ATCS Seminar, 27.01.2026

Small Empty Cycles in Simple Drawings of K_n

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Simple Drawings

A simple drawing of a graph G in the plane:

- vertices: pairwise distinct points
- edges: Jordan arcs between the respective end-vertices s.t.
  - do not pass through other vertices
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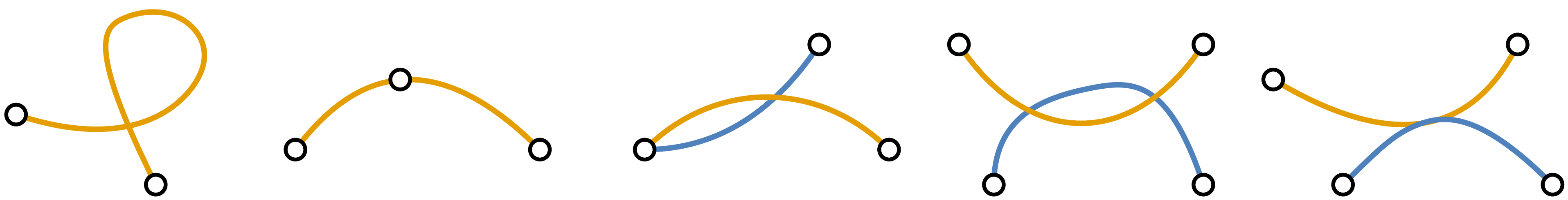

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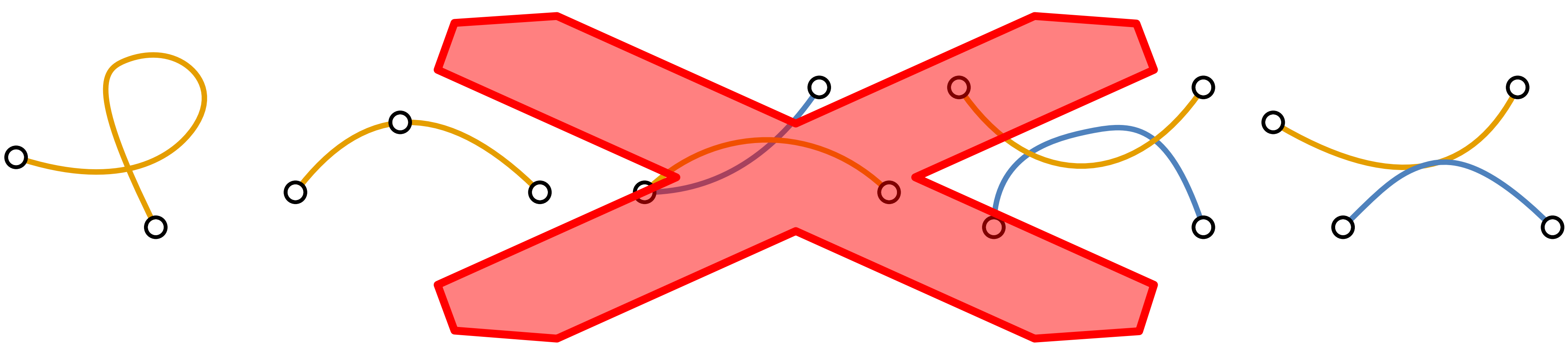

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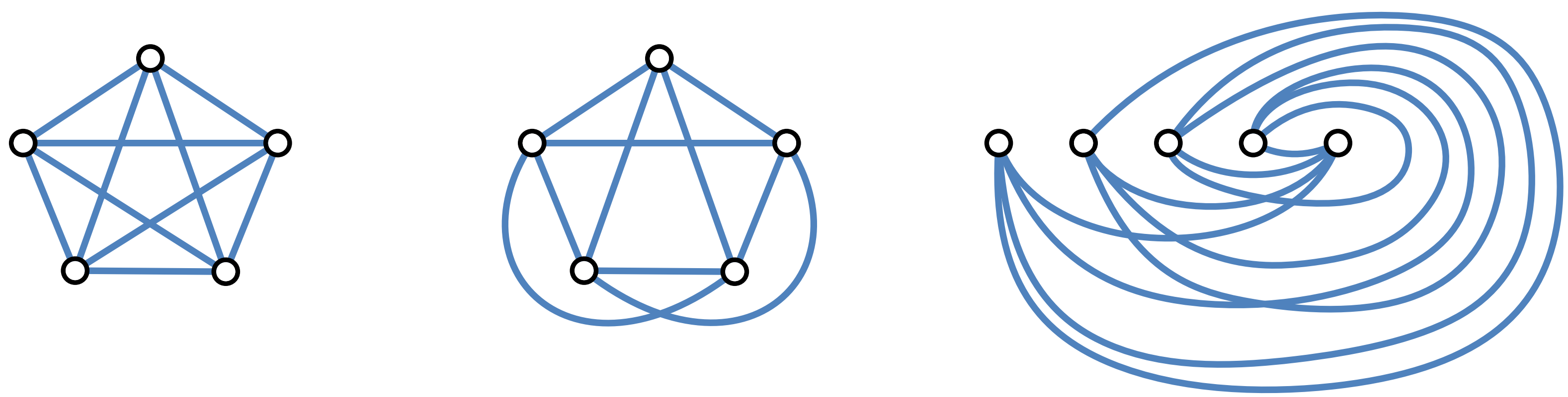
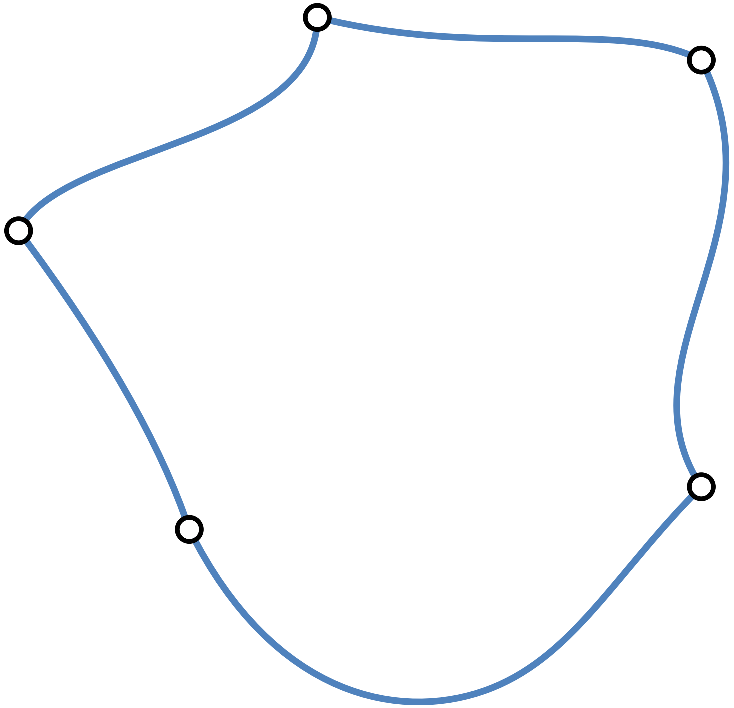


Empty Plane k-Cycles

Crossing-free cycle on k vertices (for $k \geq 3$)



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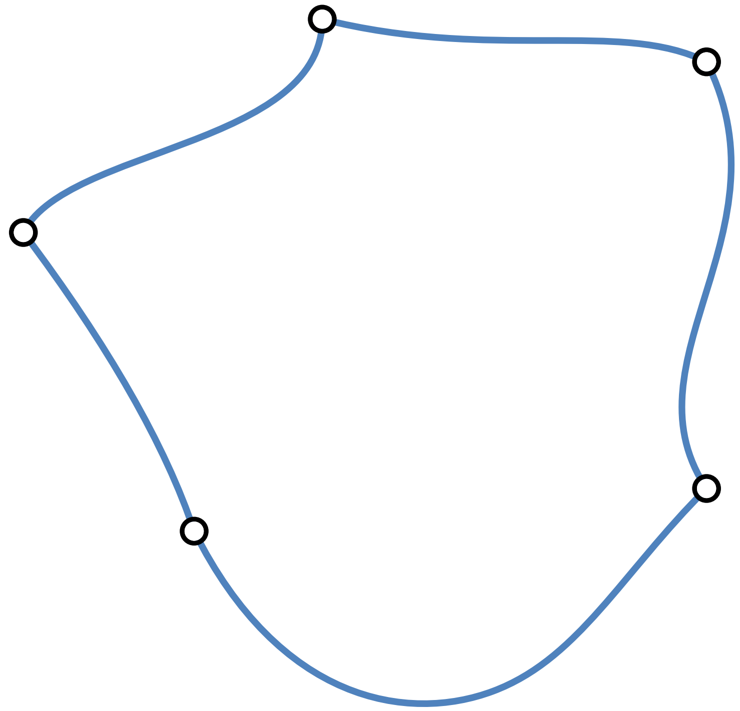

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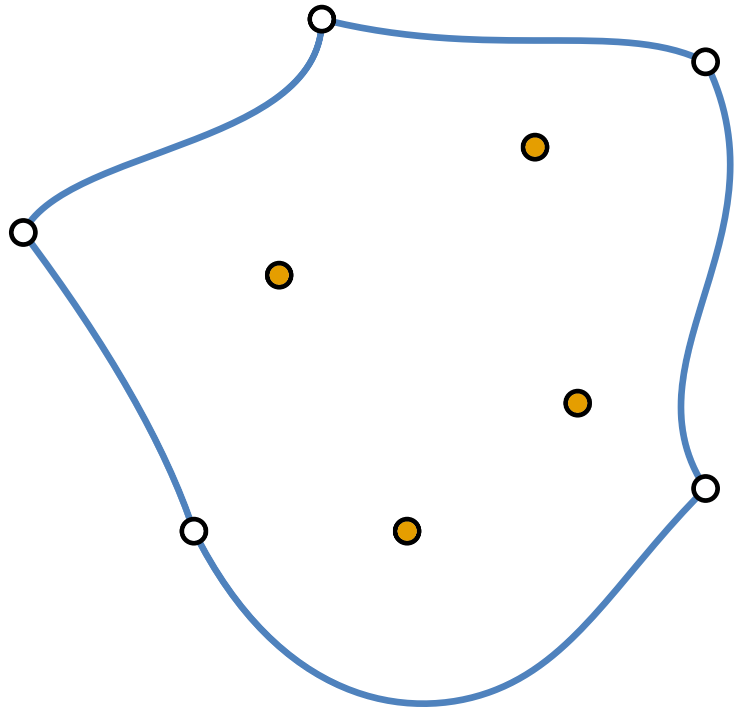

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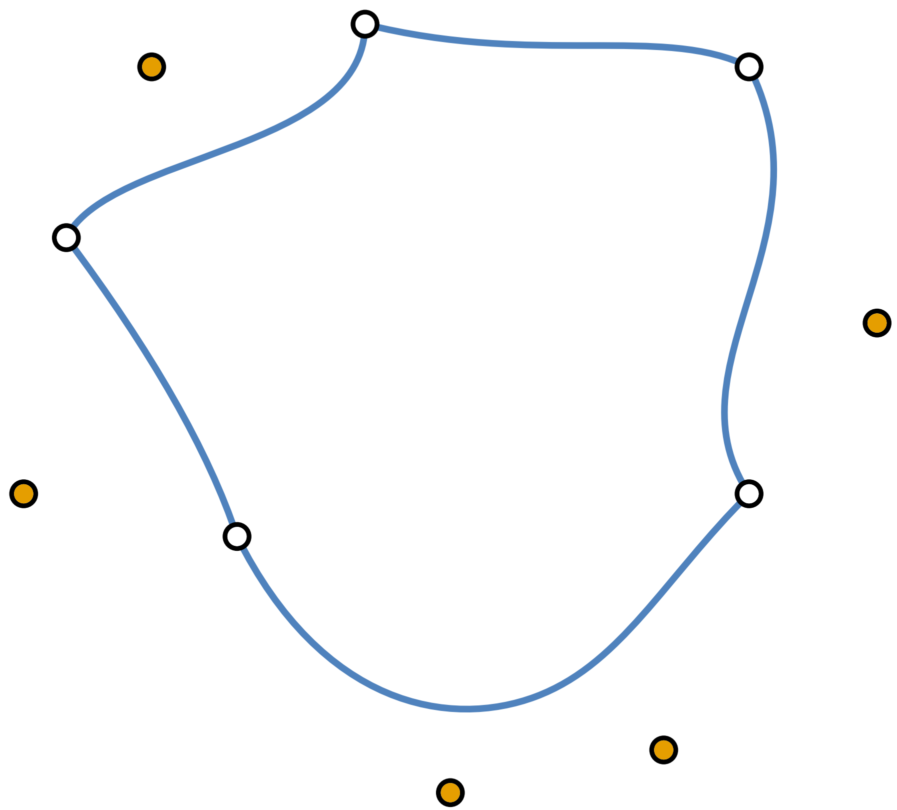

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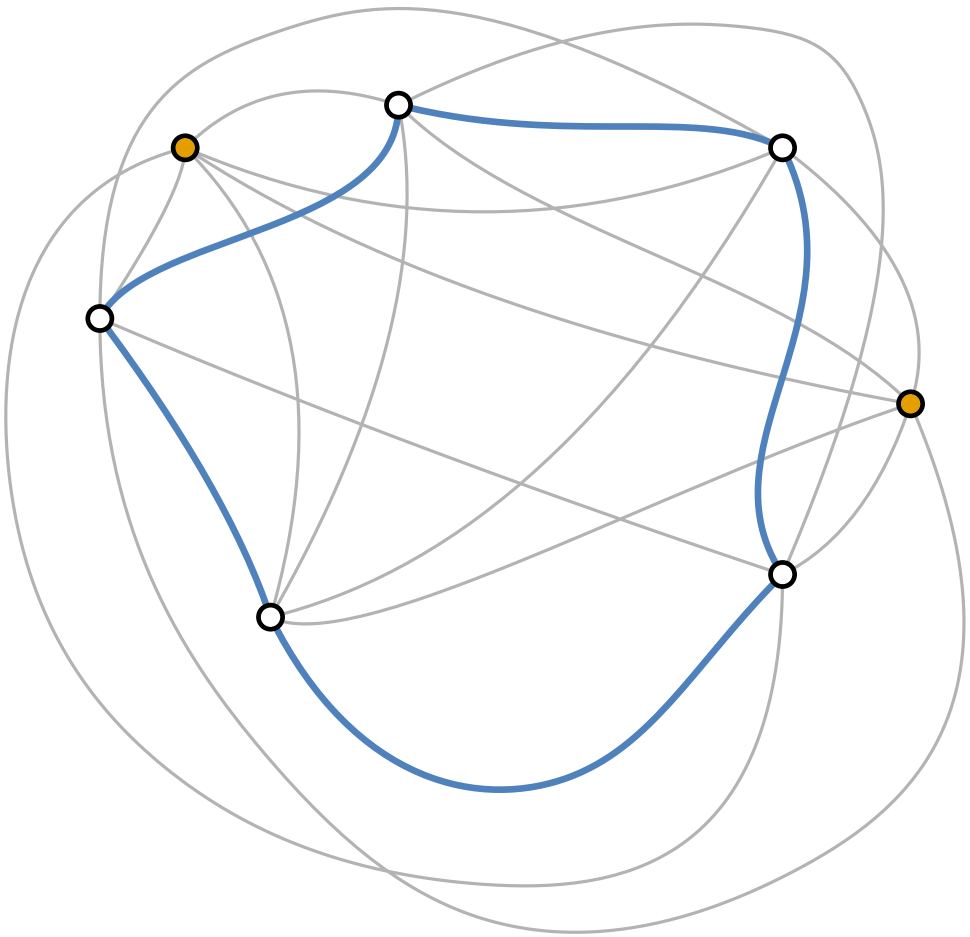

