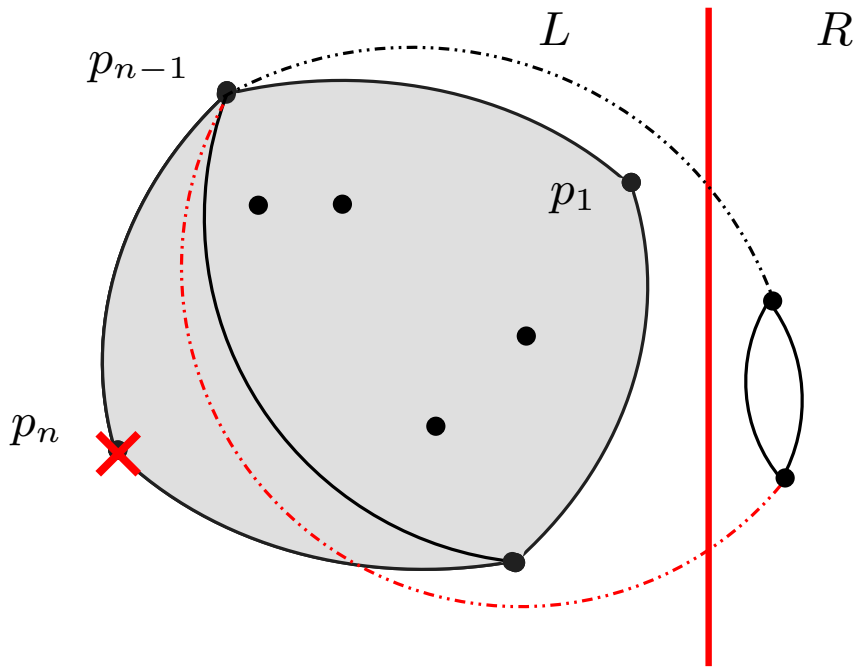

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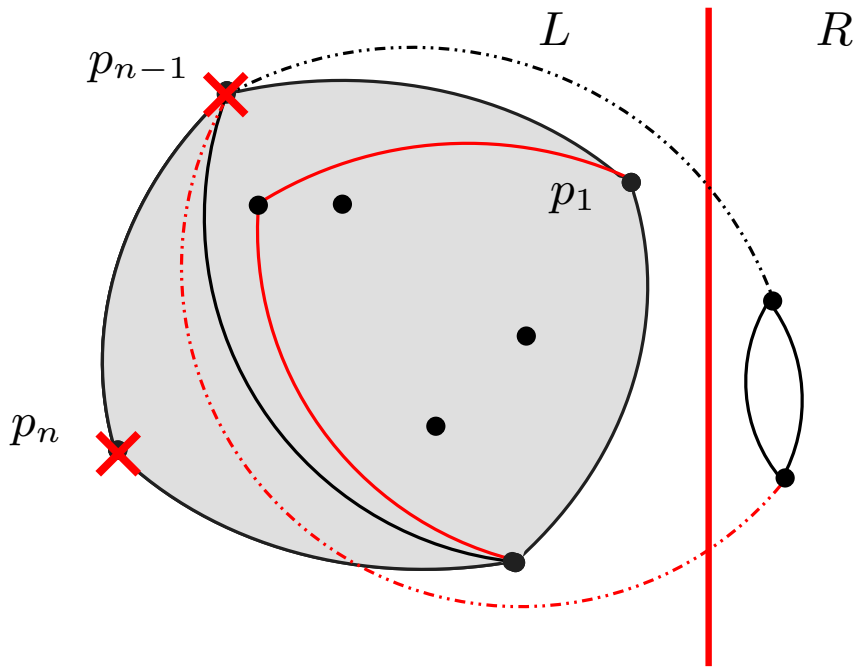

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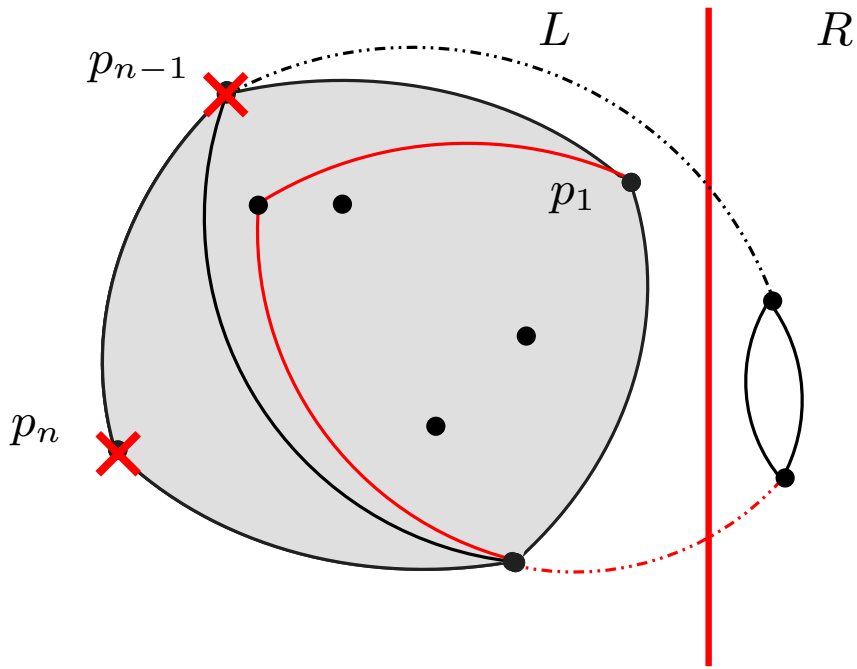

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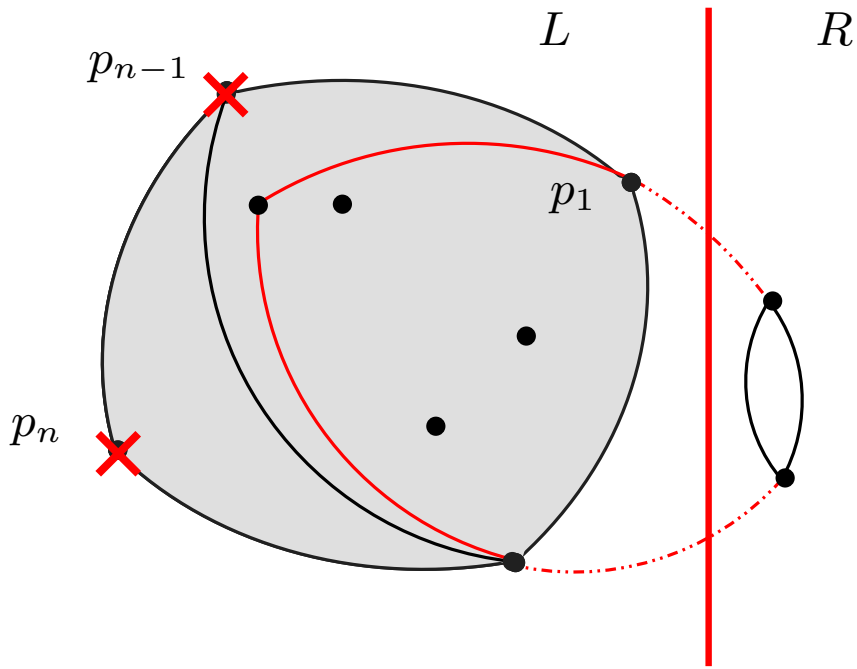

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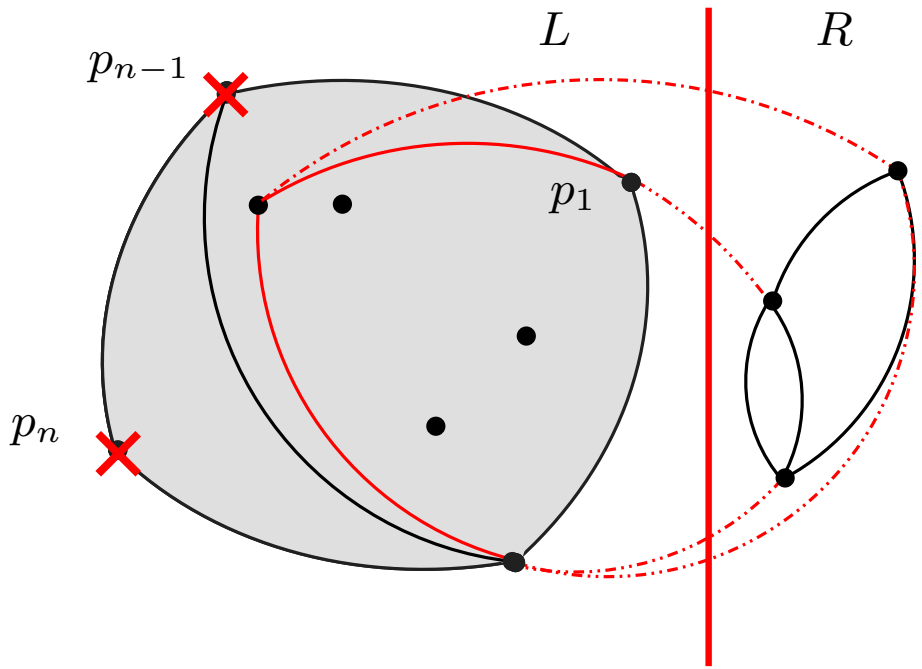

