Algebraic Approach to Promise Constraint Satisfaction

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Constraint Satisfaction Problems

Constraint Satisfaction Problems

Definition

A **CSP template** $\mathbb{A}$ consists of

- a finite domain A
- a set of relations $R_1, \ldots, R_m$ on A.

An **instance** of $\mathbb{A}$ consists of

- a finite variable set X
- for each relation R_i of arity k_i a clause set $\mathcal{C}_i \subseteq X^{k_i}$

Is there a map $X \to A$ that satisfies all clauses?

Examples

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Let $B = \{0, 1\}$ and for $i, j, k \in \{0, 1\}$ let $R_{i,j,k} = \{0, 1\}^3 \setminus \{(ijk)\}$
Then $(x_1, x_2, x_3) \in R_{001} \Leftrightarrow x_1 \vee x_2 \vee \overline{x}_3$

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SAT is not equivalent to any CSP template.

Polymorphisms

Definition

Let $\mathbb{A}$ be a CSP template and $f : A^n \to A$ a map. We call f a **polymorphism** of $\mathbb{A}$ if for every relation R of $\mathbb{A}$ and every $a_1, \ldots, a_n \in R$ we have that

$$f(a_1, \ldots, a_n) \coloneqq (f(a_1^1, a_2^1, \ldots, a_n^1), f(a_1^2, a_2^2, \ldots, a_n^2), \ldots) \in R$$

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Examples

Projections are polymorphisms of every CSP template.

Let n be odd and consider the **majority operation** $f : \{0, 1\}^n \to \{0, 1\}$ which returns its most repeated argument. Then f is a polymorphism for 2-SAT.

Basic LP

Let $\mathbb{X} = (X, \mathcal{C}_1, \ldots, \mathcal{C}_m)$ be an instance of a CSP template $(A, R_1, \ldots, R_m)$. Consider the following ILP:

$$
\begin{aligned}
w_x(a) &\in \{0,1\} && \forall x \in X, a \in A && \text{Is } x \text{ assigned value } a\text{?} \\
p_C(y) &\in \{0,1\} && \forall C \in \mathcal{C}_i, y \in R_i && \text{Is } C \text{ satisfied by } y\text{?} \\
\sum_{a \in A} w_x(a) &= 1 && \forall x \in X && \\
\sum_{y \in R_i} p_C(y) &= 1 && \forall C \in \mathcal{C}_i && \\
\sum_{\substack{y \in R_i \\ y_{|x} = a}} p_C(y) &= w_x(a) && \forall x \in X, a \in A, C \in \mathcal{C}_i &&
\end{aligned}
$$

ILP has a solution $\Leftrightarrow$ $\mathbb{X}$ is satisfiable

Basic LP

Idea for polynomial time algorithm: accept iff LP relaxation of the ILP has a solution

Theorem 1

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Proof: Let w, p be a rational solution to the LP. Choose $k \in \mathbb{N}^+$ such that $\hat{w} := k \cdot w, \hat{p} := k \cdot p$ are integral.

LP rounding

We now have a solution to

$$\begin{aligned}
\hat{w}_x(a) &\in \mathbb{N} && \forall x \in X, a \in A \\
\hat{p}_C(y) &\in \mathbb{N} && \forall C \in \mathcal{C}_i, y \in R_i \\
\sum_{a \in A} \hat{w}_x(a) &= k && \forall x \in X \\
\sum_{y \in R_i} \hat{p}_C(y) &= k && \forall C \in \mathcal{C}_i \\
\sum_{\substack{y \in R_i \\ y_{|x} = a}} \hat{p}_C(y) &= \hat{w}_x(a) && \forall x \in X, a \in A, C \in \mathcal{C}_i
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\end{aligned}$$

Let f be a k-ary symmetric polymorphism of $\mathbb{A}$. For $x \in X$ let $a_x^1, a_x^2, \ldots, a_x^k$ be such that any a occurs $\hat{w}_x(a)$ many times. Assign x the value $f(a_x^1, a_x^2, \ldots, a_x^k)$.
This yields a satisfying assignment.

CSP dichotomy

Theorem 2 [Bulatov 2017, Zhuk 2017]

CSP($\mathbb{A}$) is in P iff $\mathbb{A}$ admits a k-ary polymorphism f with $k \geq 2$ satisfying the identities

$$f(y, x, x, \ldots, x) = f(x, y, x, x, \ldots, x) = f(x, x, x, \ldots, x, y).$$

Otherwise it is NP-hard.

In particular, CSP($\mathbb{A}$) never has intermediate complexity.