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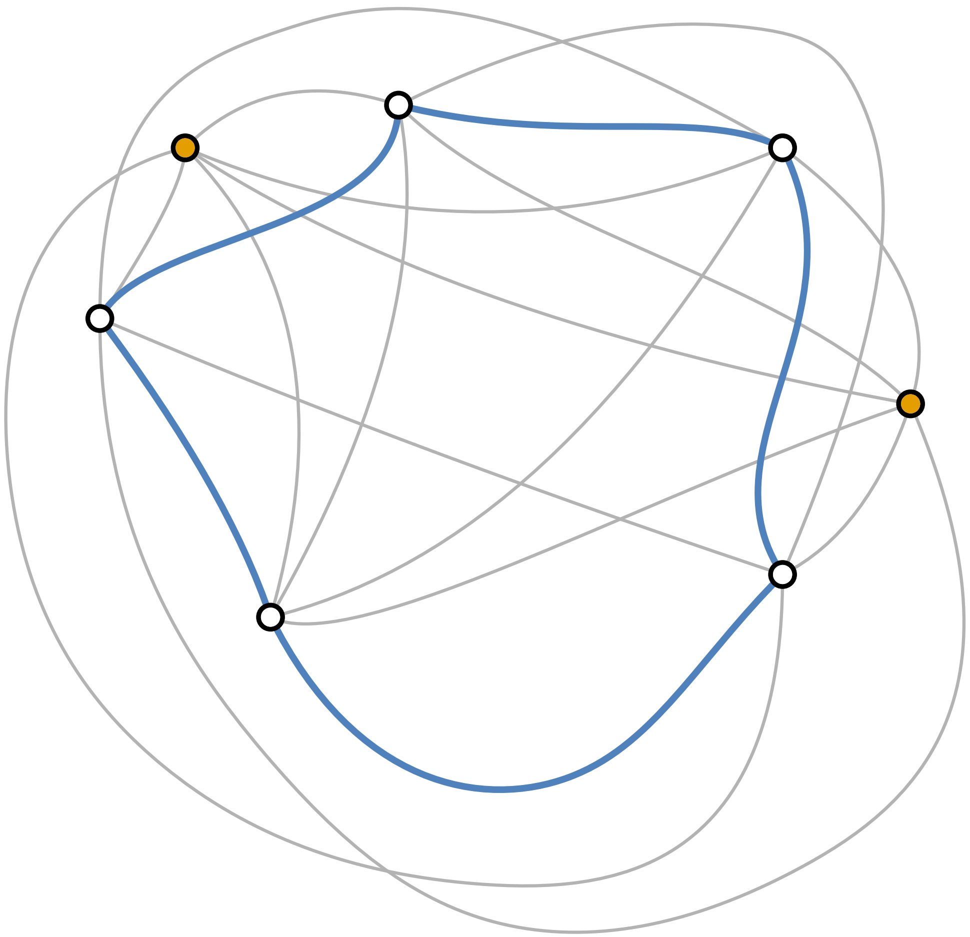

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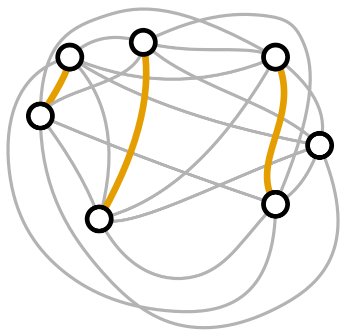
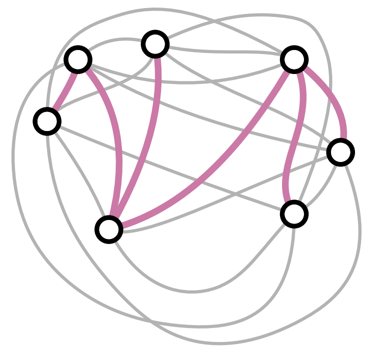
Invariance under (weak) isomorphism!

Plane Sub-drawings in Simple Drawings of K_n

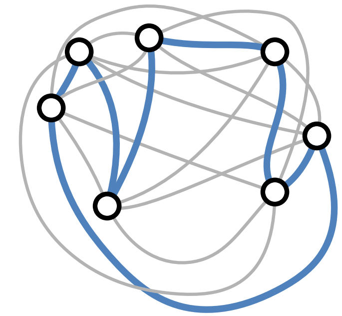
matchings



trees

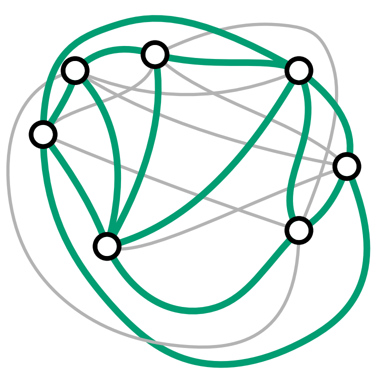


cycles & paths



and many more...

maximal plane sub-drawings



Questions:

- existence
- (asymptotic) size
- number
- ...

Plane Empty Cycles

Conjecture 1 [Rafla 1988]:
Every simple drawing of K_n contains a plane Hamiltonian cycle.

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Conjecture 2 [BFROS 2024]:
Every simple drawing of K_n contains an empty plane k-cycle for every $k = 3, \ldots, n$.

Conjecture 2 holds for subclasses of straight-line, x-monotone, and g-convex drawings. For simple drawings in general, it has been shown for $k = 3$ and $k = 4$.

Conjecture 2 $\Rightarrow$ Conjecture 1

Small values of k

Theorem: Every simple drawing of K_n for $n \geq 5$ contains an empty plane k-cycle of length $k = 5$ or $k = 6$ *at every vertex*.

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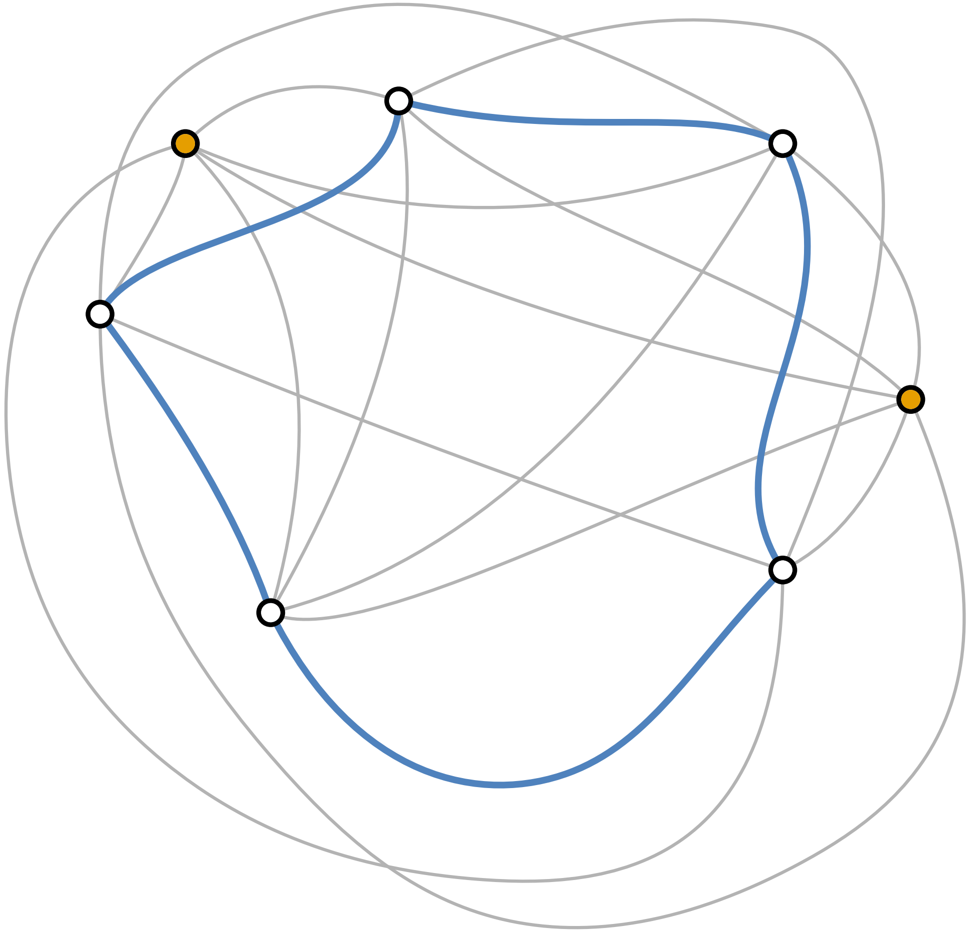

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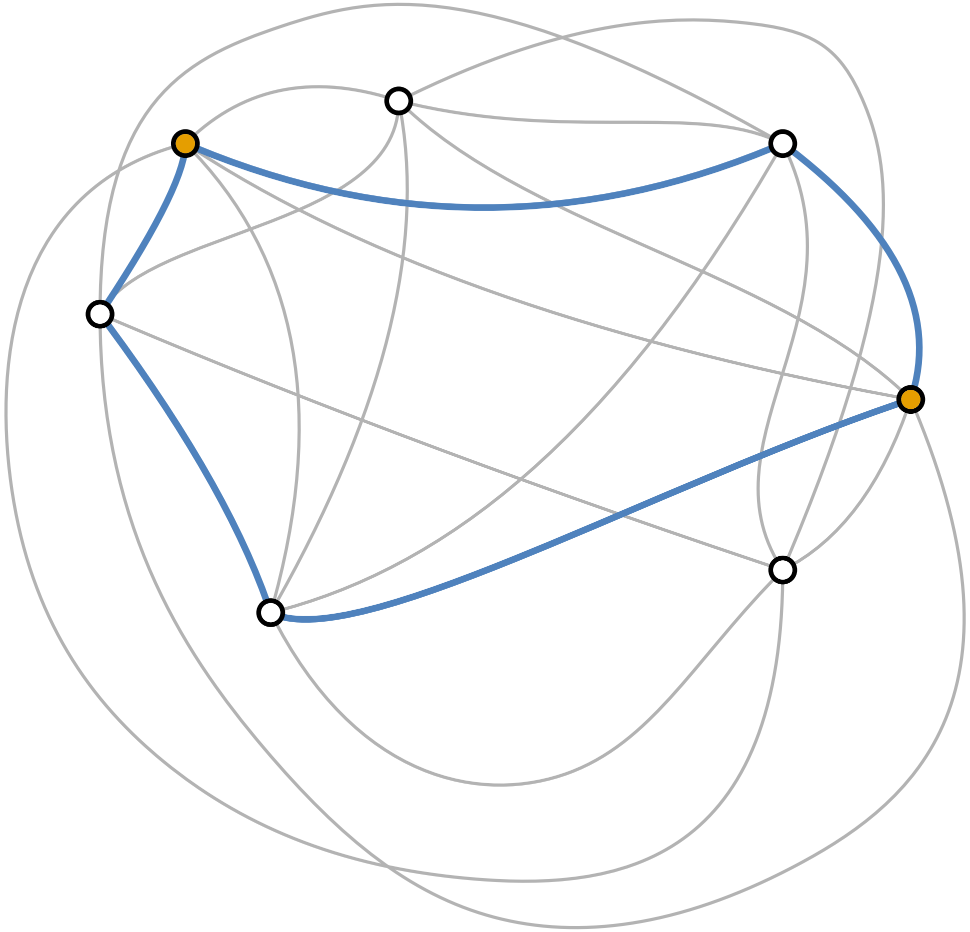
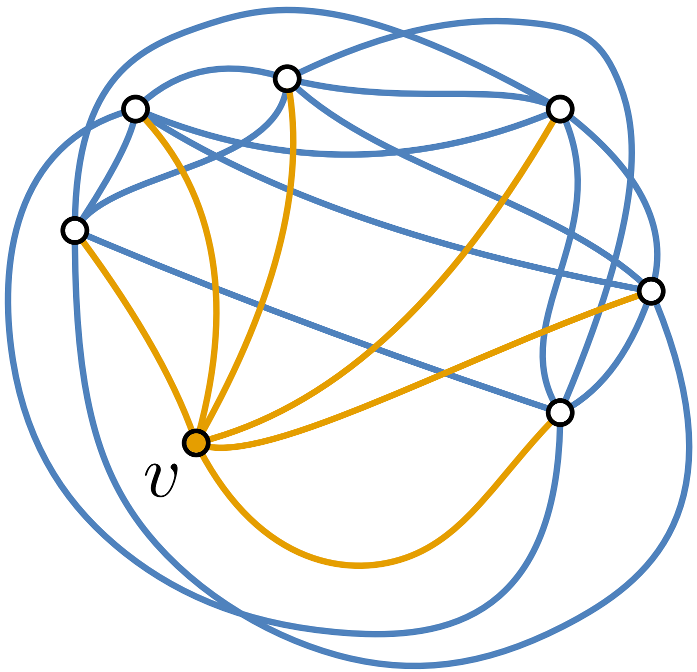


Preliminaries

simple drawing D of K_n



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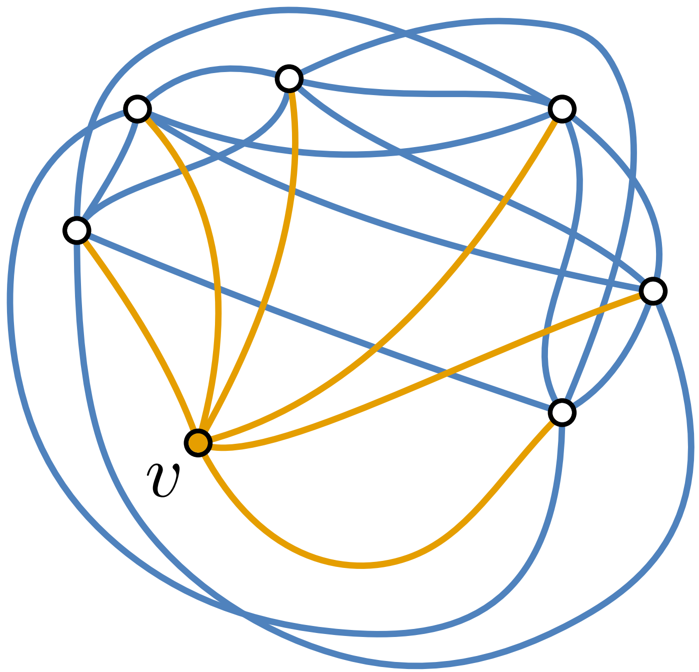
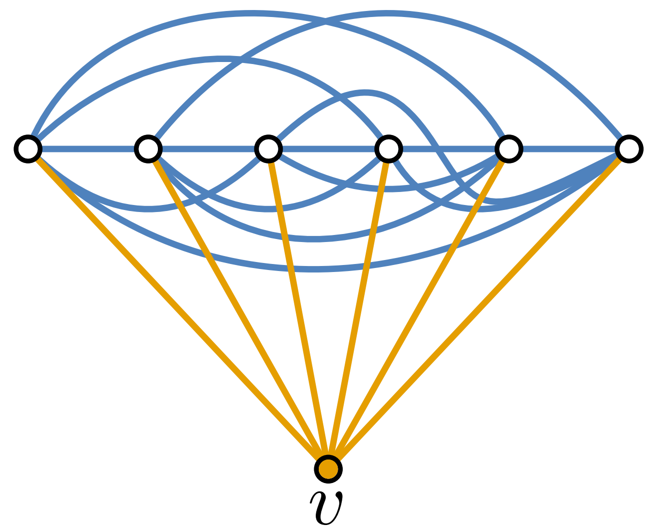


$\rightsquigarrow$



star(v) is the spanning subdrawing of all edges incident to v

an ice-cream cone drawing at v

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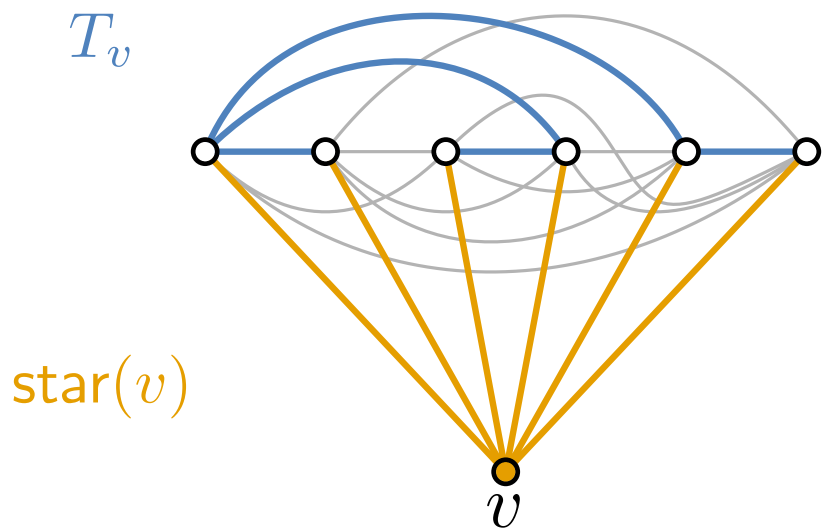
Lemma [GPT 2021]: Given a simple drawing D of K_n and a vertex v, there exists a spanning sub-tree T_v on $V \setminus v$ s.t. $D' := \textit{star}(v) \cup T_v$ is a plane sub-drawing of D.