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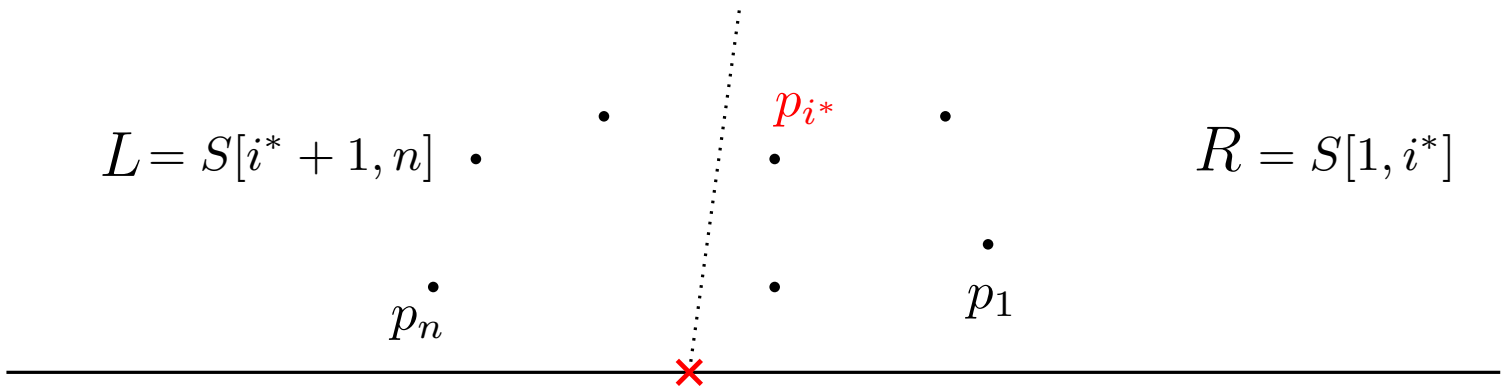
[Wang 20] Decision for Near-by-Case



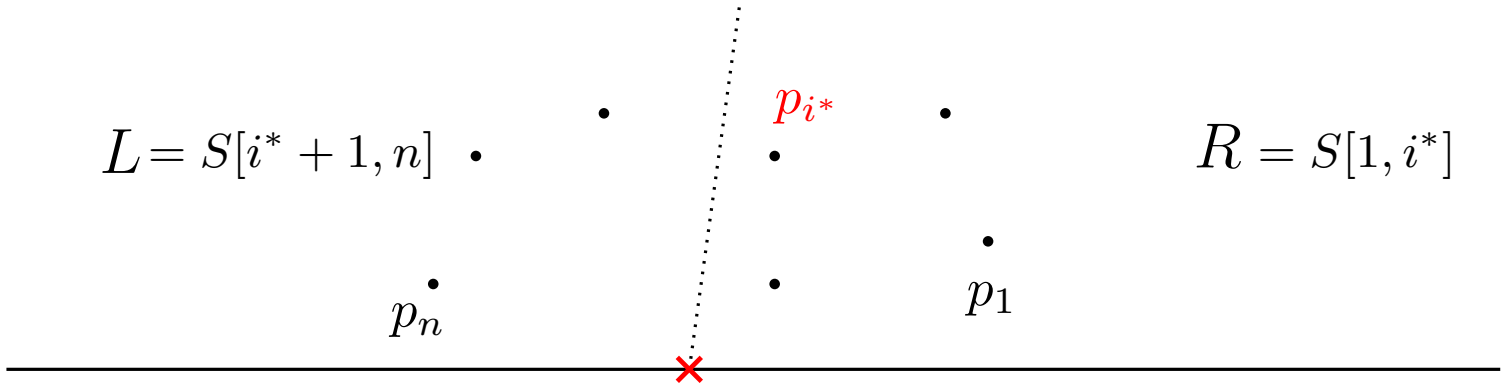
Easy case: If all points p have $y(p) > 0$?

Angular sorting $S = \{p_1, \ldots, p_n\}$ (preprocessing)

Line sweeping:
Find the maximum i^* s.t. $S[1 : i^*]$ is covered by a disk of radius r

Check $S[i^* + 1 : n]$ is covered by a disk of radius r

[Wang 20] Decision for Near-by-Case



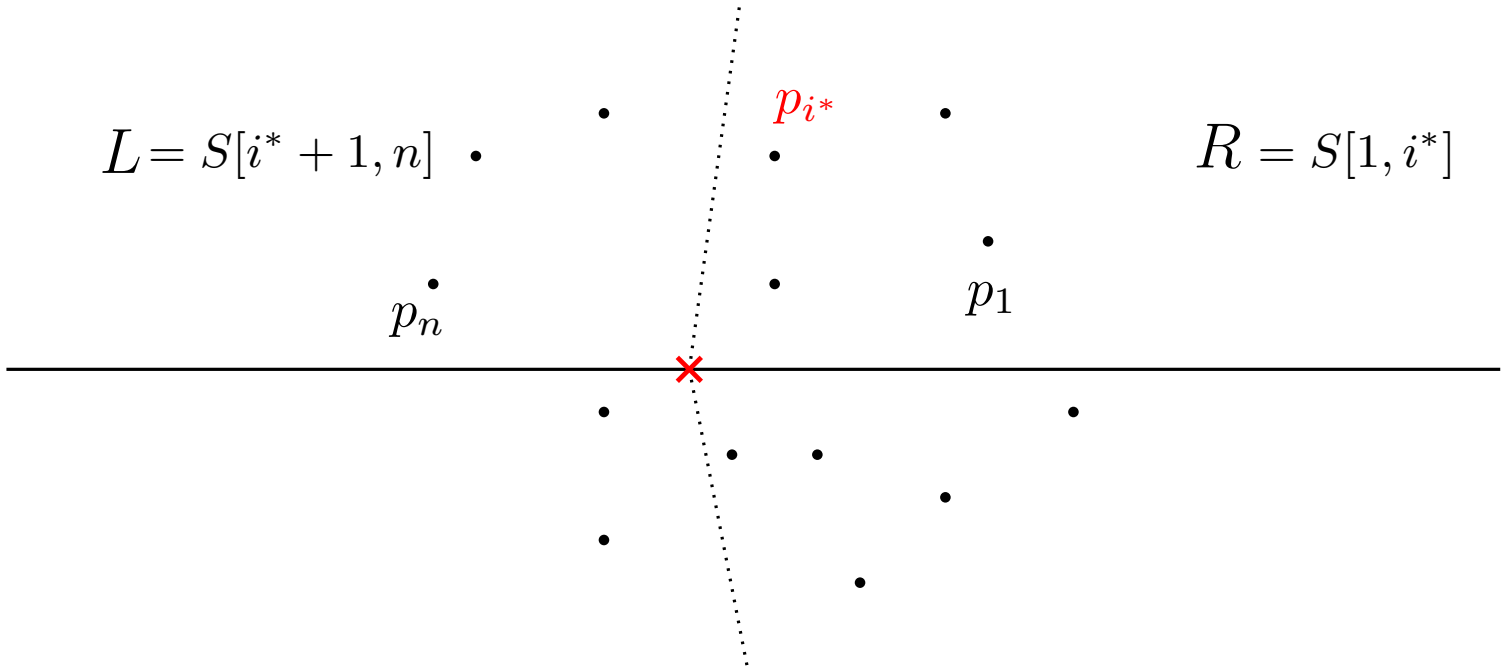
Easy case: If all points p have $y(p) > 0$?

Angular sorting $S = \{p_1, \ldots, p_n\}$ (preprocessing)

Line sweeping:
Find the maximum i^* s.t. $\alpha_r(S[1 : i^*])$ is not empty

Check $S[i^* + 1 : n]$ is covered by a disk of radius r

[Wang 20] Decision for Near-by-Case



General case ?