Promise Constraint Satisfaction

PCSP

Definition

Let $\mathbb{A} = (A, R_1^A, \ldots, R_m^A)$ and $\mathbb{B} = (B, R_1^B, \ldots, R_m^B)$ be CSP templates such that for each i we have $\mathrm{ar}(R_i^A) = \mathrm{ar}(R_i^B)$. A **homomorphism** from $\mathbb{A}$ to $\mathbb{B}$ is a map h such that for each i

$$\text{if } (a_1, \ldots, a_{\mathrm{ar}(R_i^A)}) \in R_i^A, \text{ then } (h(a_1), \ldots, h(a_{\mathrm{ar}(R_i^A)})) \in R_i^B.$$

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Notation: $\mathbb{A} \to \mathbb{B}$ if a homomorphism exists.
A **PCSP template** consists of a pair $(\mathbb{A}, \mathbb{B})$ such that $\mathbb{A} \to \mathbb{B}$. Given an instance $\mathbb{X}$, decide whether $\mathbb{X}$ is satisfiable over $\mathbb{A}$ or unsatisfiable over $\mathbb{B}$.

Examples

- $\mathrm{PCSP}(\mathbb{A}, \mathbb{A}) = \mathrm{CSP}(\mathbb{A})$
- Let $\mathbb{A} = (\{0, 1\}; \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\})$ be 1-in-3-SAT
  and $\mathbb{B} = (\{0, 1\}; \{0, 1\}^3 \setminus \{(0, 0, 0)\,(1, 1, 1)\})$ be NAE-SAT
- Let $\mathbb{K}_k = (\{1, \ldots, k\}; \{(i, j) \mid i \neq j\})$.
  For $k < c$, $\mathrm{PCSP}(\mathbb{K}_\mathrm{k}, \mathbb{K}_\mathrm{c})$ is the approximate graph colouring problem.

Polymorphisms

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Example

Consider 1-in-3-SAT vs NAE-SAT. For $k \geq 1$, define

$$f_k : \{0,1\}^{3k-1} \to \{0,1\}, (a_1, \ldots, a_{3k-1}) \mapsto \begin{cases} 1, & \text{if at least } k \text{ of the } a_i \text{ are } 1 \\ 0, & \text{else} \end{cases}$$

Hardness of PCSP

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Not all results from CSP apply to PCSP. In particular, there is no (known) dichotomy.

Approximate Graph Colouring

Known results

Deciding whether a graph is k-colourable or not c-colourable is NP-hard for

- k sufficiently large, $c \leq 2^{\Theta(k^{\frac{1}{3}})}$ [Huang 2013]
- $k \geq 3, c \leq 2k - 2$ [Brakensiek, Guruswami 2016]
- $k \geq 3, c = 2k - 1$ [Barto, Bulín, Krokhin, Opršal 2019]

In polynomial time, one can colour 3-colourable n-vertex graphs using $\mathcal{O}(n^{0.1999})$ colours [Kawarabayashi, Thorup 2017]

Olšák polymorphisms

Definition

An **Olšák polymorphism** is a 6-ary polymorphism o satisfying the identities

$$o(x,x,y,y,y,x) = o(x,y,x,y,x,y) = o(y,x,x,x,y,y).$$

Lemma

Let $(\mathbb{A},\mathbb{B})$ be a PCSP template that does not admit an Olšák polymorphism. Then $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ is NP-hard.

Polymorphisms for approximate graph colouring

Recall: $\mathbb{K}_k = (\{1, \ldots, k\}; \{(i, j) \mid i \neq j\})$.

A homomorphism $\mathbb{K}_k \to \mathbb{K}_c$ corresponds to a graph homomorphism $K_k \to K_c$. Similarly, an n-ary polymorphism f of $(\mathbb{K}_k, \mathbb{K}_c)$ corresponds to a graph homomorphism $f' : K_k^n \to K_c$.

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Assume f is an Olšák polymorphism. It maps tuples of the form $(x, x, y, y, y, x), (x, y, x, y, x, y)$, and (y, x, x, x, y, y) to the same value
$\implies$ we can glue the corresponding vertices of K_k^6 to get a c-colourable graph G.

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Claim: G is not $(2k-1)$-colourable.

Hardness of approximate graph colouring

Claim: G obtained by glueing vertices of the form $(x, x, y, y, y, x), (x, y, x, y, x, y)$, and (y, x, x, x, y, y) is not $(2k-1)$-colourable.

Consider the $2k$ vertices $a_i = (i, i+1, i+2, i+1, i+2, i)$ and $b_i = (i+1, i, i, i, i+1, i+1) = (i, i+1, i, i+1, i, i+1) = (i, i, i+1, i+1, i+1, i)$.

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For $i \neq j$, a_i is a neighbour of a_j and b_i of b_j.
For any i, j, a_i and b_j are neighbours.
$\implies$ G contains a $2k$-clique.

Hardness of approximate graph colouring

Theorem 3 [Barto, Bulín, Krokhin, Opršal 2019]

$\mathrm{PCSP}(\mathbb{K}_k, \mathbb{K}_{2k-1})$ is NP-hard.

Remark

$\mathrm{PCSP}(\mathbb{K}_k, \mathbb{K}_{2k})$ has an Olšák polymorphism.

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