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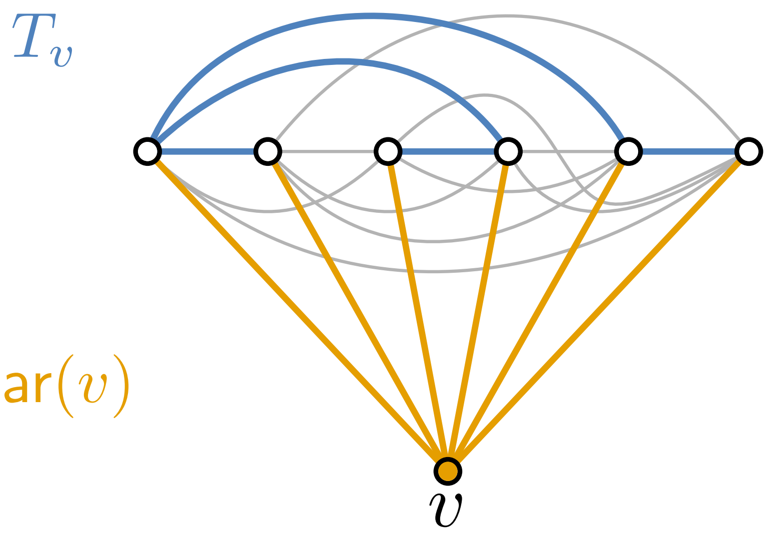

$\rightarrow$ All faces of D' are incident to v.

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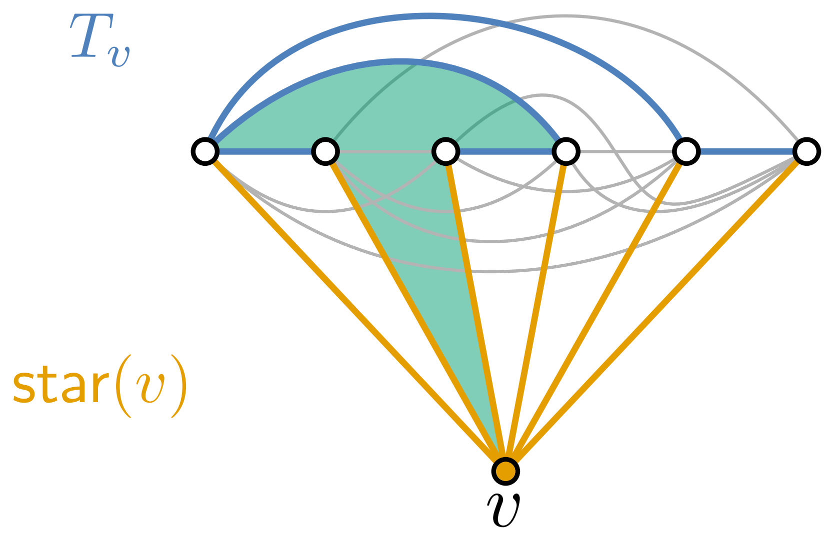

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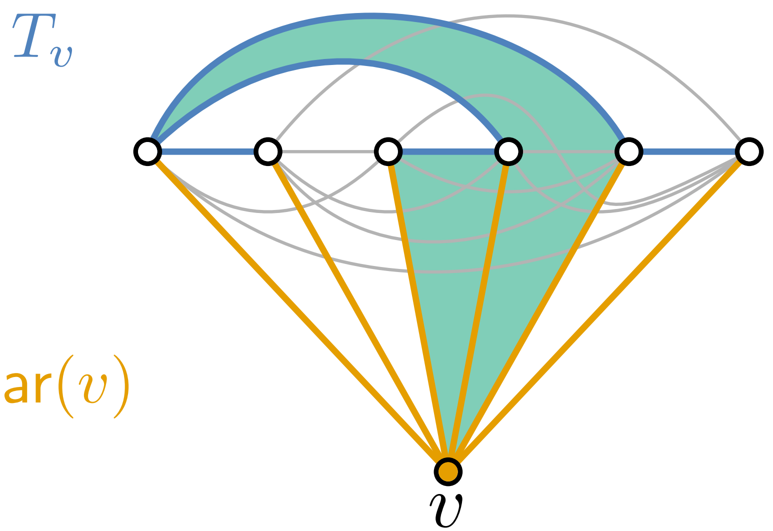

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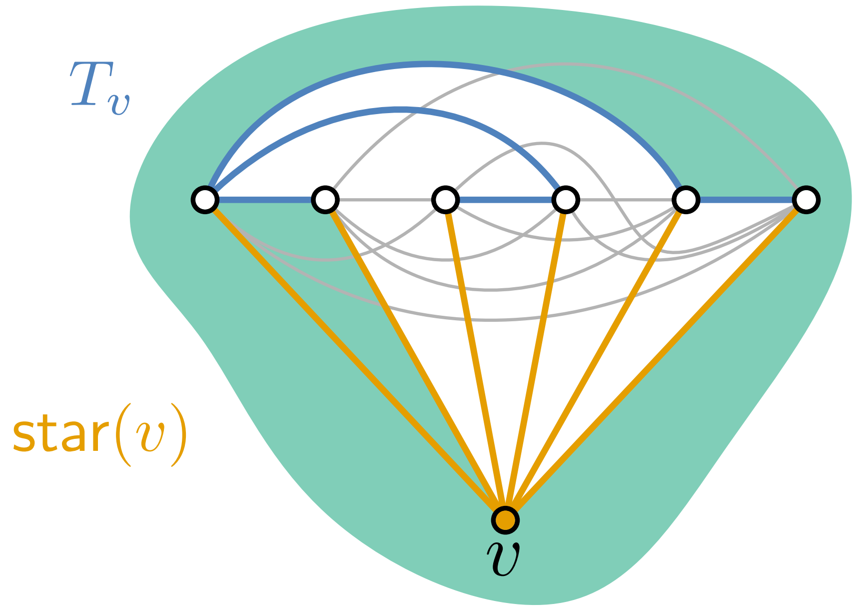

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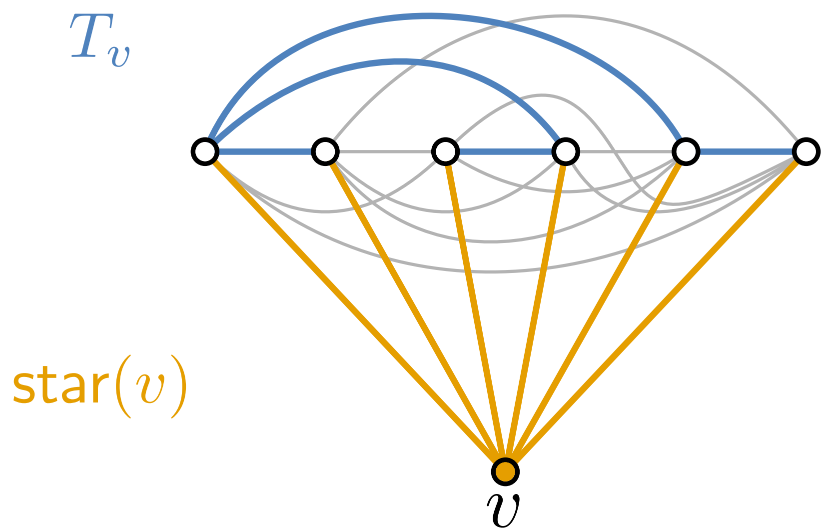

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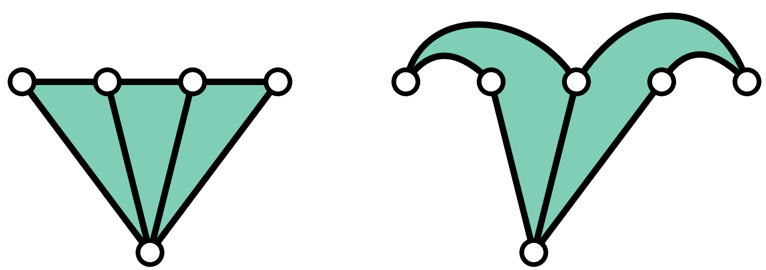

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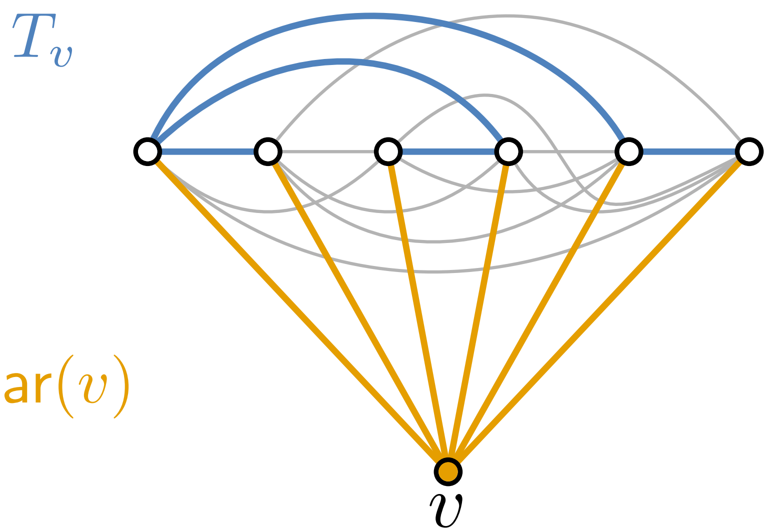

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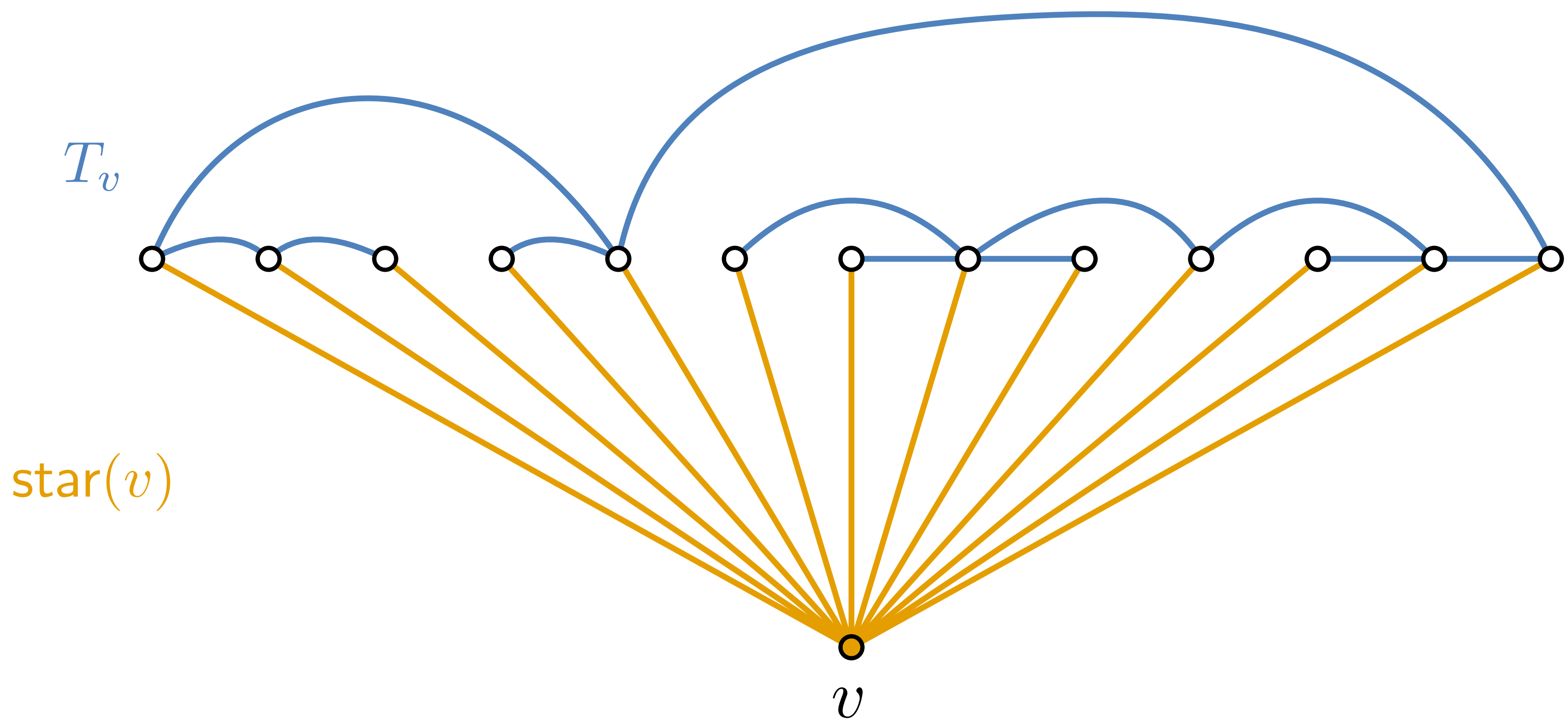
Can we always find such good (combinations of) faces in star(v) $\cup T_v$?