Angular sorting $S = \{p_1, \ldots, p_n\}$ (preprocessing)

Line sweeping:
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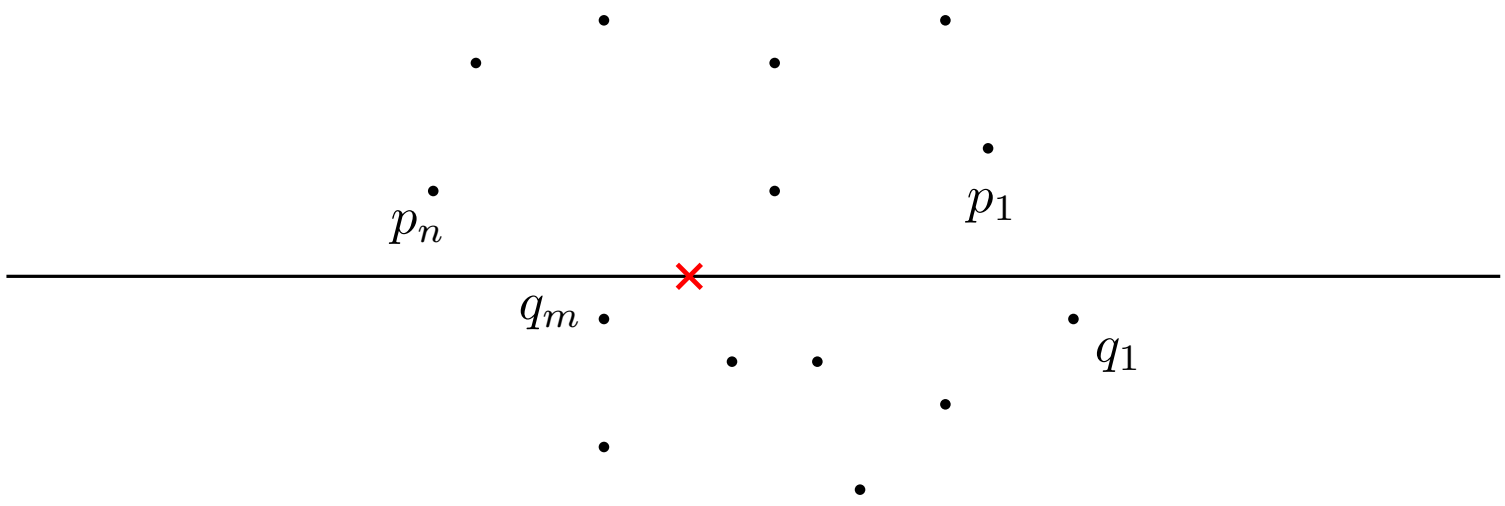
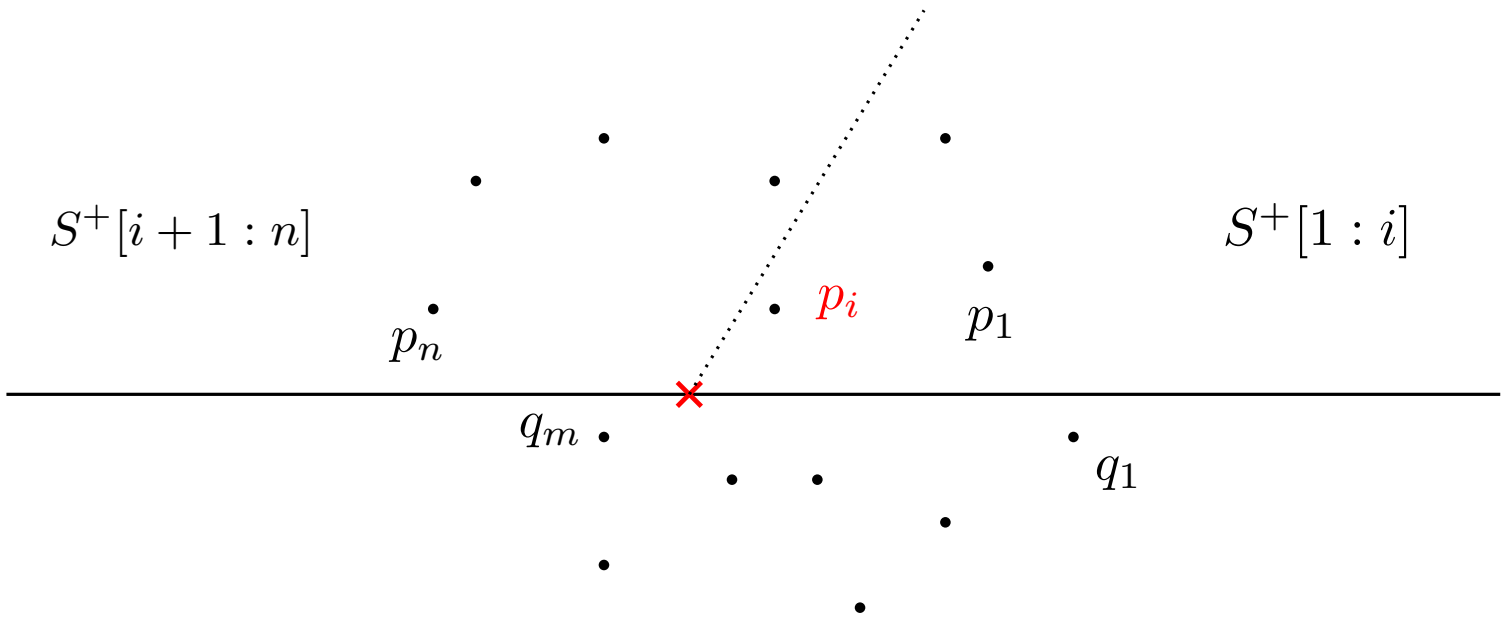
[Wang 20] Decision for Near-by-Case



Angular sorting $S^+ = \{p_1, \ldots, p_n\}$ and $S^- = \{q_1, \ldots, q_m\}$ (preprocessing)

[Wang 20] Decision for Near-by-Case



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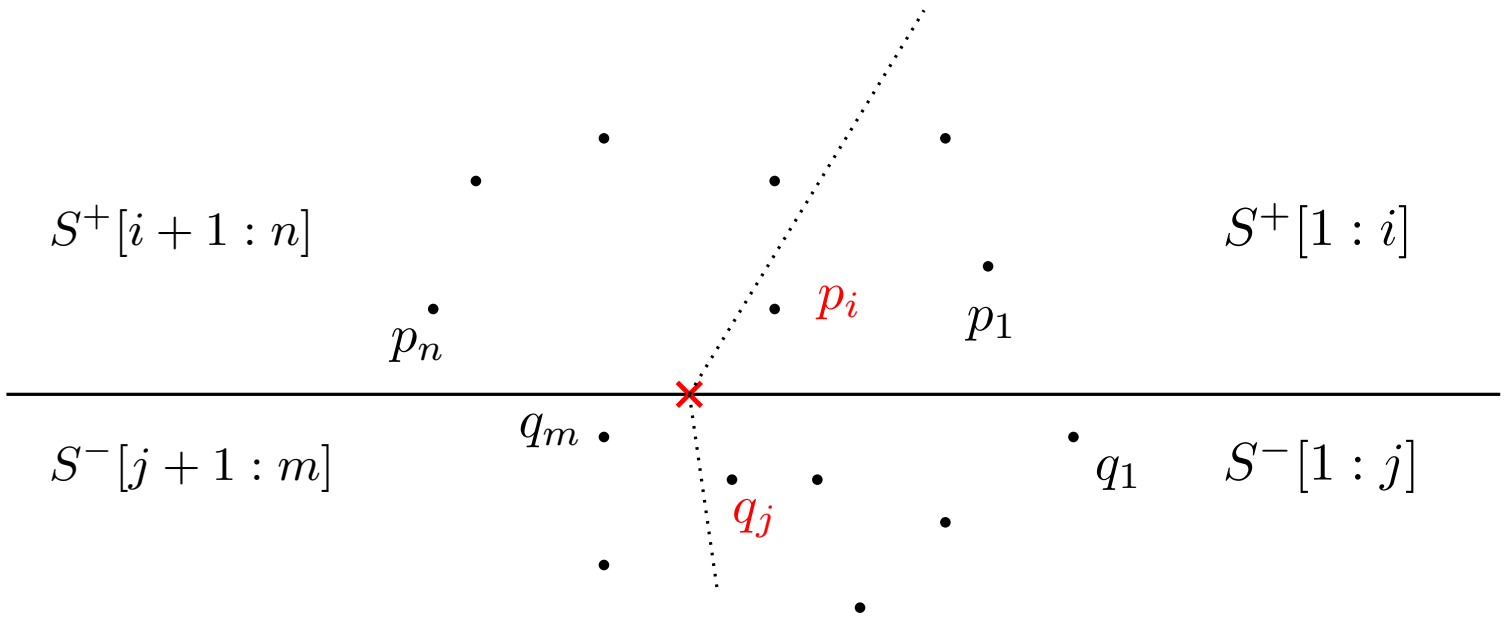
Fixing i

