Tree-Packing Revisited: Faster Fully Dynamic Min-Cut and Arboricity

SODA 2025

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Der Wissenschaftsfonds.

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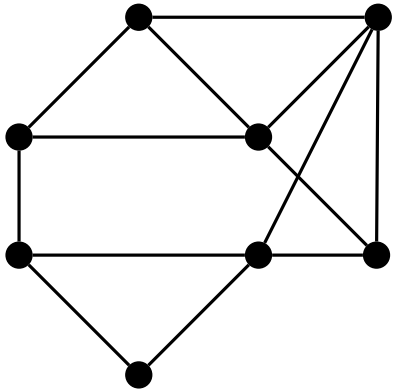
- Graph $G = (V, E)$ with edge updates: insertions/deletions.



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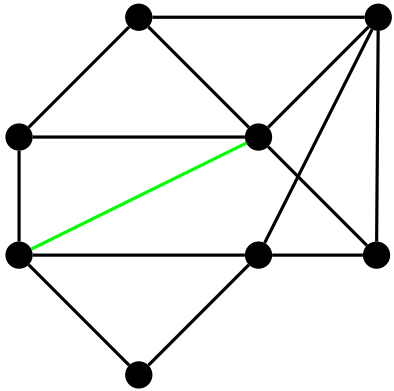

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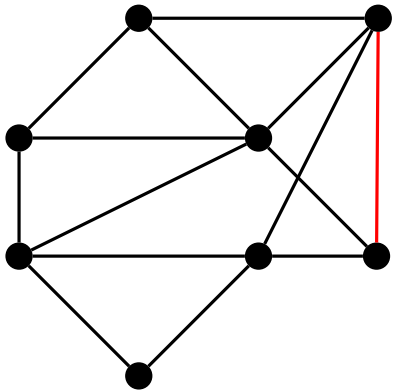

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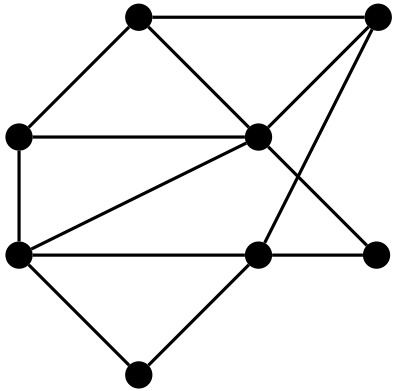

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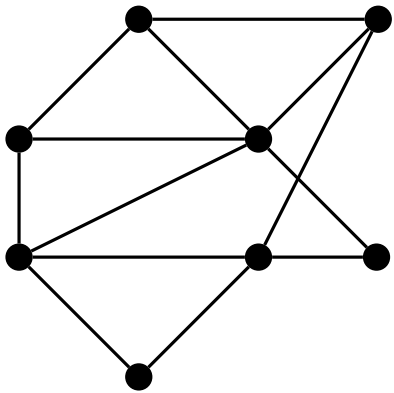

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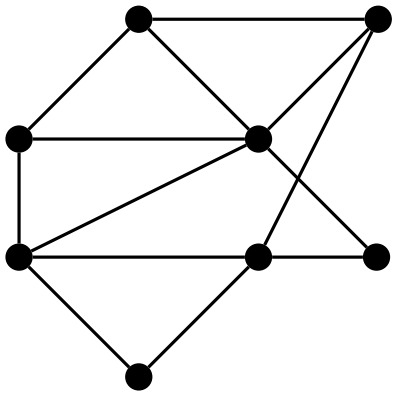

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- Maintain the answer to a problem.
- Goal: low update time.
  - Worst-case.
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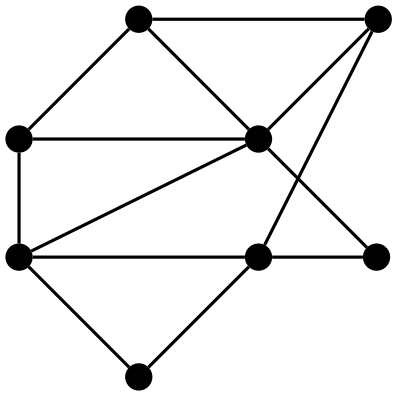
- Adversary determines updates:
  - Oblivious Adversary: updates fixed beforehand.
  - Adaptive Adversary: updates can depend on (random) choices of the algorithm.