Preliminaries - Umbrellas and Pockets

We consider a sub-drawing $D' = \text{star}(v) \cup T_v$ of D as ice-cream cone drawing at v.

umbrella of size k: a bounded face of size $k \geq 4$



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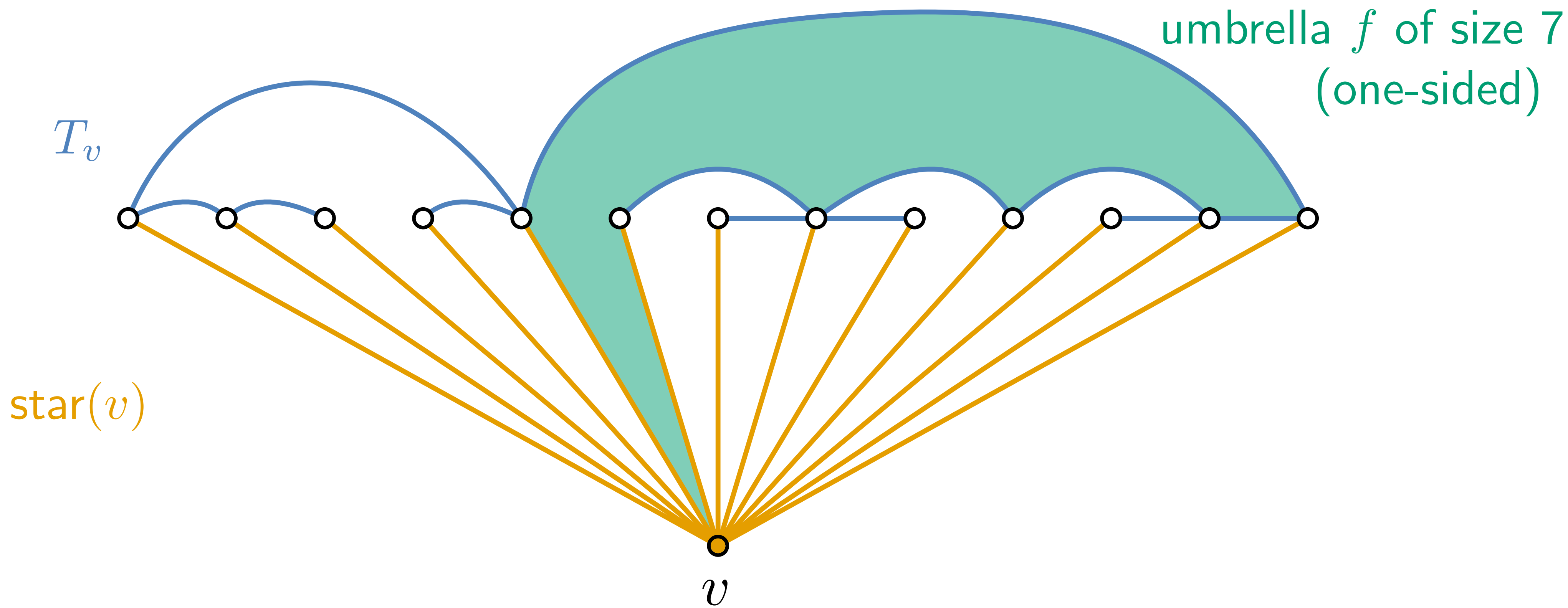

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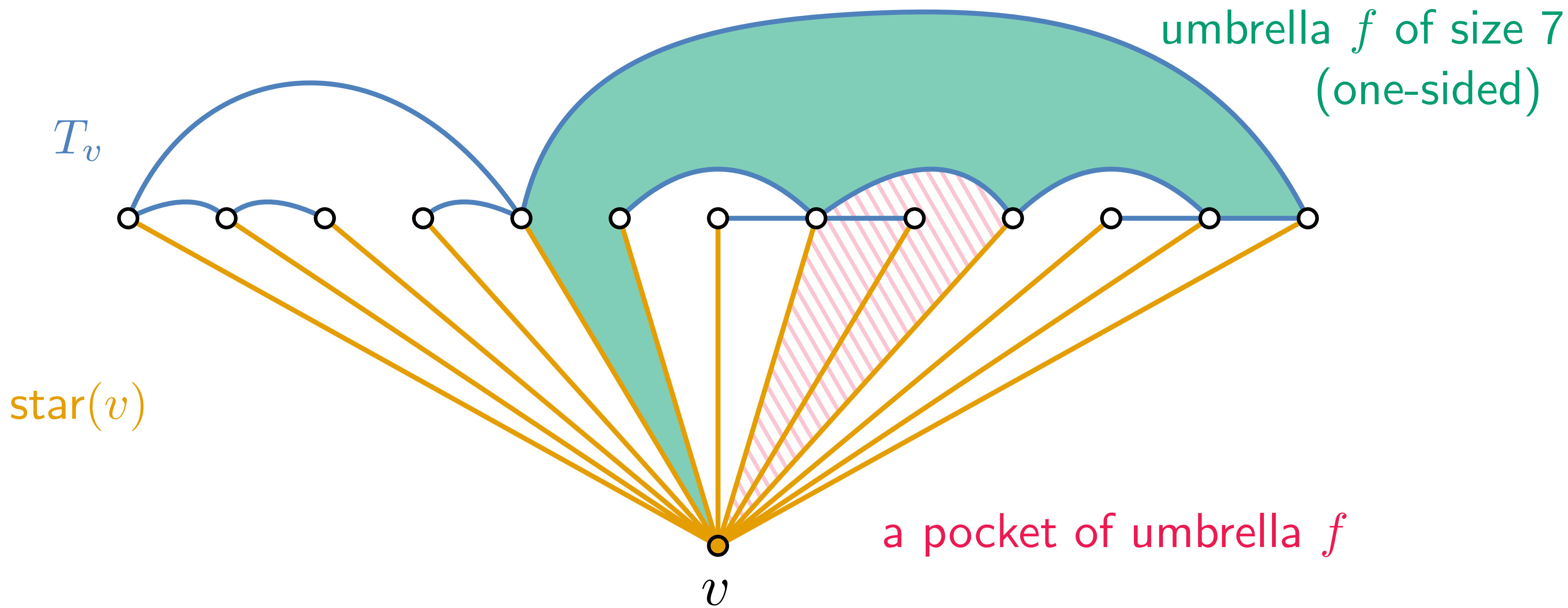

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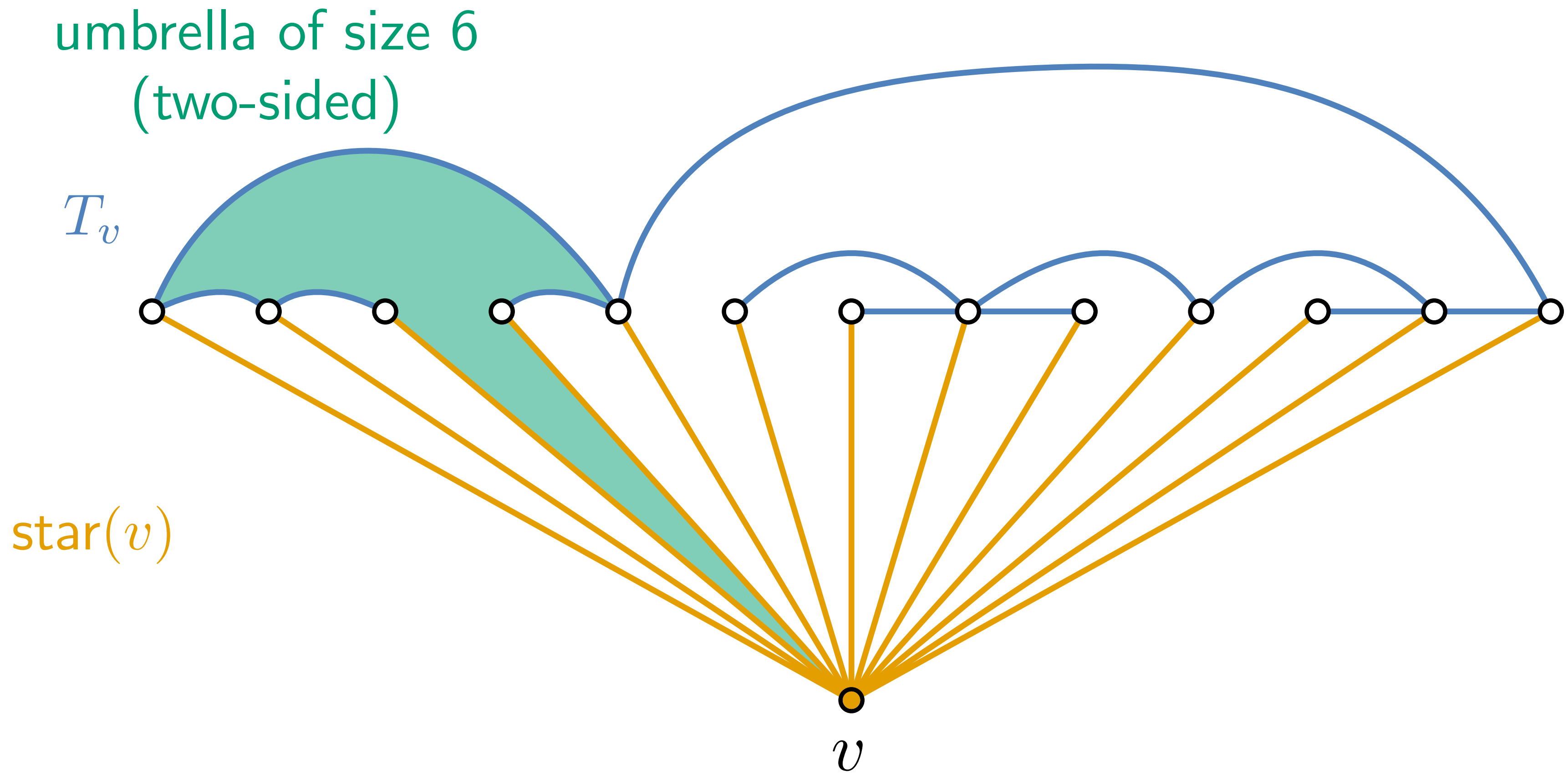

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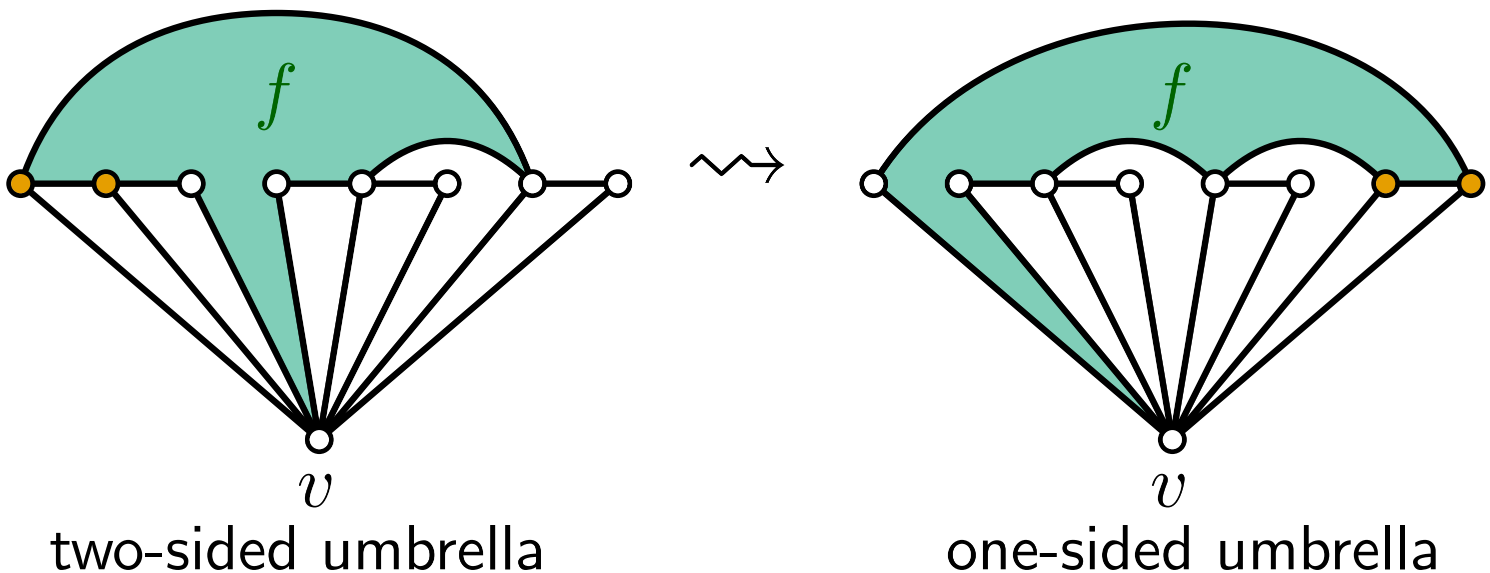
$\rightarrow$ w.l.o.g. assume that f is a one-sided umbrella.



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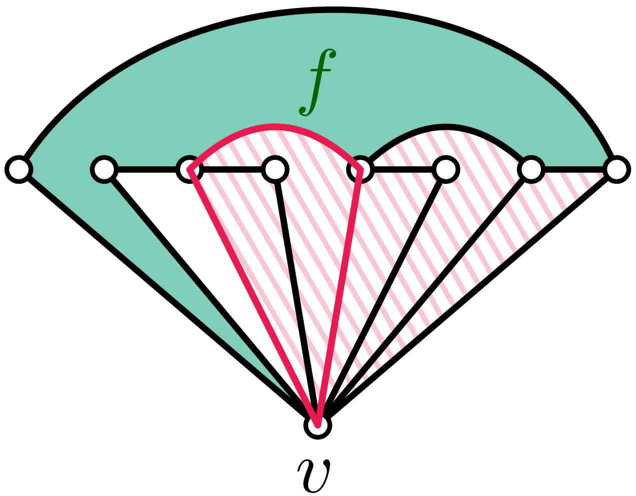

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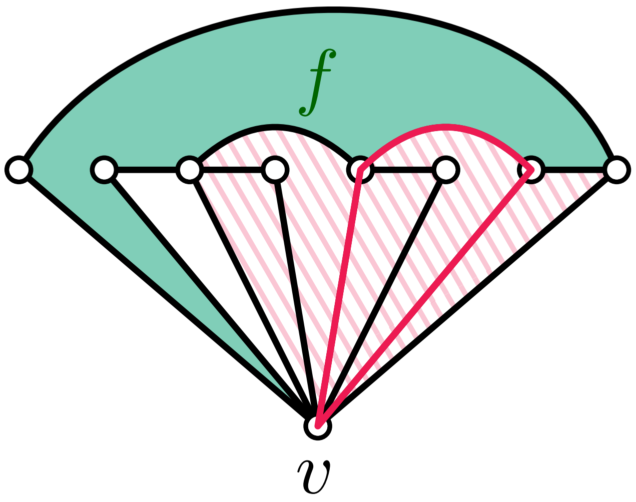

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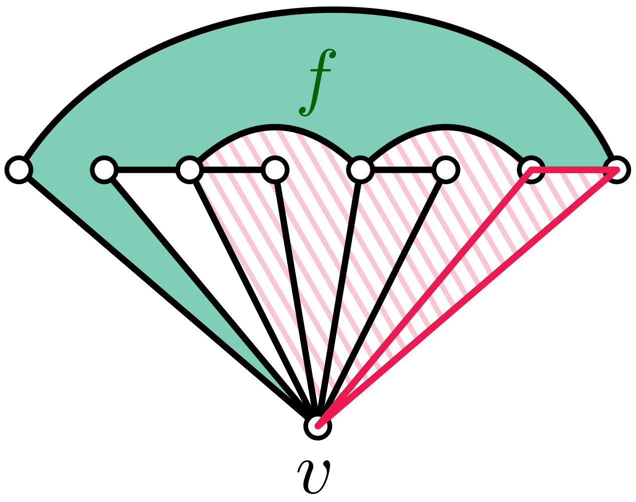

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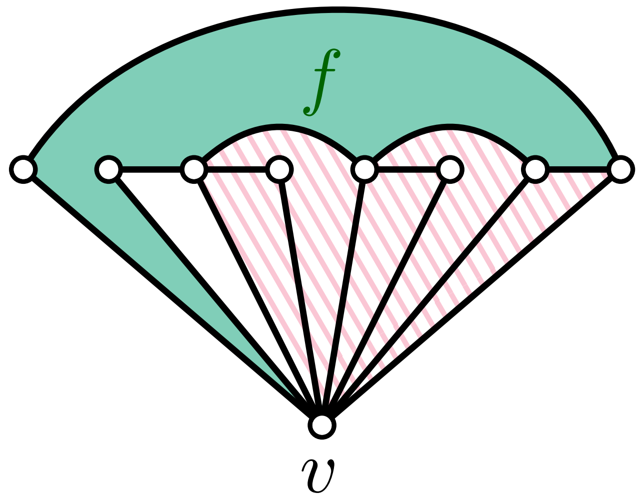

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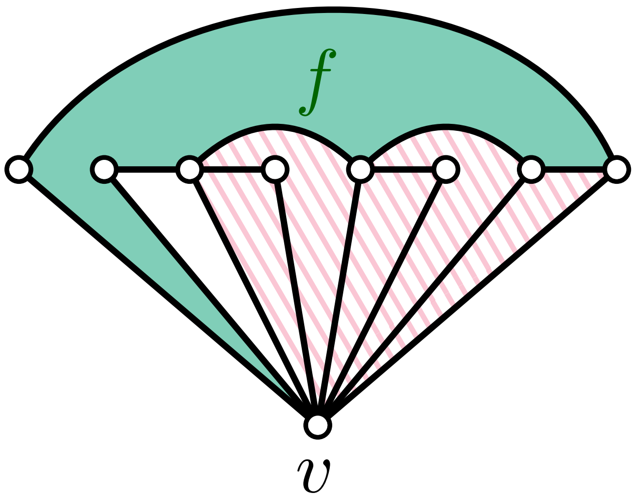

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We find an empty plane 5- or 6-cycle!