[Wang 20] Decision for Near-by-Case



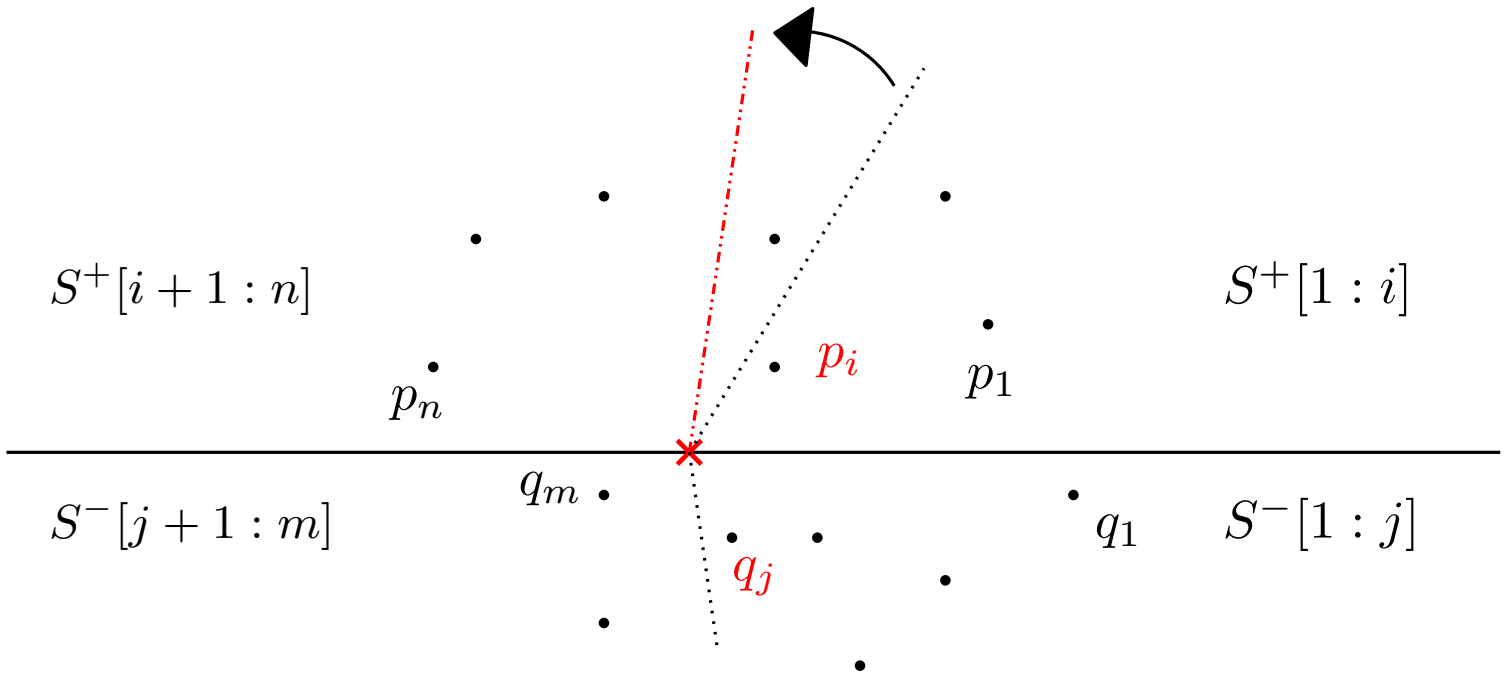
Angular sorting $S^+ = \{p_1, \ldots, p_n\}$ and $S^- = \{q_1, \ldots, q_m\}$ (preprocessing)

Fixing i

Find the maximum j s.t. $S^+[1:i] \cup S^-[1:j]$ is covered by a disk of radius r

Check $S^+[i+1:n] \cup S^-[j+1:m]$ is covered by a disk of radius r

[Wang 20] Decision for Near-by-Case



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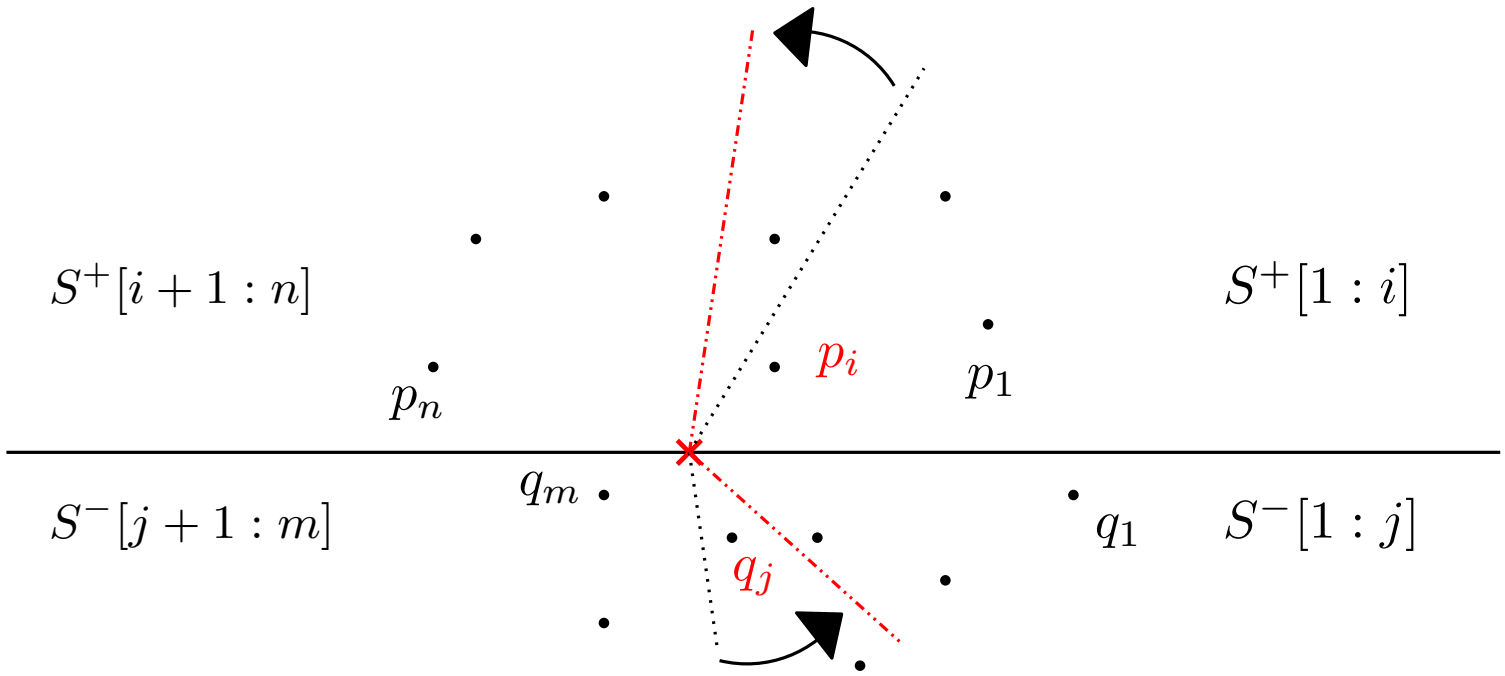

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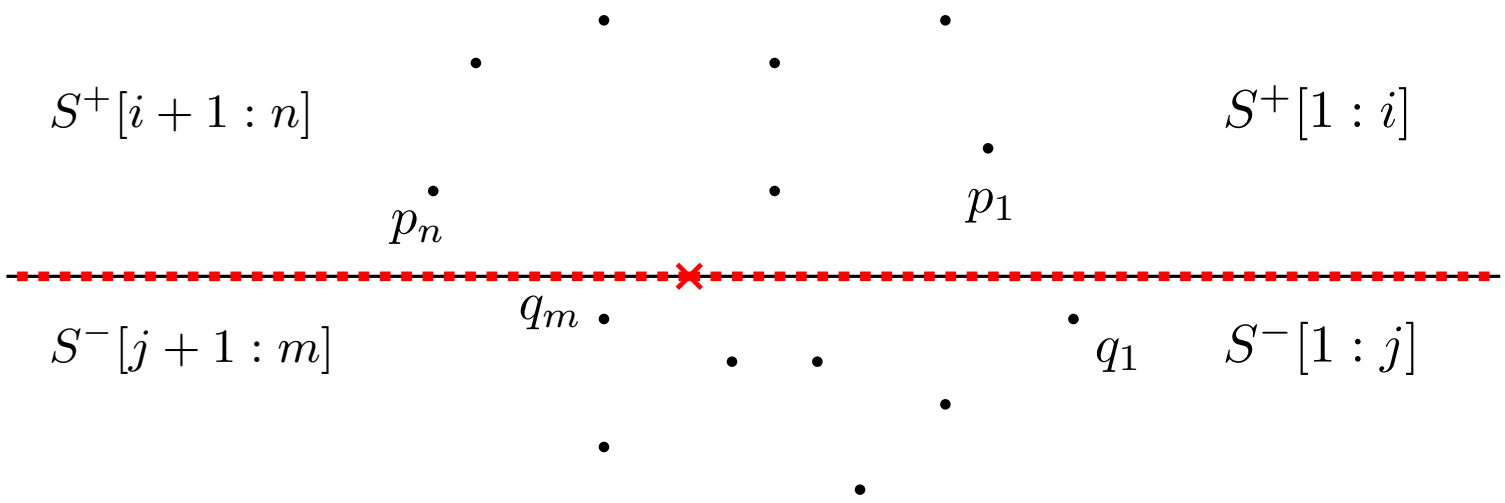
[Wang 20] Decision for Near-by-Case



Angular sorting $S^+ = \{p_1, \ldots, p_n\}$ and $S^- = \{q_1, \ldots, q_m\}$ (preprocessing)