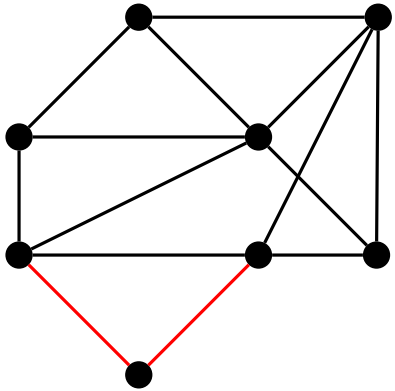
Part 1: Dynamic Min-Cut

Part 2: Dynamic Arboricity

Part 3: More on Tree-Packings

Problem

- Min-Cut (connectivity): find λ, the minimum number of edges disconnecting the graph.



Dynamic Min-Cut: Main Result

Theorem

Unweighted, undirected (multi-)graph $G = (V, E)$. There exists a deterministic dynamic algorithm that maintains the exact min-cut value λ if $\lambda \leq \lambda_{\max}$ in $\tilde{O}(\lambda_{\max}^{5.5}\sqrt{n})$ worst-case update time.

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Corollary

Simple, unweighted, undirected graph $G = (V, E)$. There exists a deterministic dynamic algorithm that maintains the exact min-cut value λ with amortized update time

$$\tilde{O}(\min\{m^{1-1/12}, m^{11/13}n^{1/13}, n^{3/2}\}).$$

Previously: $\tilde{O}(m^{1-1/31})$ [Goranci/Henzinger/Nanongkai/Saranurak/Thorup/Wulff-Nilsen '23].

Tree-Packing and Min-Cut

Definitions

- Tree-packing $\mathcal{T} = \{T_1, T_2, \dots\}$: a family of trees in G such that [...].
- Load $L^{\mathcal{T}}(e)$: the number of trees in $\mathcal{T}$ that contain an edge e.
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- Static [Karger '00] + [...].
- Distributed [Daga/Henzinger/Nanongkai/Saranurak '19, Dory/Efron/Mukhopadhyay/Nanongkai '21].
- Dynamic [Thorup/Karger '00, Thorup '07].

Static: Disjoint Trees [Karger '00]

Theorem [Nash-Williams '61, Tutte '61]

A graph with minimum cut λ contains at least $\lambda/2$ disjoint spanning trees.

- Take $\mathcal{T} = \{T_1, T_2, \dots\}$ a maximum packing of disjoint spanning trees.

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- Sample $\lambda \to \log n$ to obtain $O(m \log^3 n)$ time.

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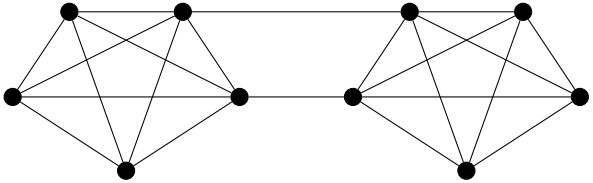

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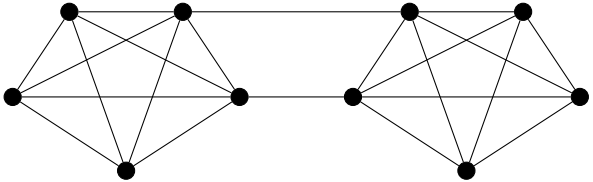

- Spoiler:

$$\text{arboricity} = \frac{1}{\min_{e \in E} \ell^*(e)}.$$

Ideal Tree-Packing – Formal

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  - For each $S \in \mathcal{P}^*$, recurse on the subgraph $G[S]$.

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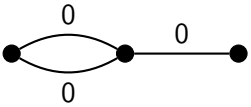

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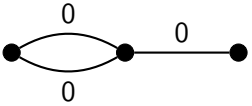

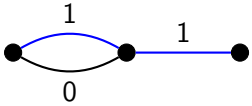

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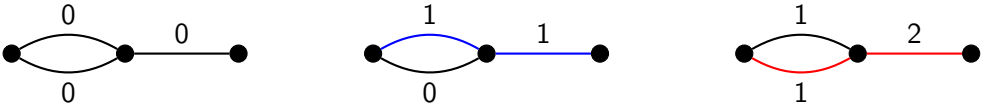

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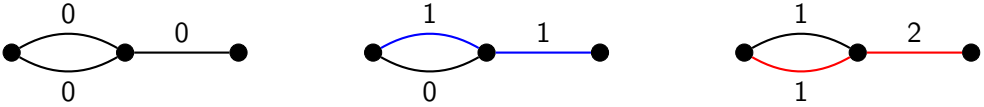