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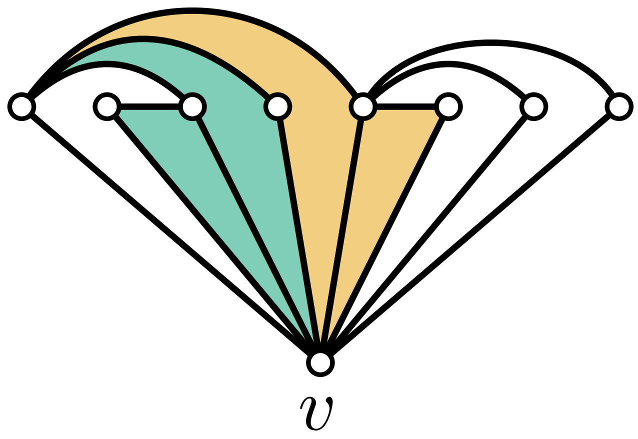

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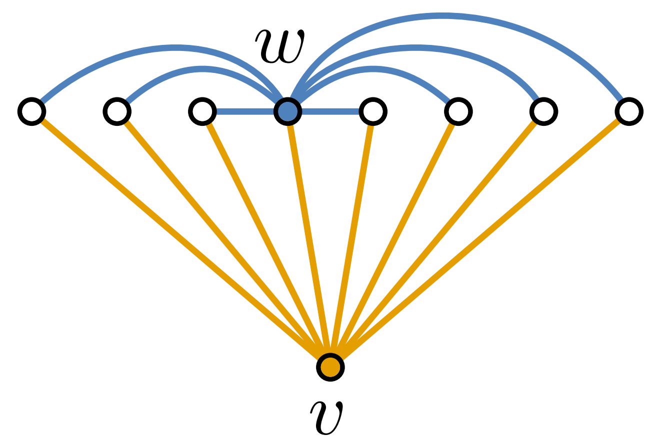

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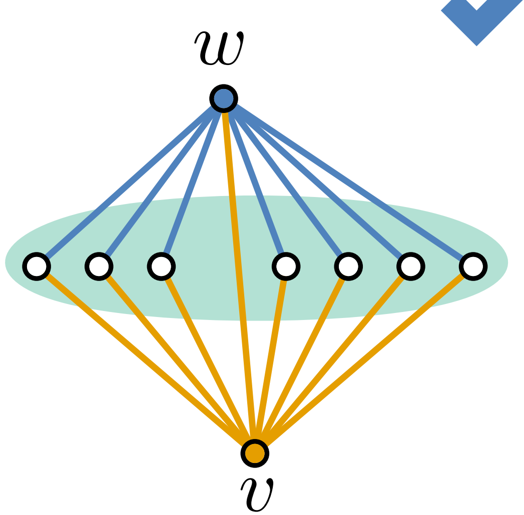

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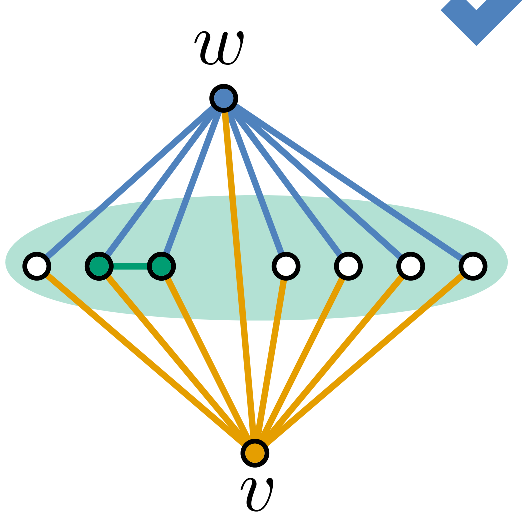

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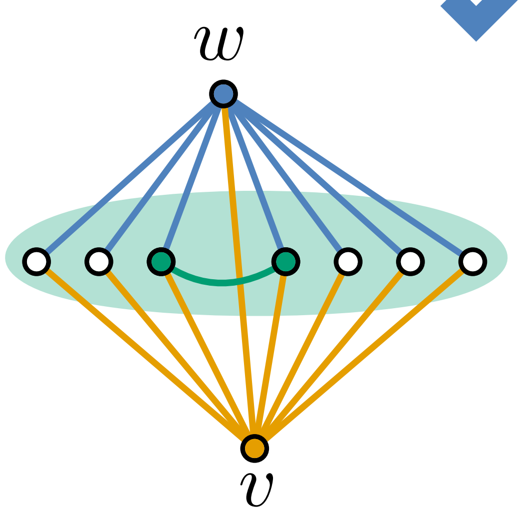

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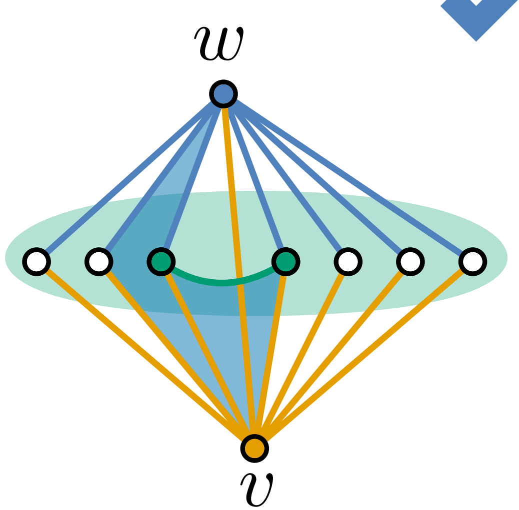

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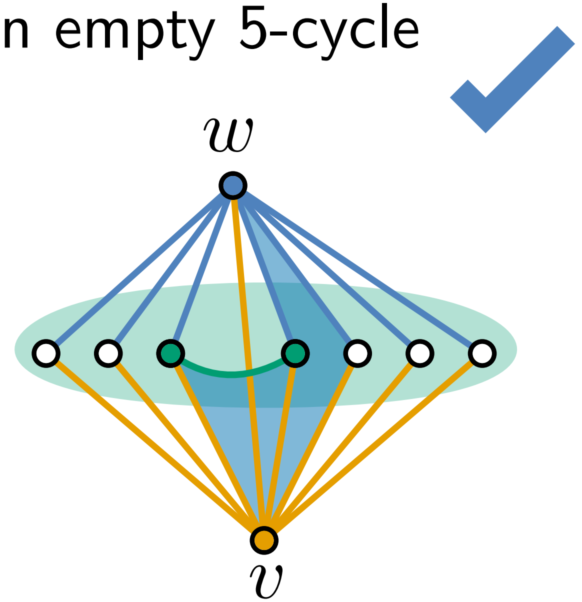

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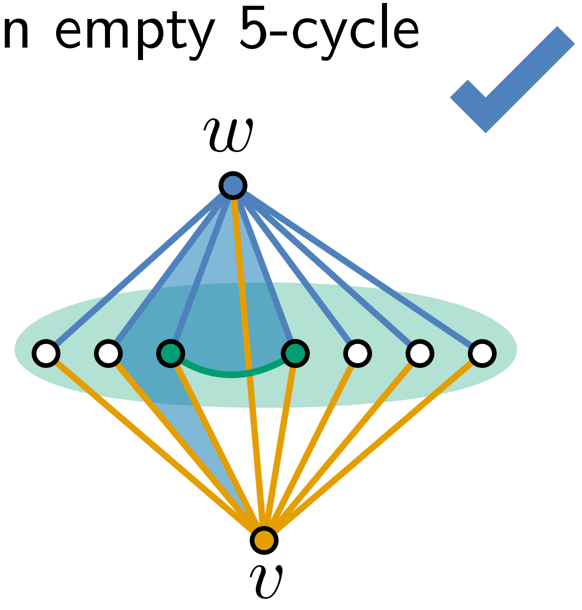

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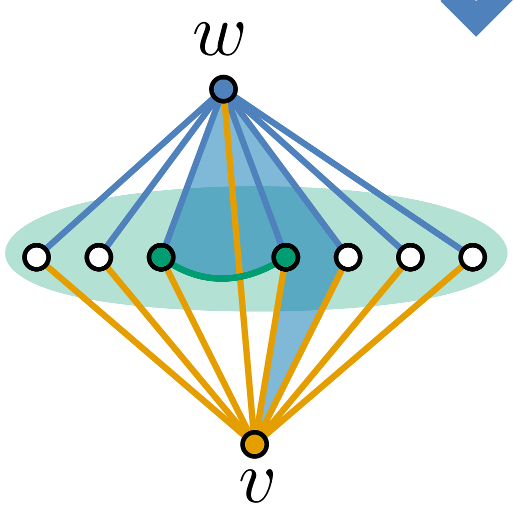

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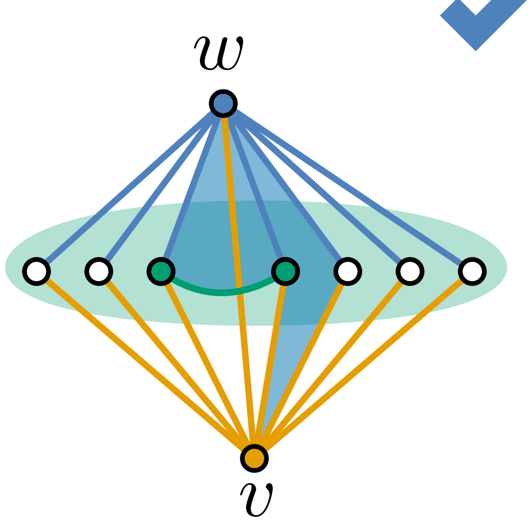
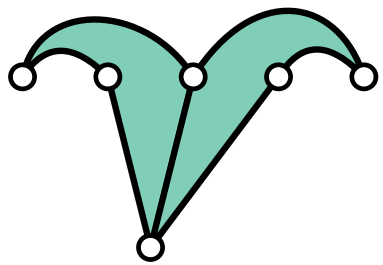


Get two empty plane 5-cycles at v!
(resp. four if $n \geq 6$) ✓

What now?

- Can we get rid of the 6-cycles?



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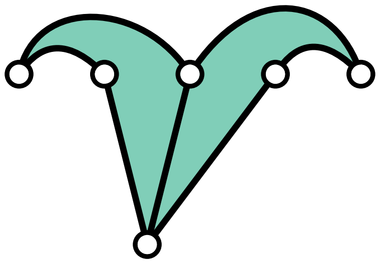

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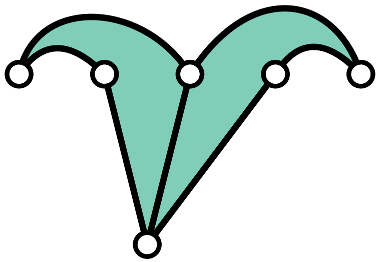

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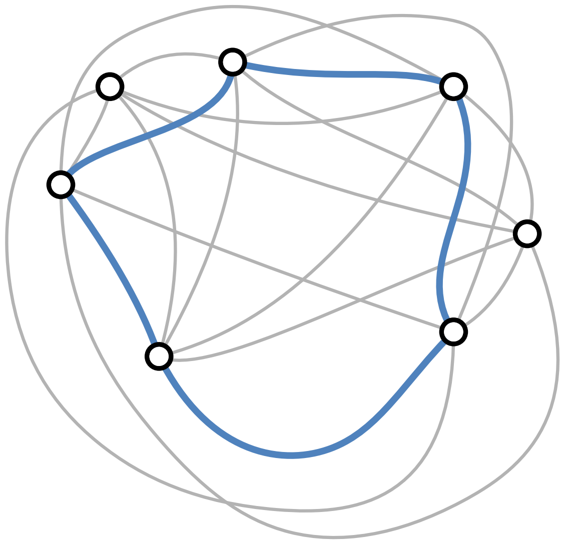
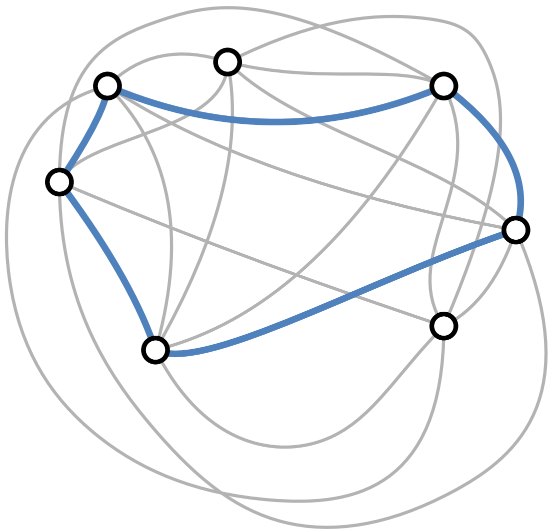
Open Problem [Orthaber 2025]:
Every simple drawing of K_n contains *two* empty plane k-cycles for $k = 3, \ldots, n$ *at every vertex*.

Conclusion

Theorem: Every simple drawing of K_n for $n \geq 5$ contains an empty plane k-cycle of length $k = 5$ or $k = 6$ *at every vertex*.