Starting from $i = 0$ and $j = n$

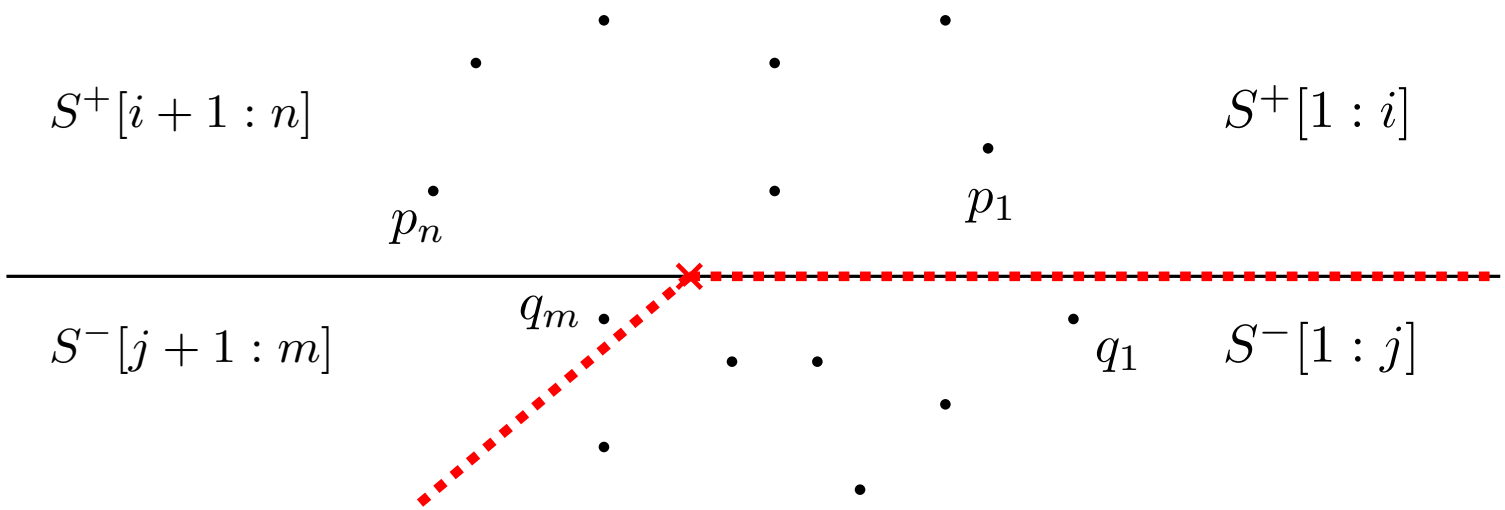
Check $S^+[1:i] \cup S^-[1:j]$ is covered by a disk of radius r

If not, $j \leftarrow j - 1$

Check $S^+[i+1:n] \cup S^-[j+1:m]$ is covered by a disk of radius r

If not, $i \leftarrow i + 1$

[Wang 20] Decision for Near-by-Case



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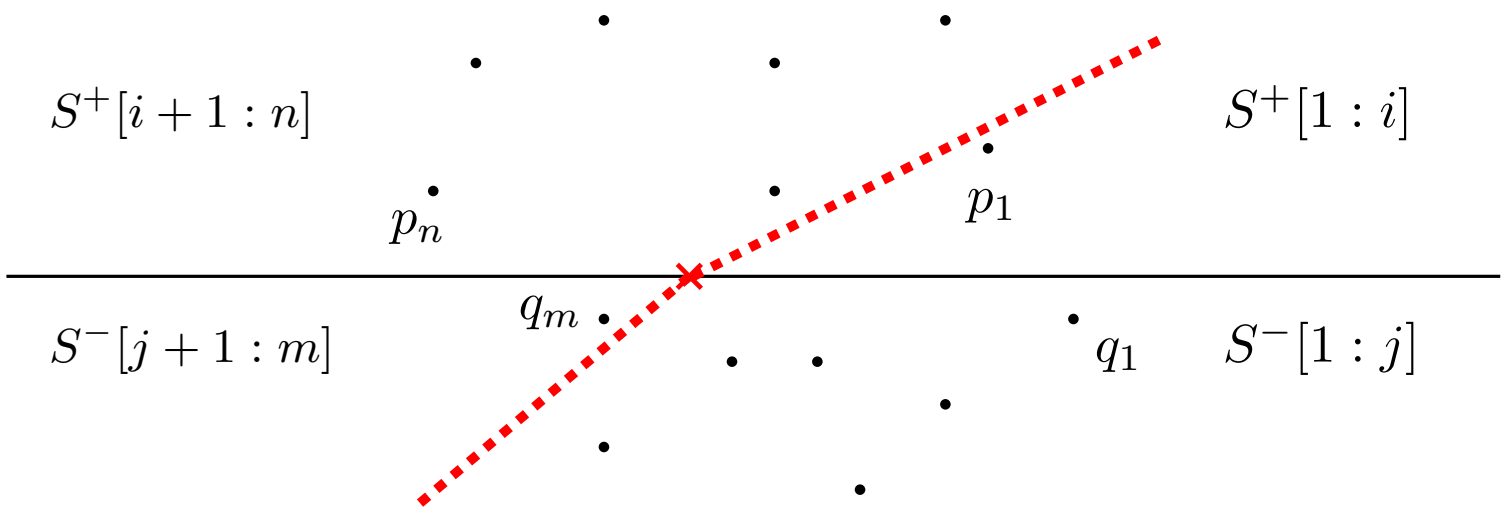
[Wang 20] Decision for Near-by-Case



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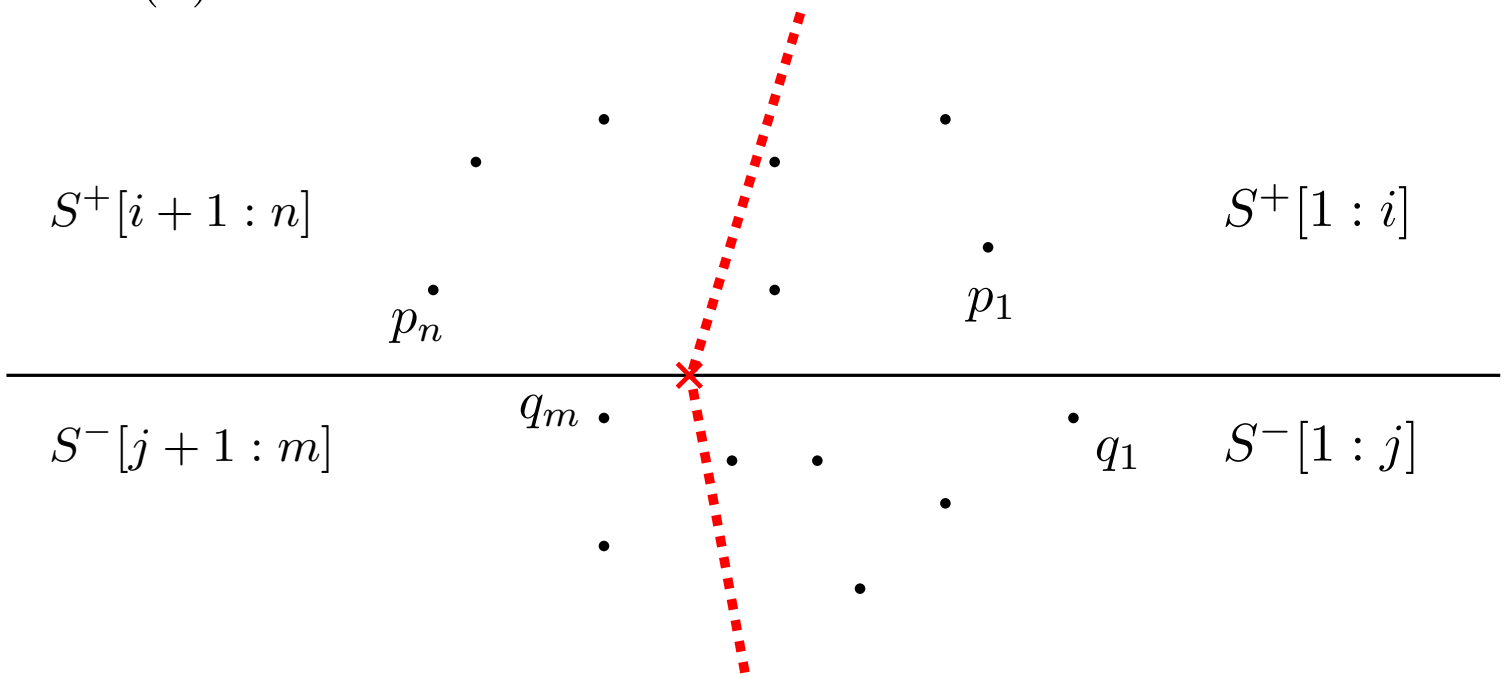
Check $S^+[i+1:n] \cup S^-[j+1:m]$ is covered by a disk of radius r