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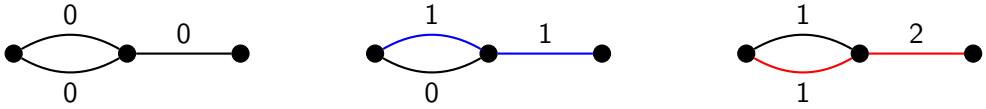

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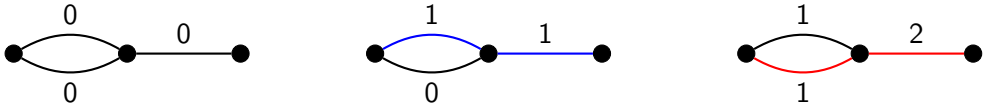

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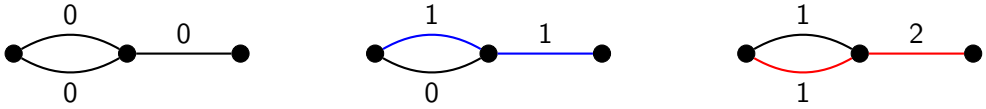

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  - Find greedy trees (easy).
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Contracted Graphs

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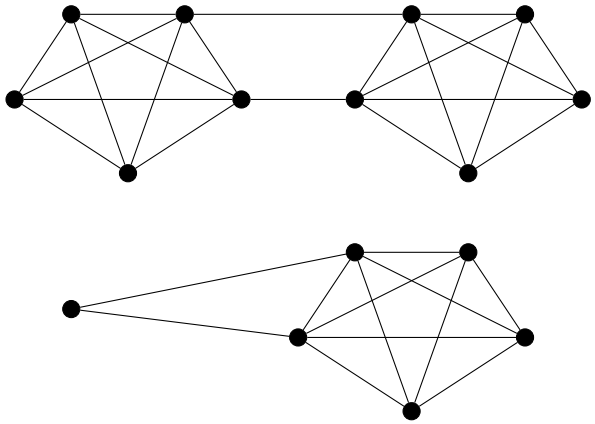

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- On each level, $E(G/\{e \in E : \ell^*(e) < c)\}$ corresponds to the minimum partition.
- We get $|E(G/\{e \in E : \ell^*(e) < c)\} \leq c^{-1}(|V(G/\{e \in E : \ell^*(e) < c\})| - 1) \approx \frac{\lambda}{2}|V(G/\{e \in E : \ell^*(e) < c\})|$.

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- Contradiction.

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- So a trivial min-cut of $G/\{e \in E : \ell^*(e) < c\}$ is a trivial min-cut of $G/\{e \in E : \ell^{\mathcal{T}}(e) < c\}$.

Algorithm

- Maintain tree-packing $\mathcal{T}$ of size $|\mathcal{T}| = \Theta(\lambda_{\max}^3 \log m)$.
  - Decreased recourse ($\lambda_{\max}^5$ instead of $\lambda_{\max}^6$).
- Maintain 1-respecting cuts for each $T \in \mathcal{T}$ in $\tilde{O}(\sqrt{\lambda_{\max} n})$ worst-case update time.
- Maintain trivial cuts in contracted graph $G/\{e \in E : \ell^{\mathcal{T}}(e) < c\}$ using top trees.

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Theorem

Unweighted, undirected (multi-)graph $G = (V, E)$. There exists a deterministic dynamic algorithm that maintains the exact min-cut value λ if $\lambda \leq \lambda_{\max}$ in $\tilde{O}(\lambda_{\max}^{5.5}\sqrt{n})$ worst-case update time.

Previously: $\tilde{O}(\lambda_{\max}^{14.5}\sqrt{n})$ [Thorup '07].

Part 1: Dynamic Min-Cut

Part 2: Dynamic Arboricity

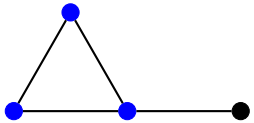
Part 3: More on Tree-Packings

Arboricity

Define

- The fractional arboricity:

$$\alpha := \max_{S \subseteq V} \frac{|E(S)|}{|S| - 1}.$$



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Theorem. We have $\tilde{\alpha} = \lceil \alpha \rceil$.

Arboricity

Theorem

Unweighted, undirected (multi-)graph $G = (V, E)$. There exists a deterministic dynamic algorithm that maintains a $(1+\varepsilon)$-approximation of the fractional arboricity α when $\alpha \leq \alpha_{\max}$ in $O(\alpha_{\max} \log^6 m/\varepsilon^4)$ amortized update time.