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Short edges in c-monotone drawings of K_n

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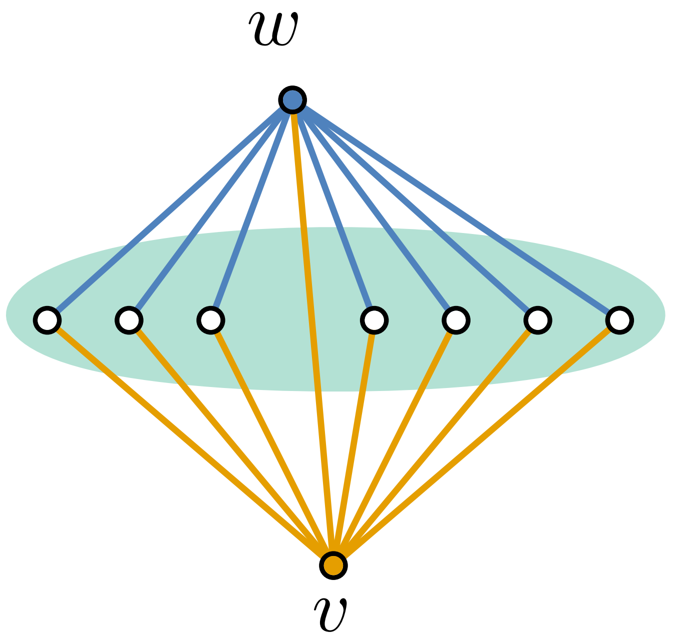

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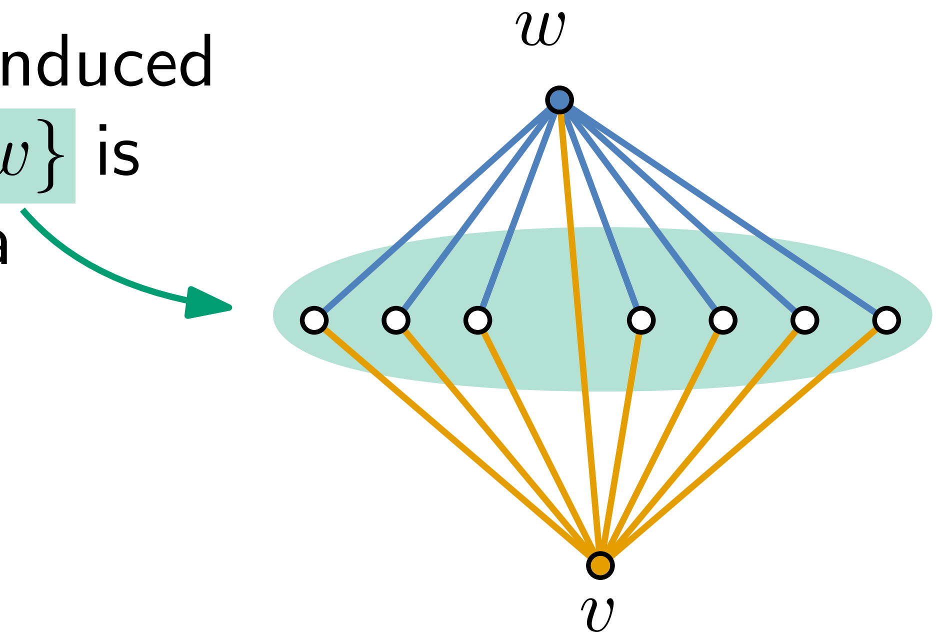

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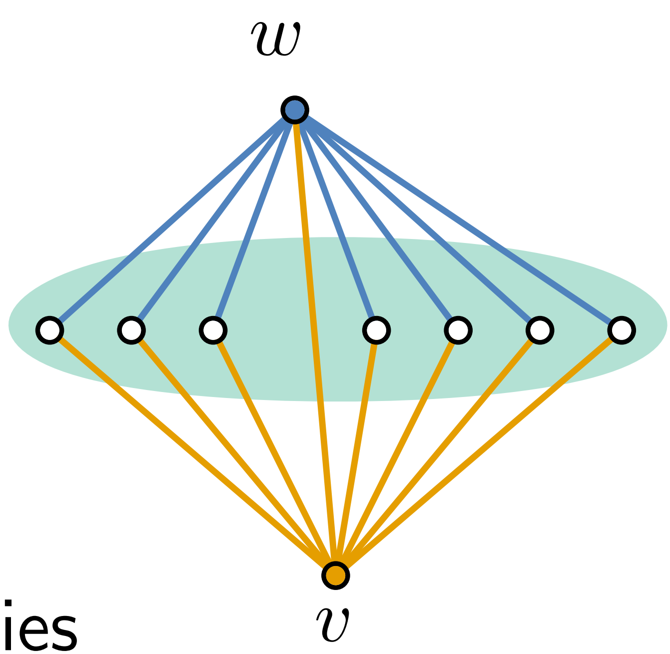
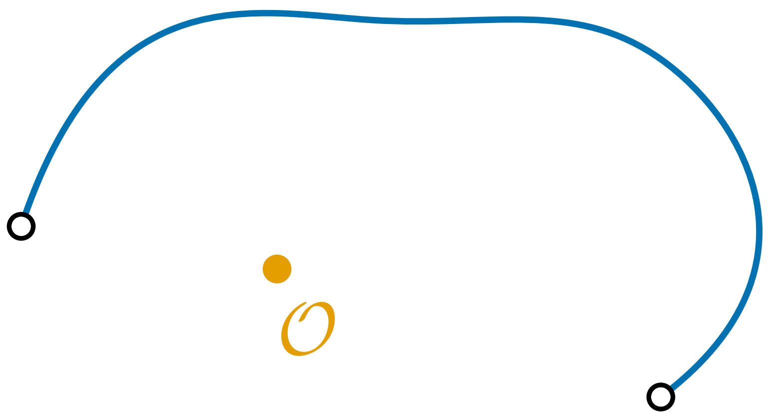


$\rightarrow$ We can use structural properties of c-monotone drawings to find a "good" edge.

c-Monotone Drawings

- An edge is c-monotone w.r.t. a point $\mathcal{O}$ if every ray emanating from $\mathcal{O}$ crosses the edge at most once.



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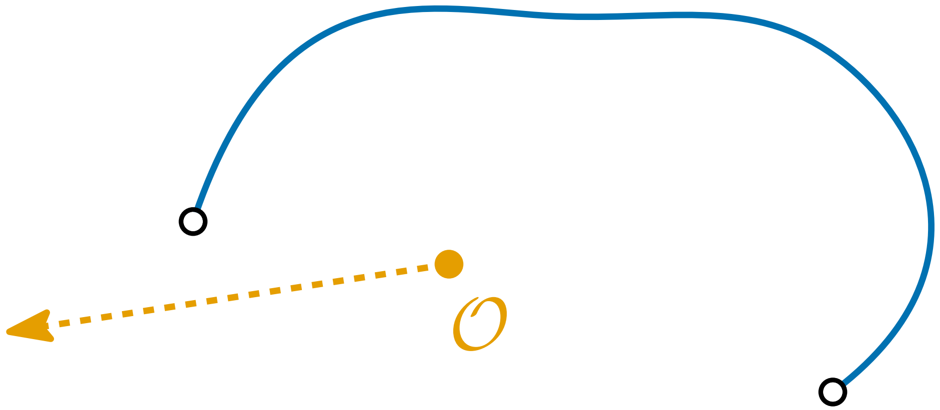

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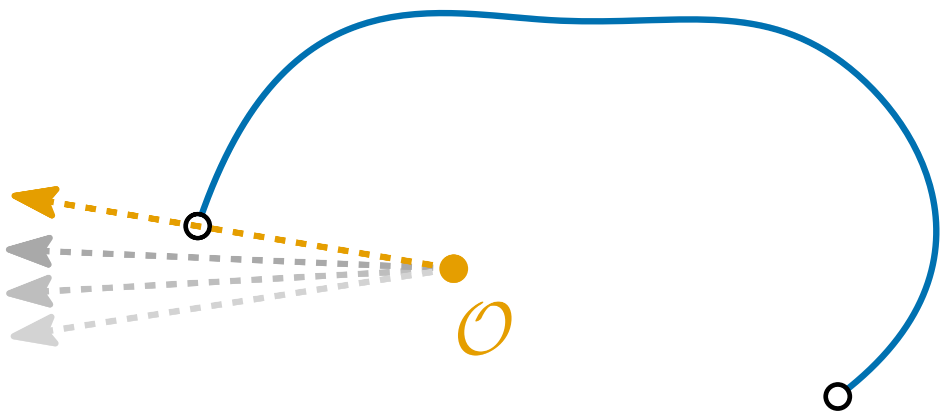

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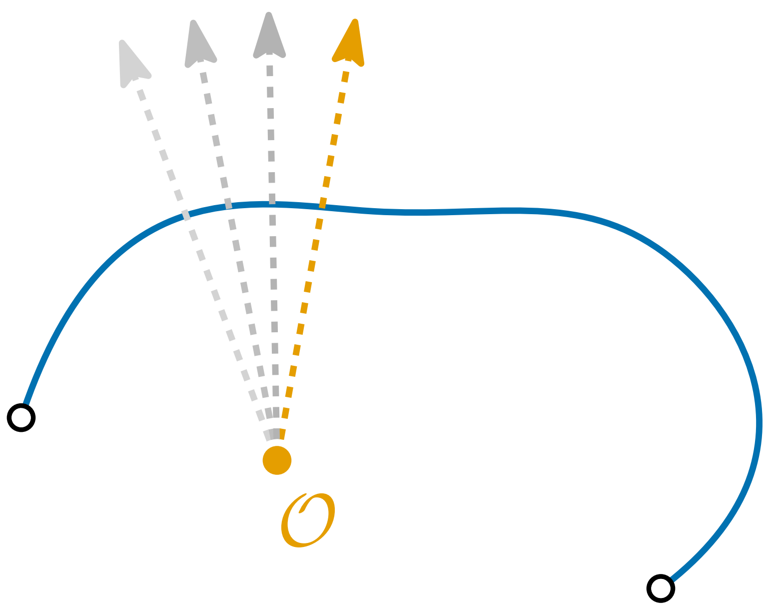

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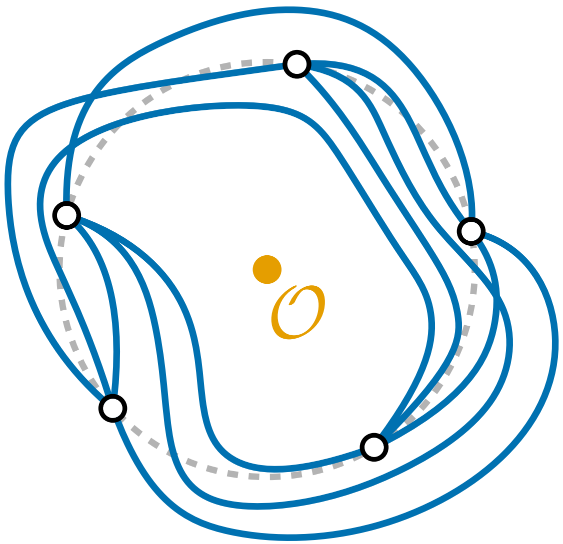

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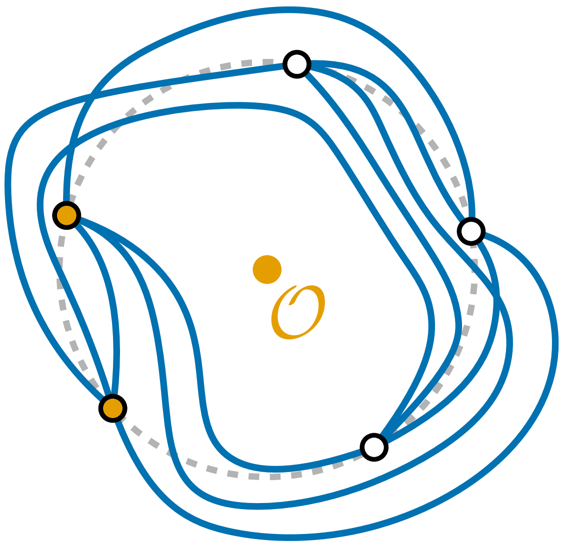

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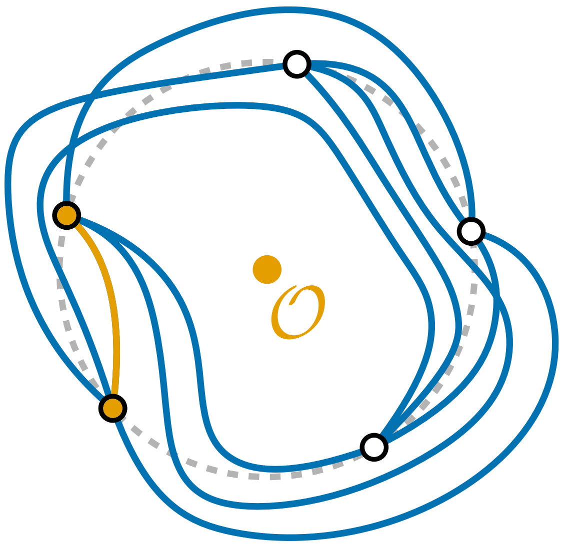

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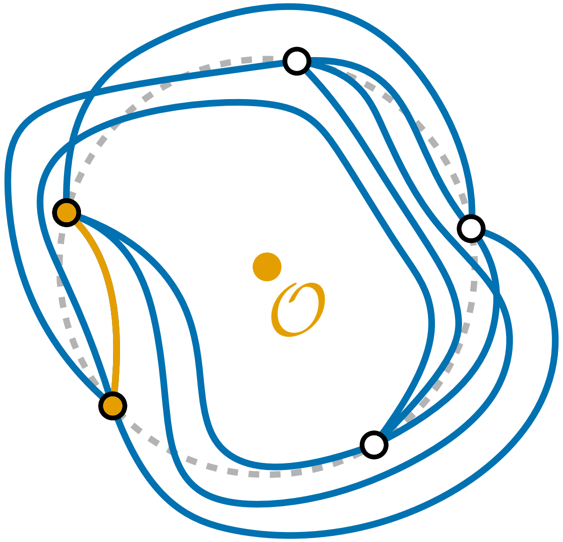
Lemma: Every c-monotone drawing of K_n for $n \geq 3$ contains a short edge between a pair of neighboring vertices.