If not, $i \leftarrow i + 1$

[Wang 20] Decision for Near-by-Case

Decision takes $O(n)$ time



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Starting from $i = 0$ and $j = n$

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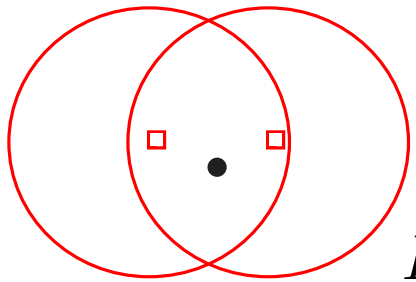
If not, $j \leftarrow j - 1$

Check $S^+[i+1:n] \cup S^-[j+1:m]$ is covered by a disk of radius r

If not, $i \leftarrow i + 1$

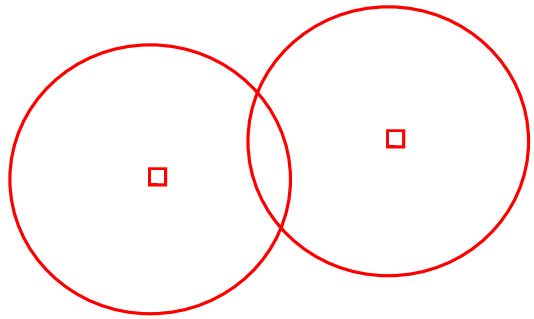
[COWX 24] Approach

Nearby 2-center problem	Distant 2-center problem



$D_1 \cap D_2 \ni p$

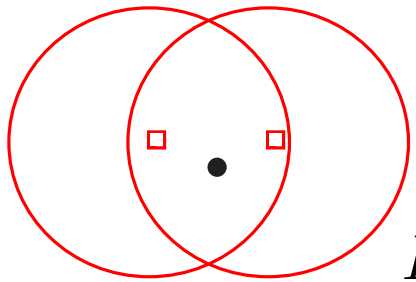
[Choi and Ahn 2021] $O(n \log n)$ time



[Eppstein 1997] $O(n \log^2 n)$ time

[COWX 24] Approach

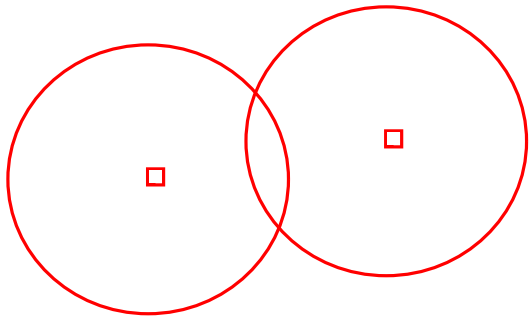
Nearby 2-center problem



$D_1 \cap D_2 \ni p$

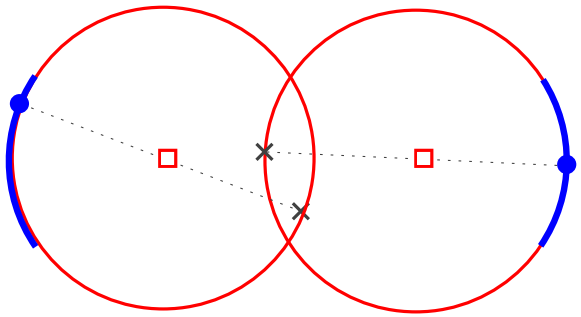
[Choi and Ahn 2021] $O(n \log n)$ time

Distant 2-center problem



[Eppstein 1997] $O(n \log^2 n)$ time

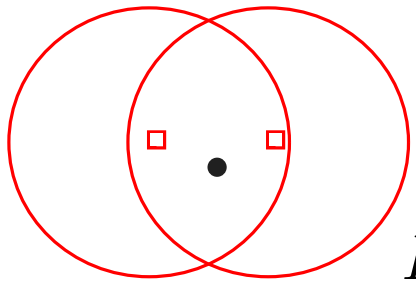
Antipodal case



$\partial(D_1)$ (and $\partial(D_2)$) has a point so that its **antipodal point** lies in $D_1 \cap D_2$

[COWX 24] Approach

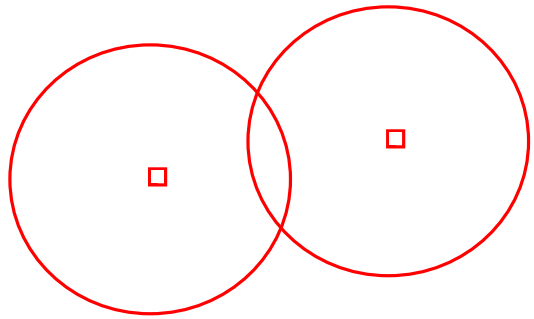
Nearby 2-center problem



$D_1 \cap D_2 \ni p$

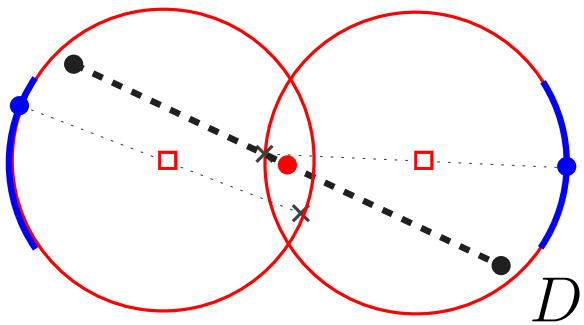
[Choi and Ahn 2021] $O(n \log n)$ time

Distant 2-center problem



[Eppstein 1997] $O(n \log^2 n)$ time

Antipodal case



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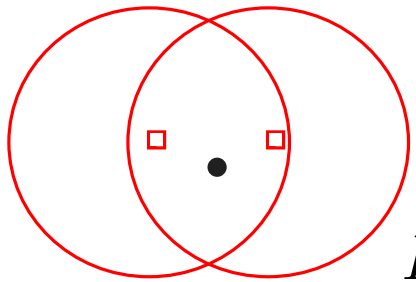

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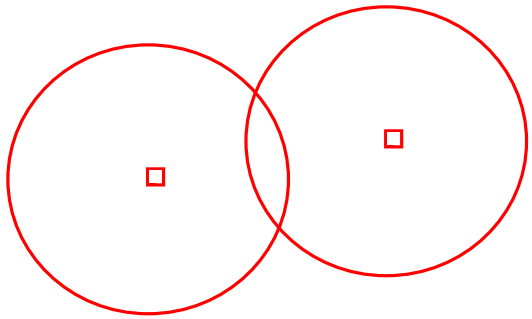

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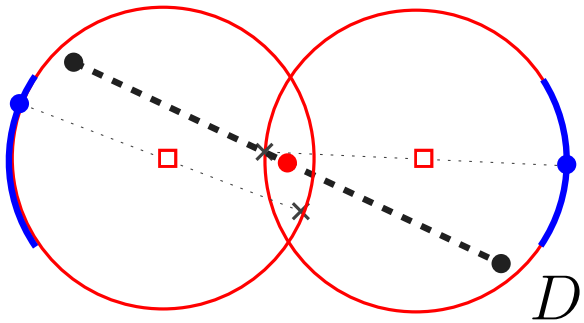

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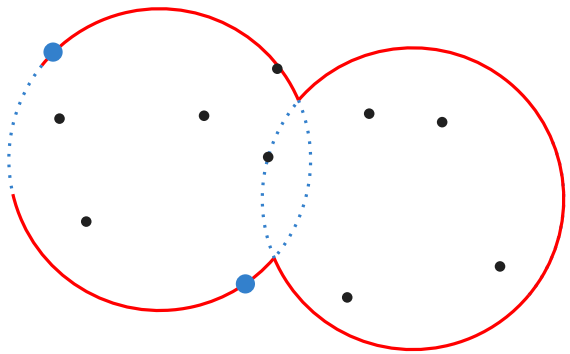
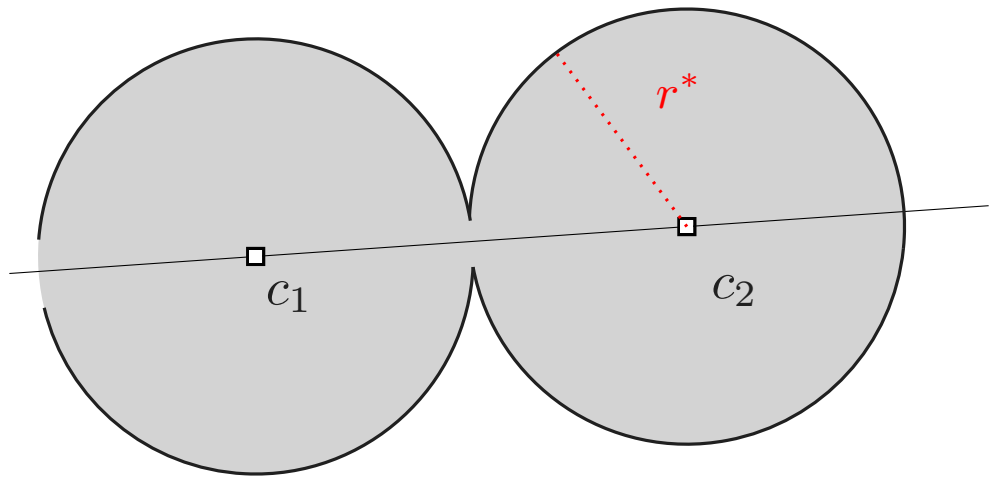
Otherwise?