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- Previously only $(2+\varepsilon)$-approximation.
- Worst-case (Las Vegas) with $m^{o(1)}$.
- We showed that $\alpha = \frac{1}{\min_{e \in E} \ell^*(e)}$ and use a greedy tree-packing.
  - [Cen/Fleischmann/Li/Li/Panigrahi '25] Used this fact to compute static arboricity faster
- Additional tricks to lower the recourse.

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Fully dynamic $(1+\varepsilon)$-approximate arboricity

- Deterministic $\sim \alpha \leq \sqrt{m}$ update time.

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- Oblivious adversary: $\text{poly}(\log n, \varepsilon^{-1})$ amortized update time
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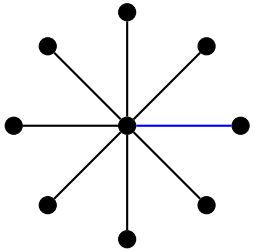
  - Standard uniform sampling $\sim \frac{\log n}{\alpha \varepsilon^2}$.
- Adaptive adversary: $\text{poly}(\log n, \varepsilon^{-1})$ amortized update time
  - Fancy sampling.

Fancy Sampling

Uniform sampling $\sim \frac{\log n}{\alpha \varepsilon^2}$.

- Problem: Adaptive adversary can delete sampled edges.



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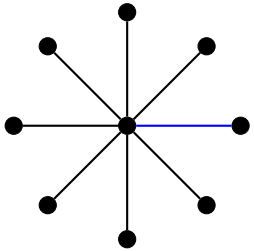

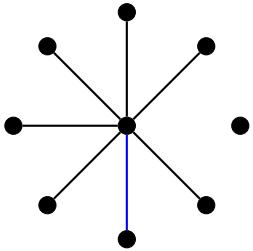

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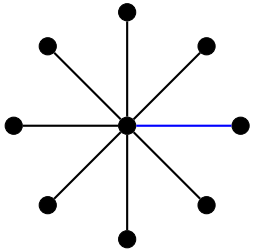

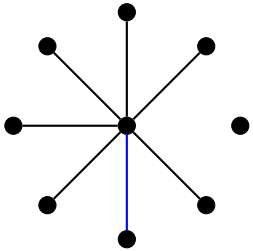

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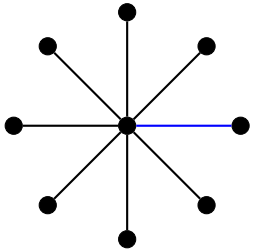
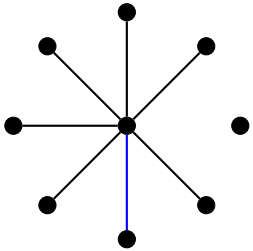
  - In sampled graph $\approx n/\alpha$.
- Solution: Use out-orientations such that each vertex 'owns' $O(\alpha)$ edges.
  - In sampled graph $O(\alpha \cdot \frac{\log n}{\alpha\varepsilon^2}) = O(\frac{\log n}{\varepsilon^2})$.





Part 1: Dynamic Min-Cut
Part 2: Dynamic Arboricity

Part 3: More on Tree-Packings

More on Tree-Packings

- Unweighted, undirected (multi-)graph $G = (V, E)$.
- Goal: $|\ell^{\mathcal{T}}(e) - \ell^*(e)| \leq \varepsilon/\lambda$ for all $e \in E$.
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There exists such a tree-packing $\mathcal{T}$ with $|\mathcal{T}| = \Theta(\lambda/\varepsilon)$ trees.

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Theorem (joint work with Mara Grilnberger)

Let $\gamma \in [\lambda, \alpha]$ and let $\mathcal{T}$ be a greedy tree-packing with $|\mathcal{T}| \geq 3\gamma \cdot \log n/\varepsilon^2$. Then

$$|\ell^{\mathcal{T}}(e) - \ell^*(e)| \leq \varepsilon \ell^*(e)$$

for all $e \in E$ with $\ell^*(e) \geq 1/\gamma$ and

$$|\ell^{\mathcal{T}}(e) - \ell^*(e)| \leq \varepsilon/\gamma$$

for all $e \in E$ with $\ell^*(e) \leq 1/\gamma$.

In particular: need $\Omega(\alpha \cdot \log n/\varepsilon^2)$ trees to approximate the arboricity instead of $\Omega(\alpha^2 \cdot \log n/\varepsilon^2)$ trees.

Conclusion

Contributions

1. New insight on relation between tree-packings and min-cut.
2. First algorithms for dynamic $(1 \pm \varepsilon)$-approximate arboricity via tree-packings.
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Thank you!