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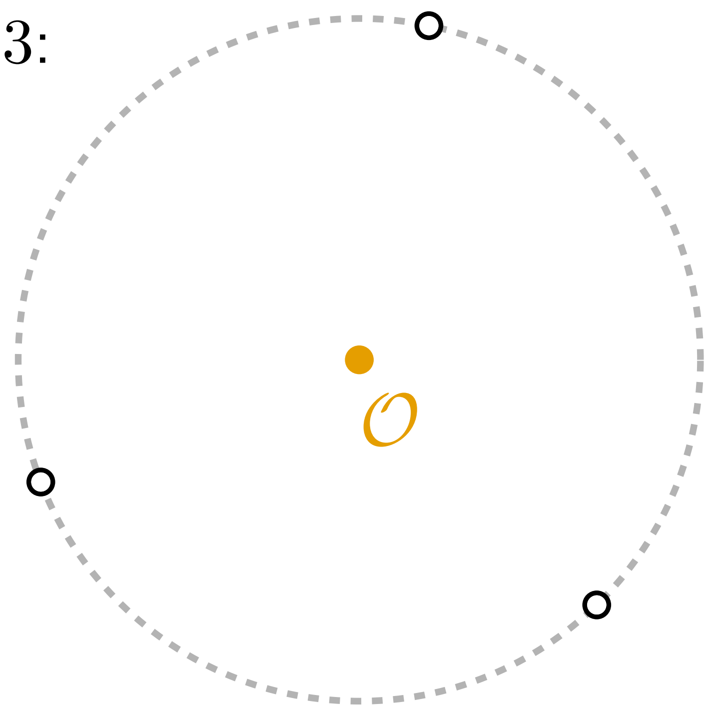
Base case for $n = 3$:



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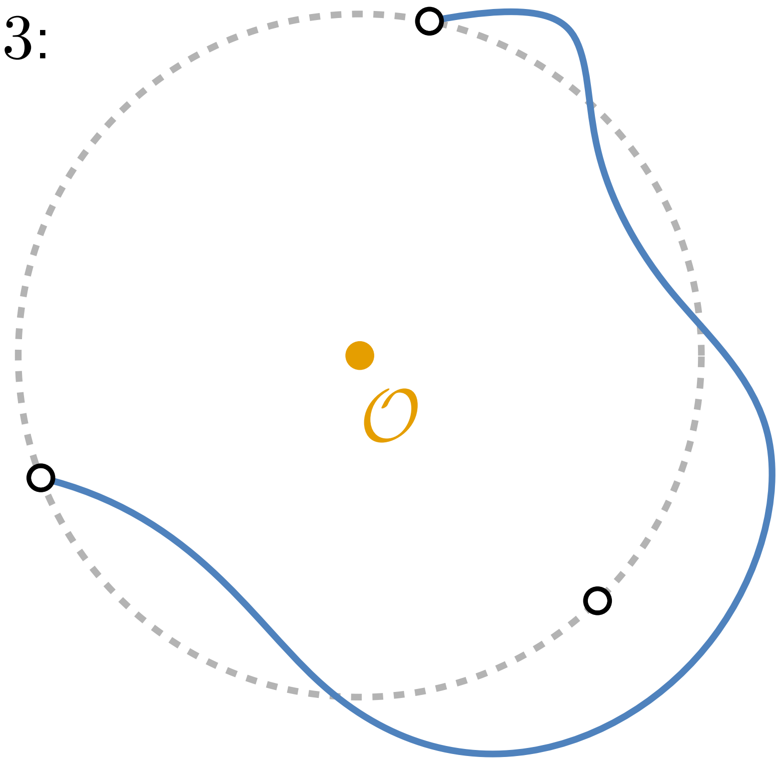

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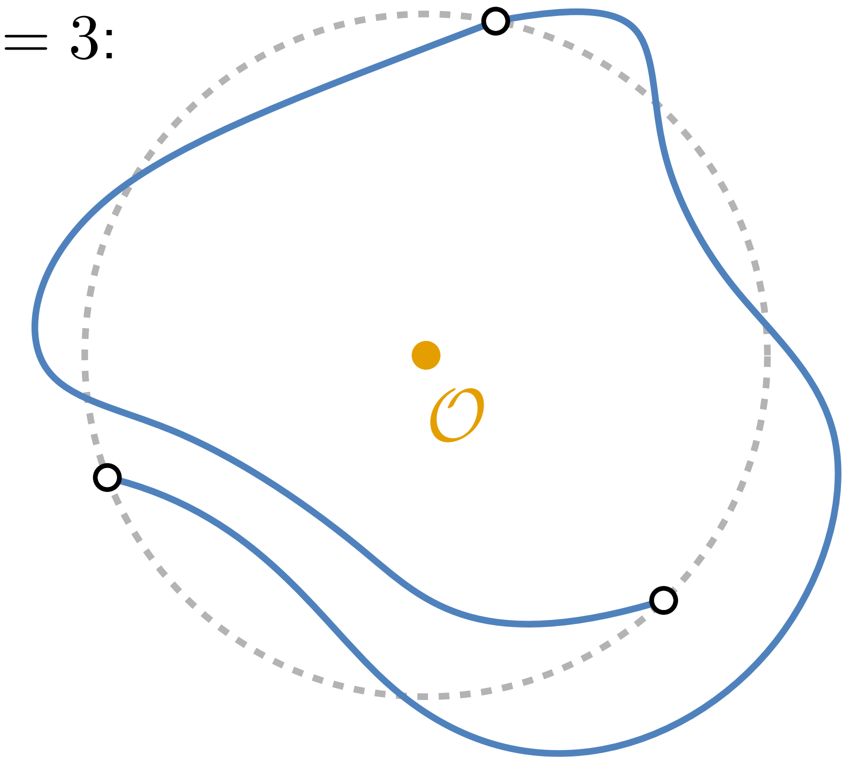

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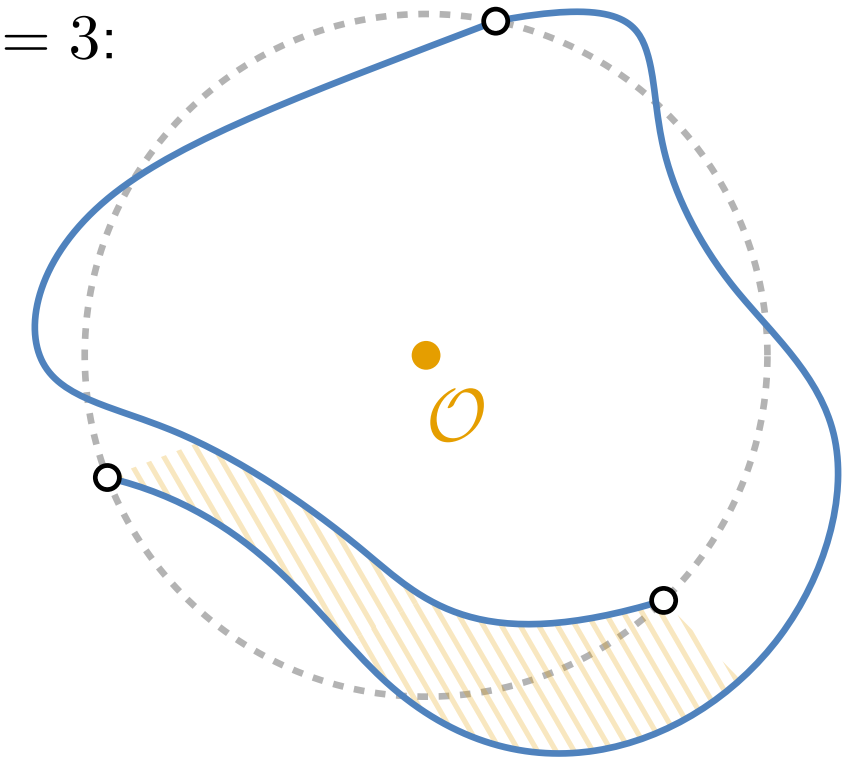

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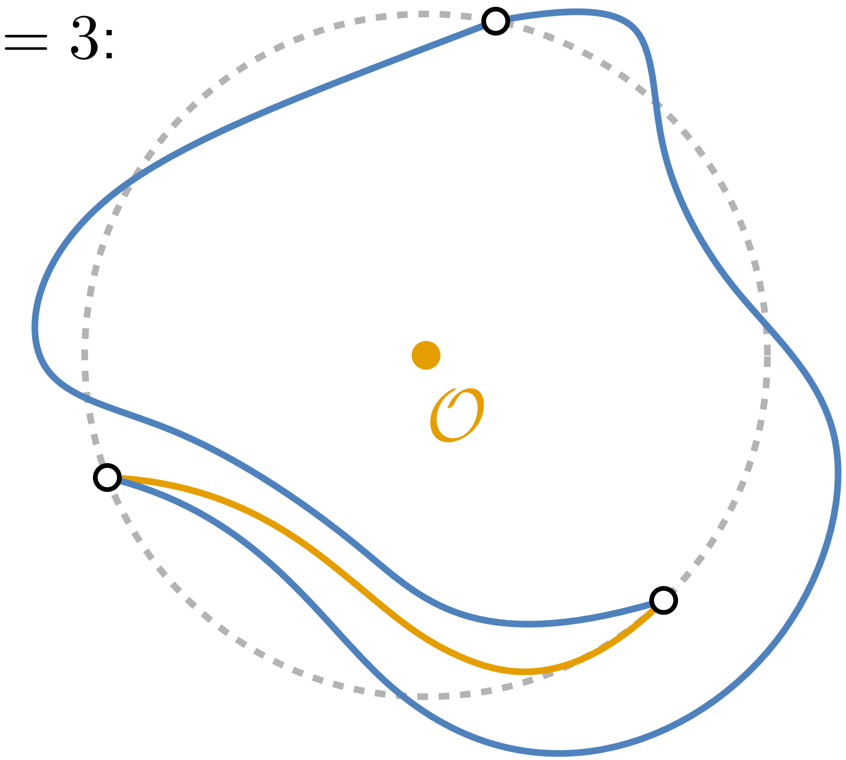

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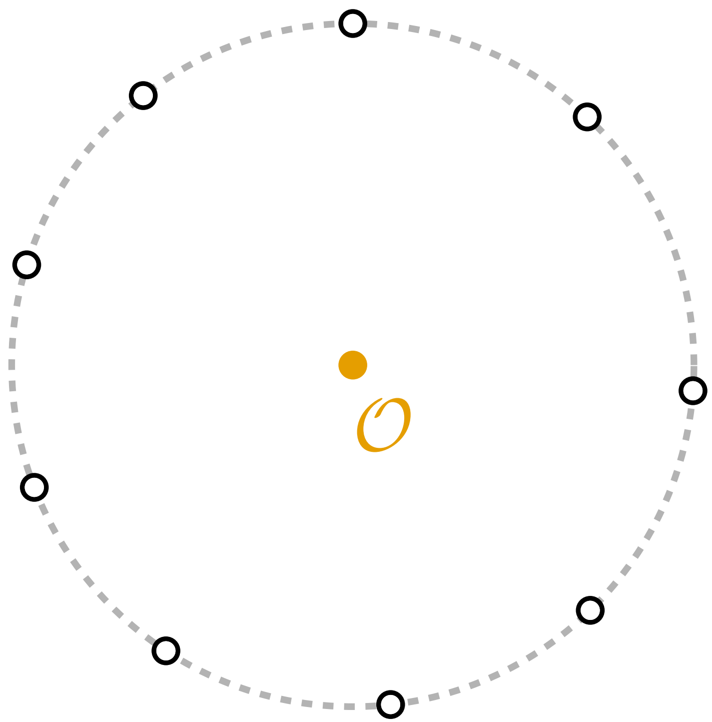

Assume we have a c-monotone drawing of K_n where all edges between neighboring vertices are *long*.

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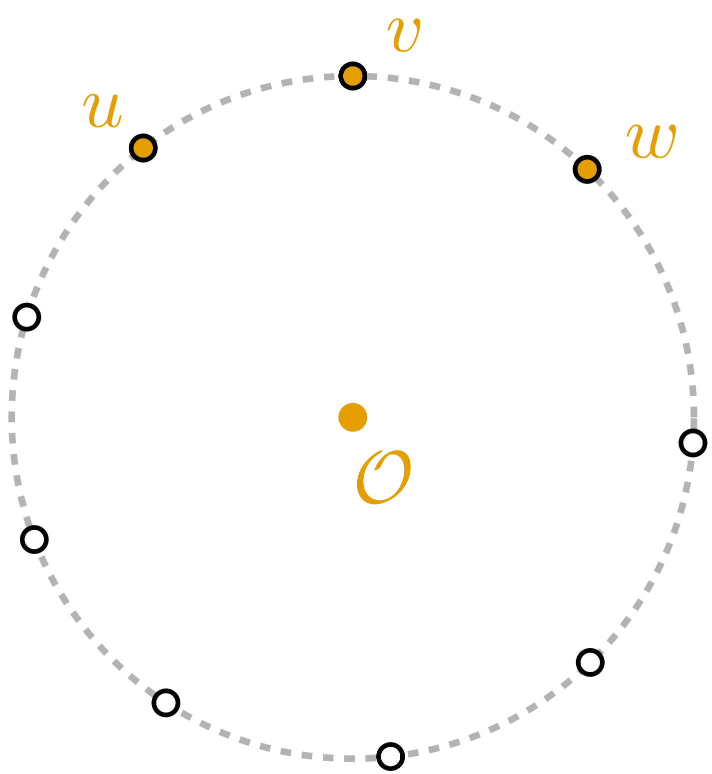

Fix a vertex v with neighbors u, w.

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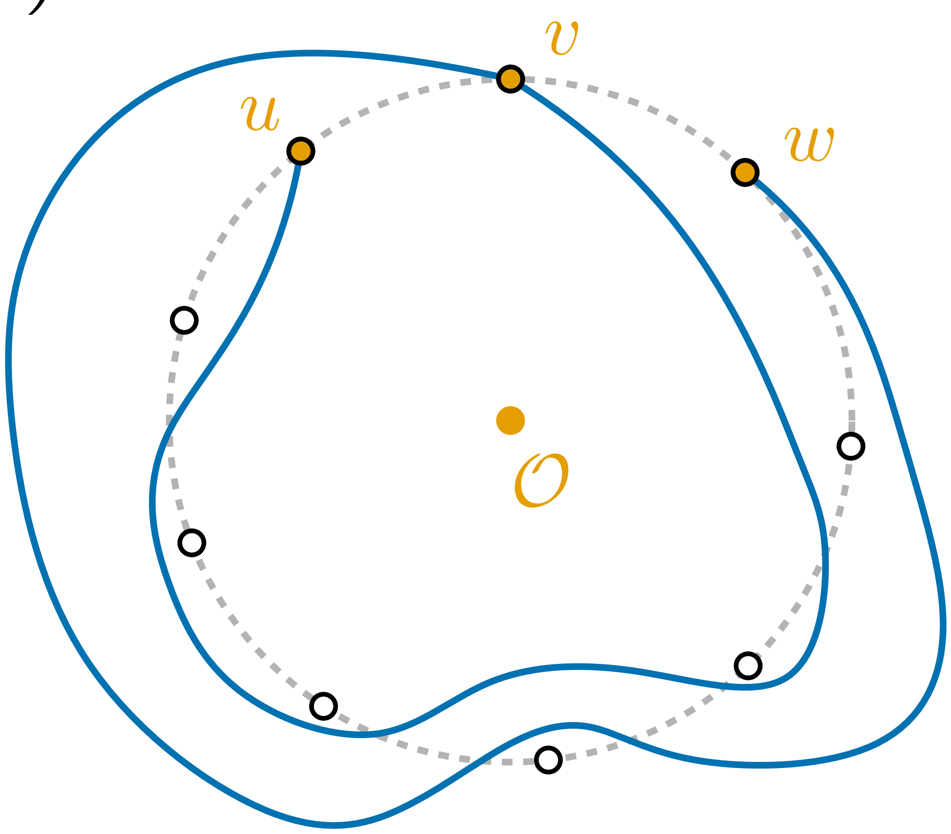

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By assumption, edges uv and vw are long.