[COWX 24] Non-Antipodal Case

$\partial(D_1)$ has **no** point so that its **antipodal point** lies in $D_1 \cap D_2$

A **tight solution** $\{D_1, D_2\}$: smallest disks covering P while minimizing $|c_1 c_2|$



P: n points in $\mathbb{R}^2$

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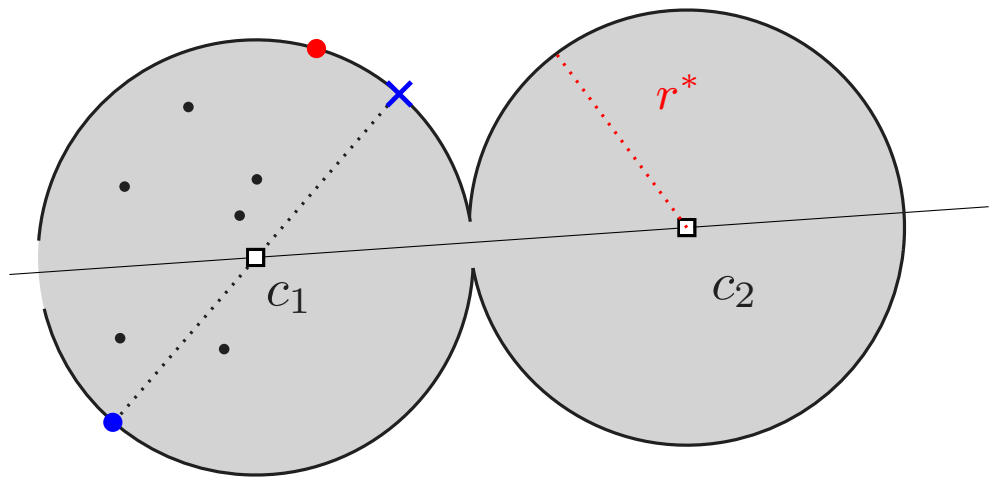

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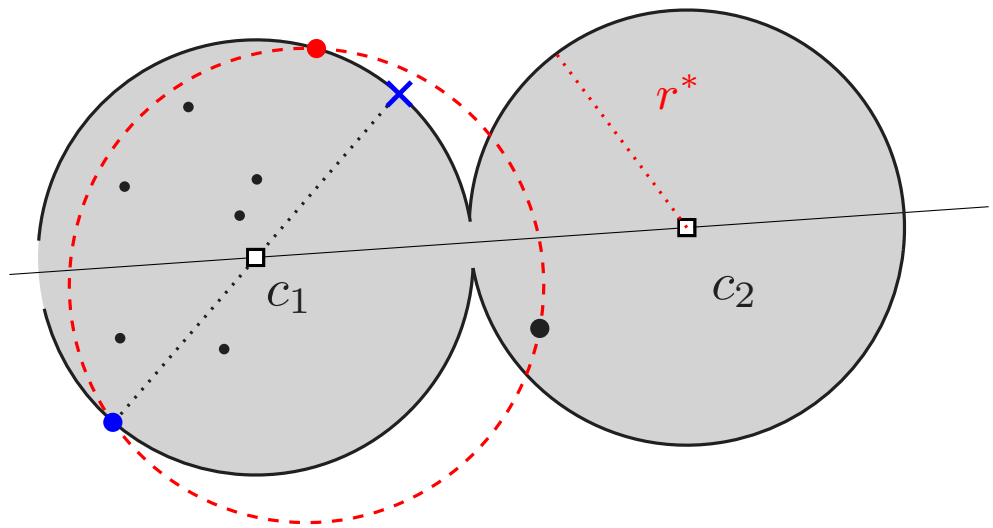

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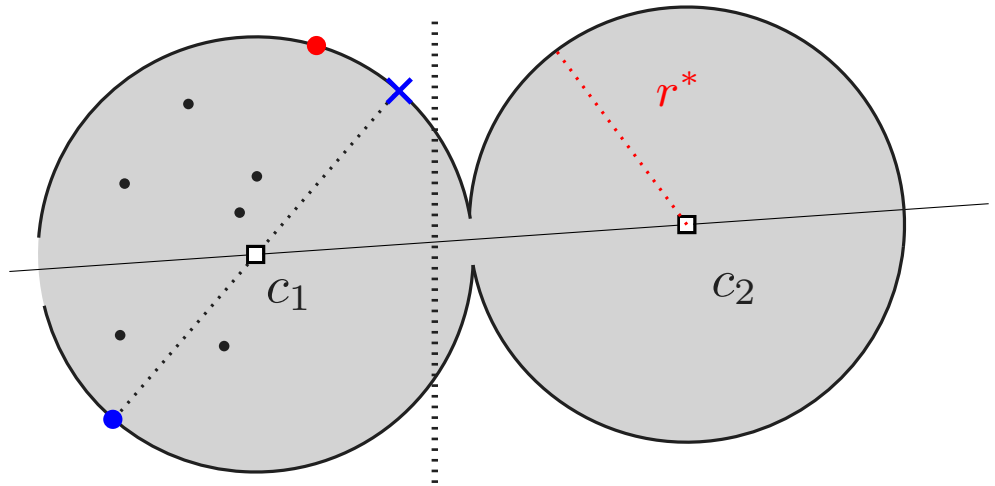
axis assumption



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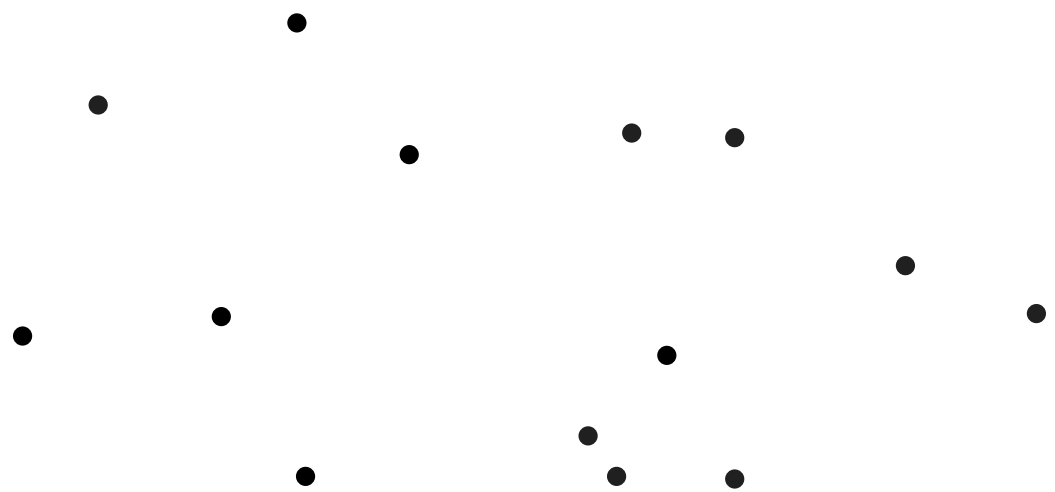
[COWX 24] Decision for Non-Antipodal Case

For given $r > 0$,
deciding whether $r \geq r^*$ or not in $O(n)$ time (after $O(n \log n)$ preprocessing time)



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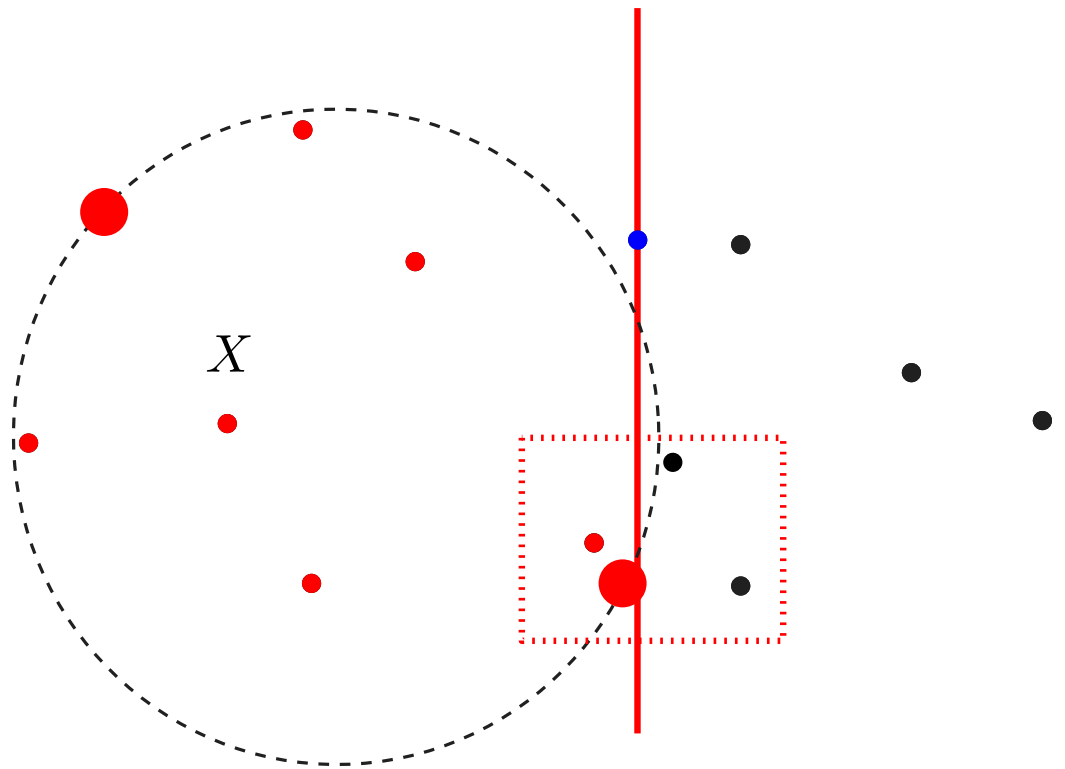
maximal r-prefix X of P



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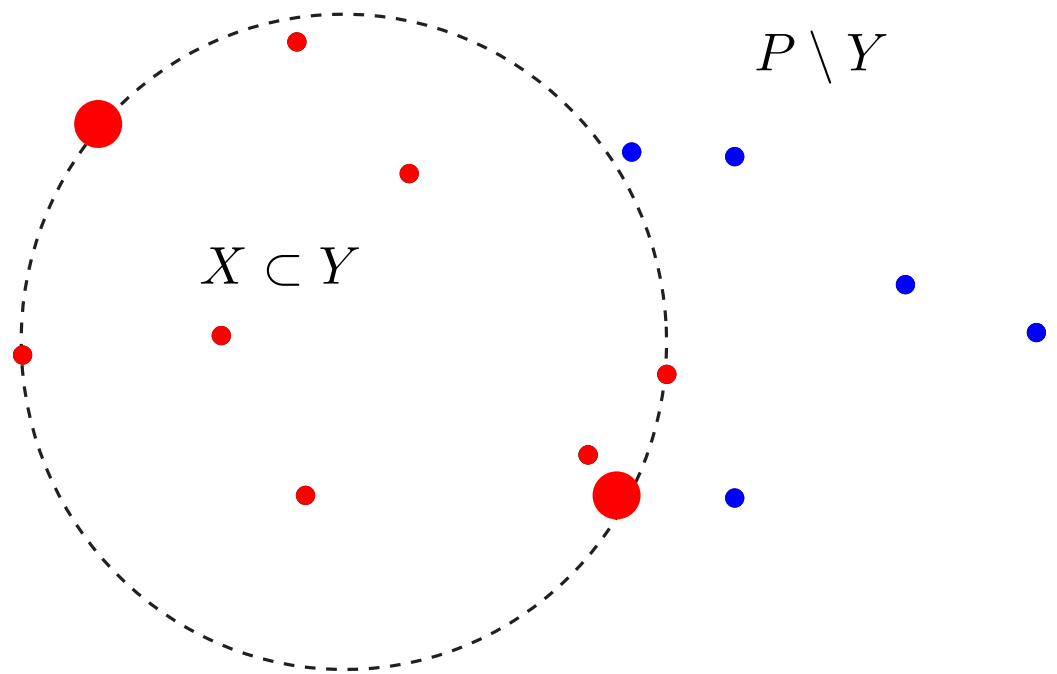

prunning $Y \subset P$: a point $q \in Y$ can be in a radius r disk along with X

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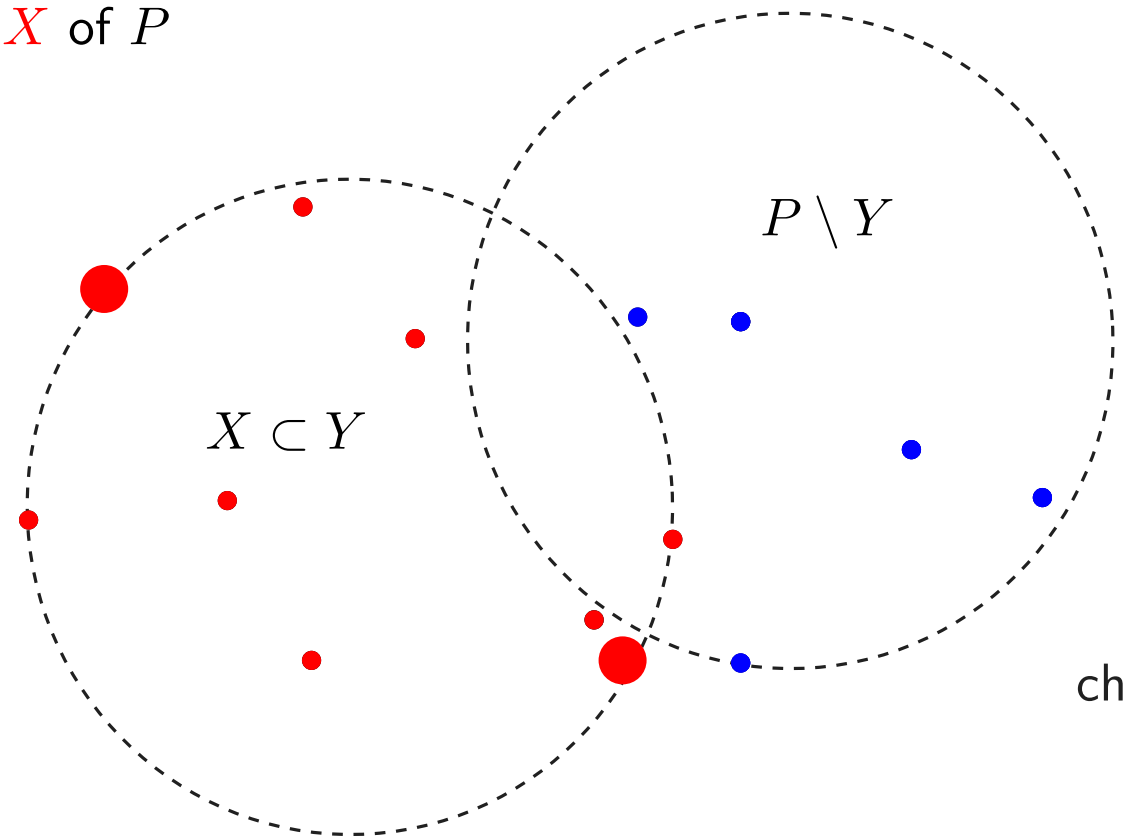


checking Y and $P \setminus Y$

prunning $Y \subset P$: a point $q \in Y$ can be in a radius r disk along with X

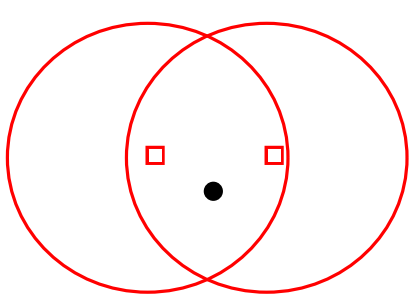
Summary

P: n points in the plane

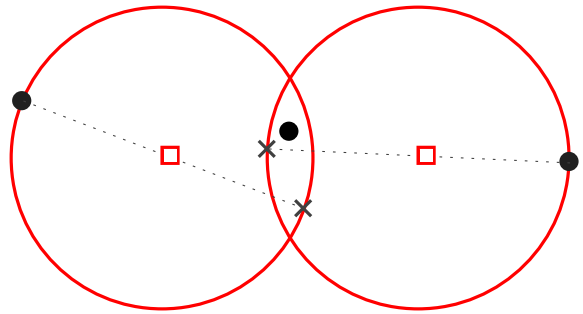
Preprocessing. $O(n \log n)$ time

- r_n: the optimal solution of the **nearby 2-center** problem of P
- r_a: the optimal solution of the **antipodal 2-center** problem of P
- P_γ: sorted P w.r.t each γ of 100 directions evenly splitting the plane

nearby 2-center problem	antipodal 2-center problem



$o \in D_1 \cap D_2$



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Decision for $r \geq$ OPT(P) or not. $O(n)$ time

- If $r \geq r_n, r_a$, then **True**.
- Otherwise, for each sorted P_γ,

Decision $r \geq r^*$ or not

ep 1) Find the maximal prefix X of P_γ

ep 2) Prune $Y \subset P$ w.r.t X

ep 3) Check Y and $P \setminus Y$

Summary

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- r_n: the optimal solution of the **nearby 2-center** problem of P
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Using the Cole's parameteric search

The planar 2-center problem can be solved in $O(n \log n)$ time