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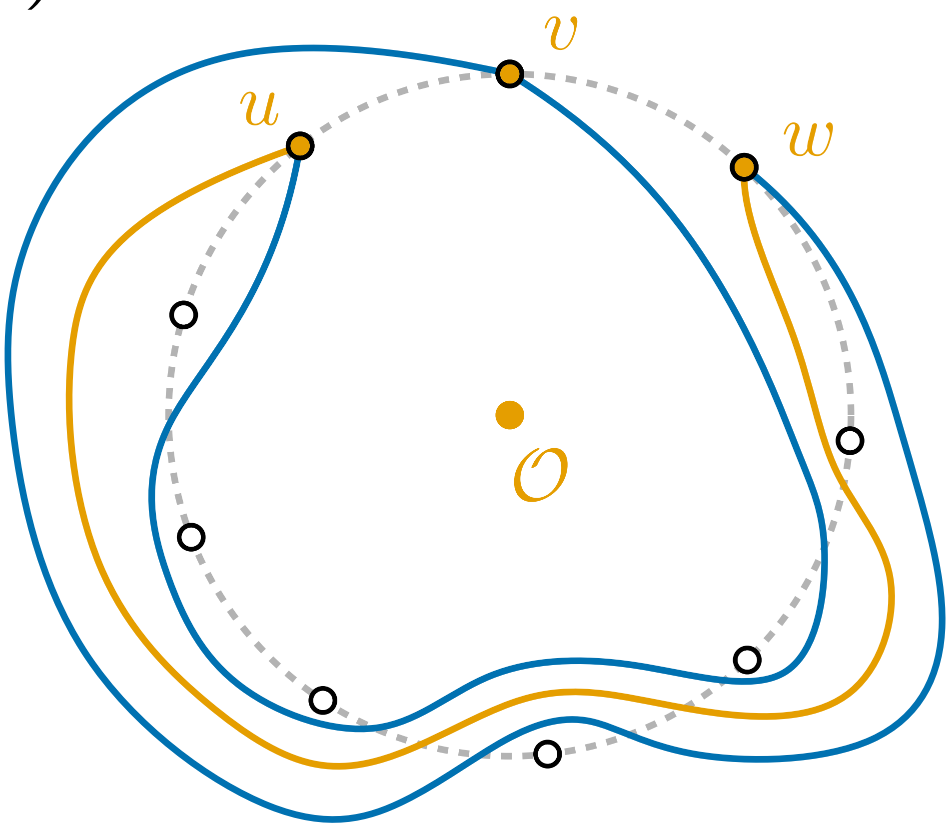

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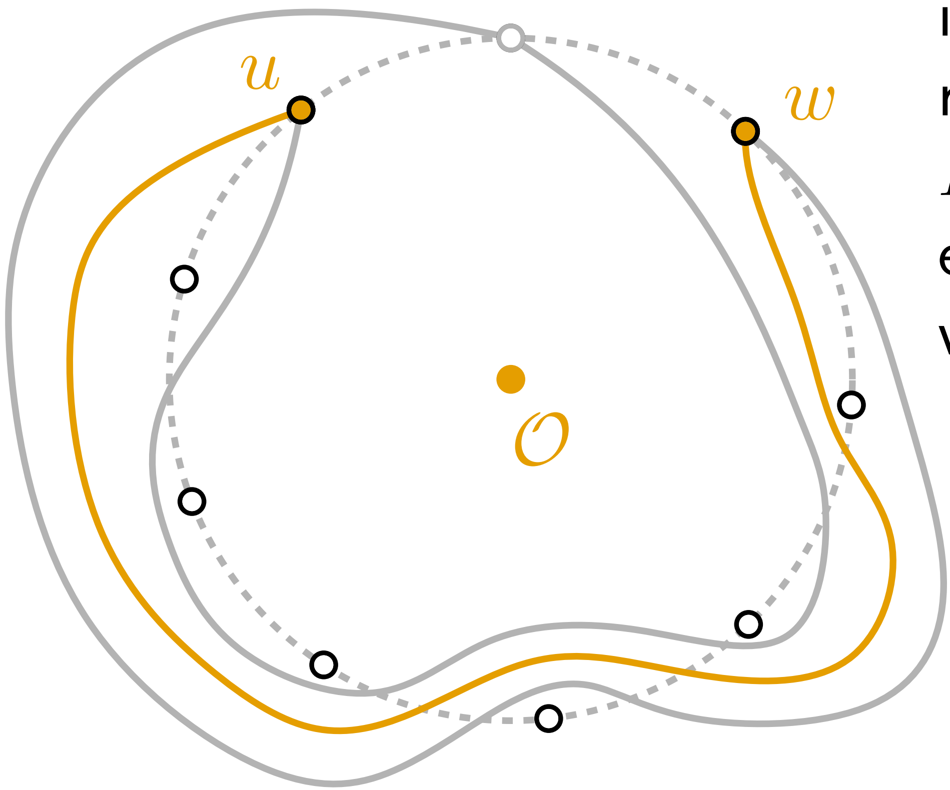

Remove vertex v and incident edges $\rightarrow$ in resulting drawing of K_{n-1} edge uw is a long edge between neighboring vertices.

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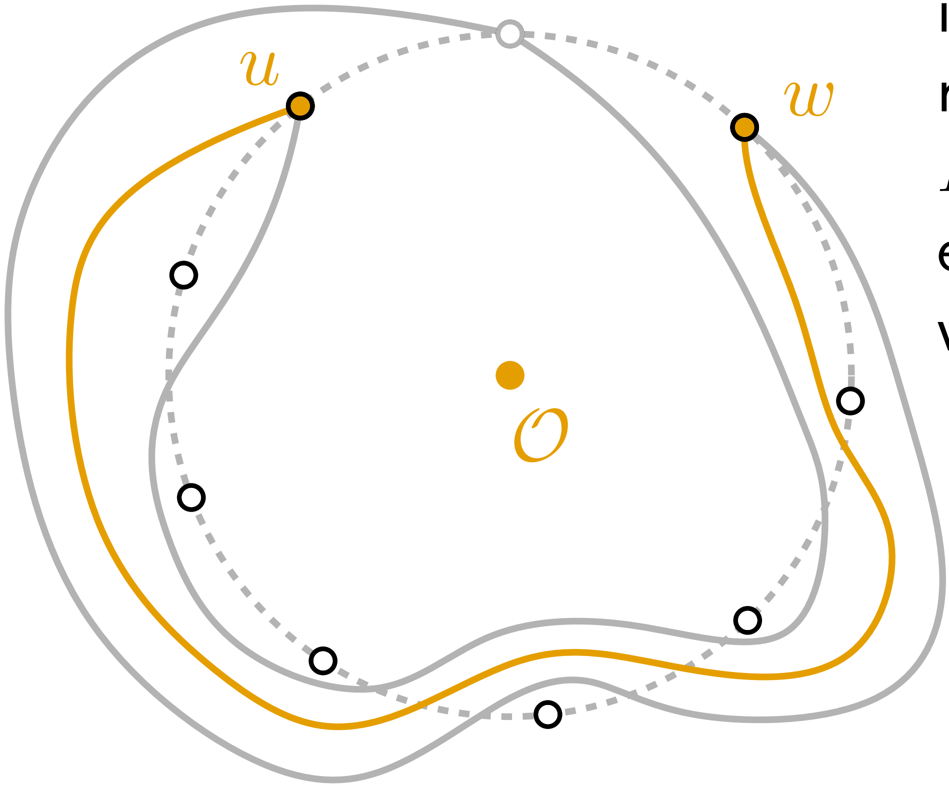

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$\Rightarrow$ Get a c-monotone drawing of K_{n-1} without any short edges between neighboring vertices.

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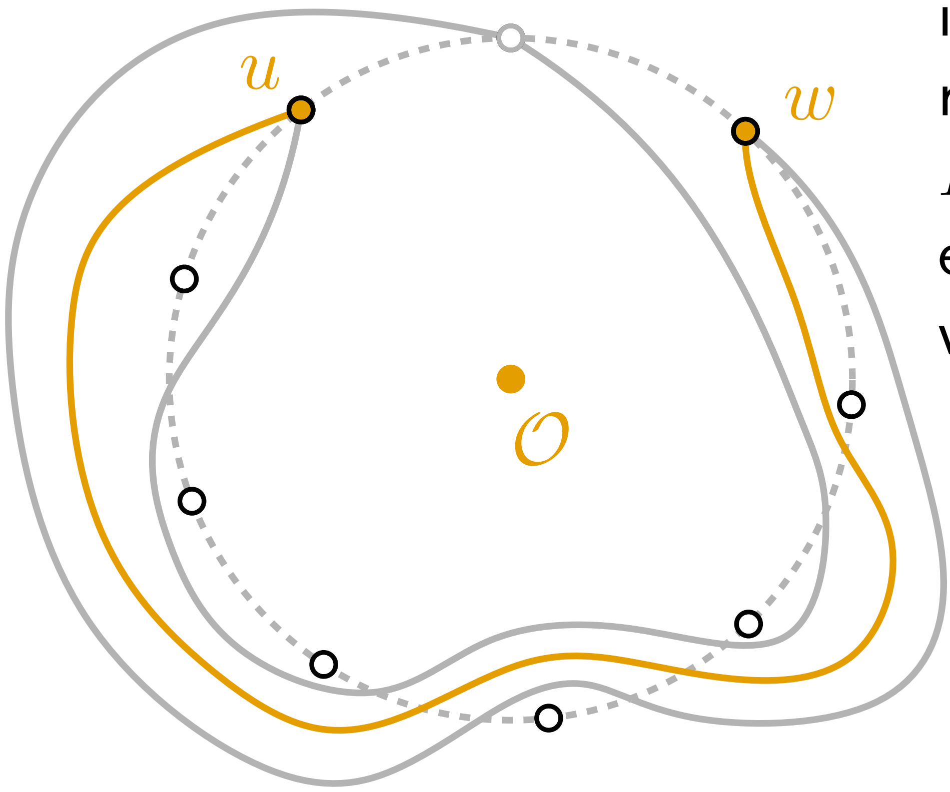

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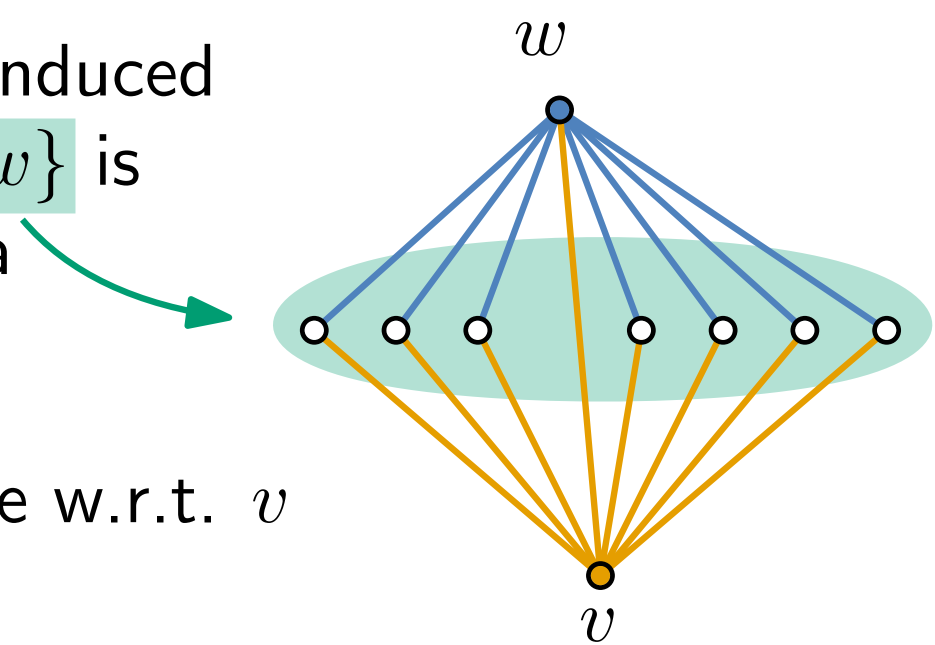
Details on how to guarantee the "good" edge in the plane double-star:

- The sub-drawing of D induced by the vertices $V \setminus \{v, w\}$ is strongly isomorphic to a c-monotone drawing.



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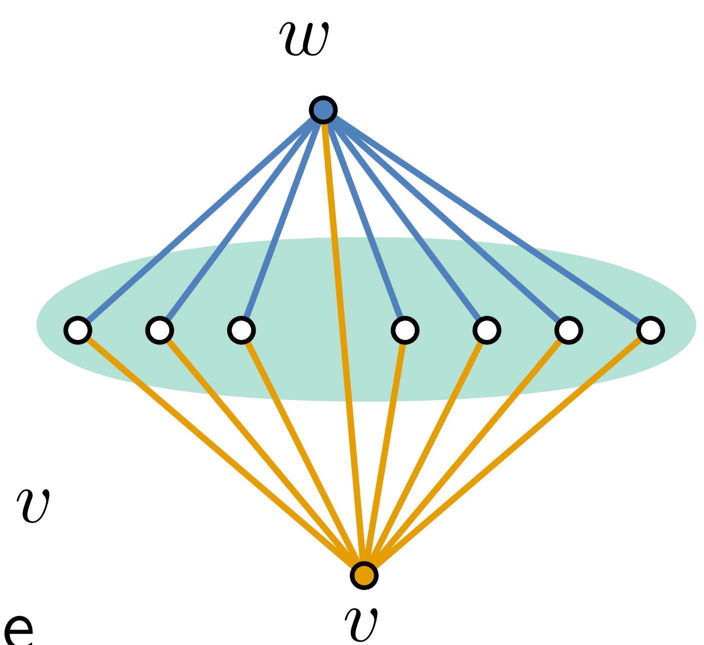

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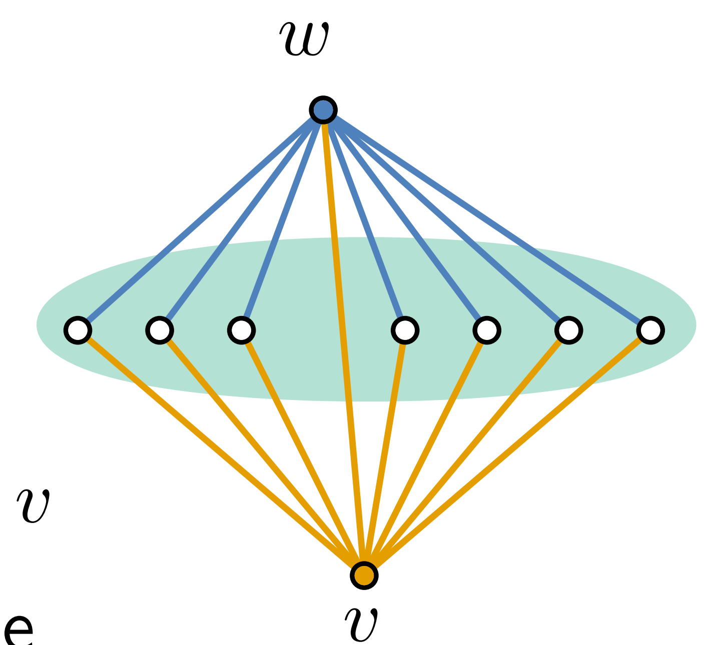

In particular: c-monotone w.r.t. v

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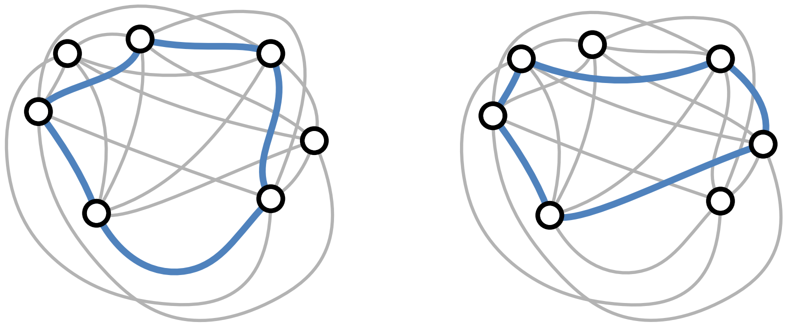
$\Rightarrow$ There exists a “good” edge, crossing at most vw

Conclusion

Theorem: Every simple drawing of K_n for $n \geq 5$ contains an empty plane k-cycle of length $k = 5$ or $k = 6$ *at every vertex*.



Open Problem [Orthaber 2025]:
Every simple drawing of K_n contains *two* empty plane k-cycles for $k = 3, \ldots, n$ *at every vertex*.

Lemma: Every c-monotone drawing (of K_n) for $n \geq 3$ contains a pair of neighboring vertices which are not joined by a long edge.

Thank you for your attention!