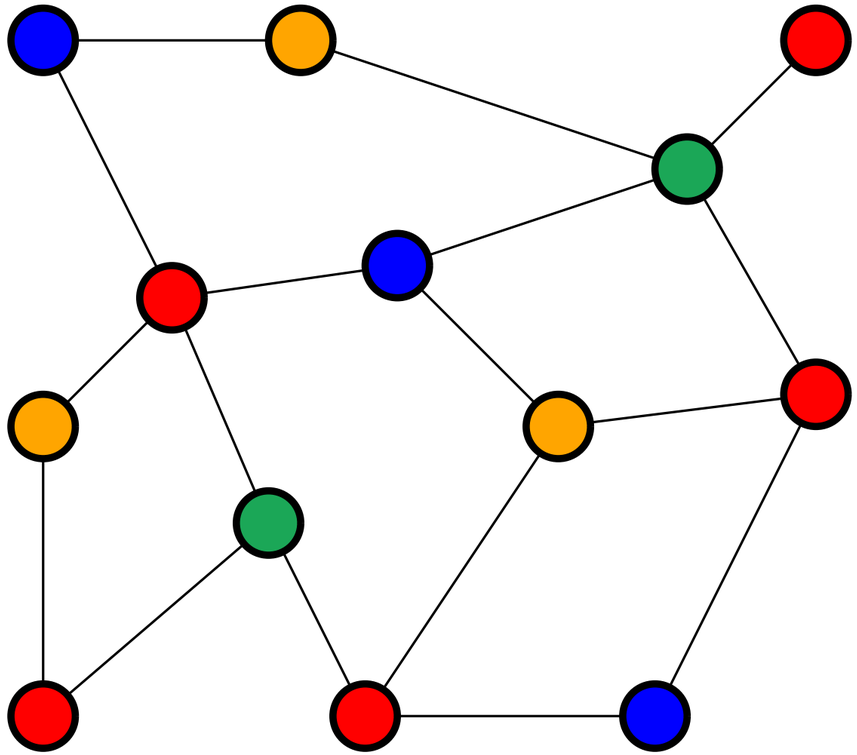
Recent Developments in Distributed Δ-coloring

Manuel Jakob

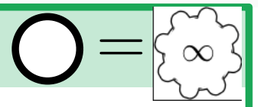
ATCS Seminar

Setup: distributed Δ-Coloring



Δ :=max. degree (here: $\Delta = 4$)

LOCAL-model



- unbounded communication in synchrounouse rounds
- each vertex needs to output its part of the solution
- complexity is the number of communication rounds

Δ-Coloring

- color vertices with colors $\{1, \ldots, \Delta\}$ s..t. no adjacent vertices get the same color

Assumptions::

- # vertices and Δ is known to all vertices
- max. degree Δ is constant

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Lower Bound:

$\Omega(\log n)$ rounds [Chang, Kopelowitz, Pettie '16]

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$O(\log^3 n)$ rounds	[Panconesi, Sirinivasan '93]
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$O(\log n \cdot \log^* n)$ rounds	[Bourreau, Brandt, Nolin '25]
$O(\log n)$ rounds on dense graphs	[**J**, Maus '25]
$O(\log n)$ rounds	[Bourreau, Brandt, Nolin '26]

Preliminaries/Tools

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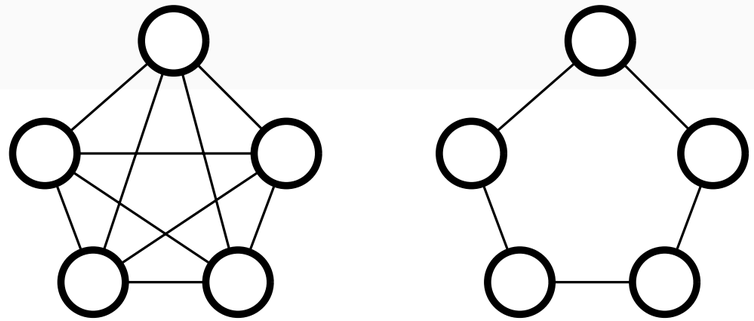
Brooks Theorem: [Brooks '41]

A graph G can be Δ-colored if it doesn't contain a clique of size $\Delta + 1$ or an odd cycle with $\Delta = 2$



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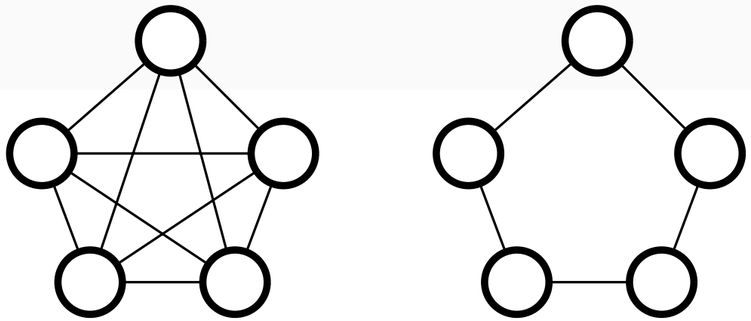
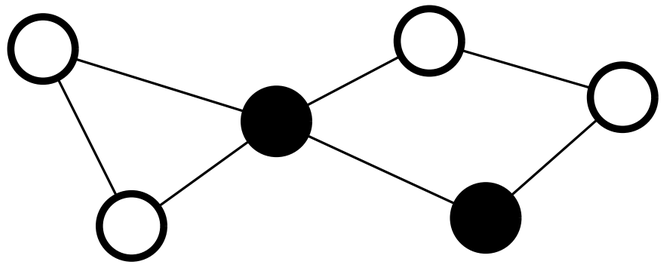


Maximal Independent Set (MIS):

A set of non-adjacent vertices that cannot be extended further. $O(\log^* n)$ [Faour, Ghaffari, Grunau, Kuhn, Rozhon '23]



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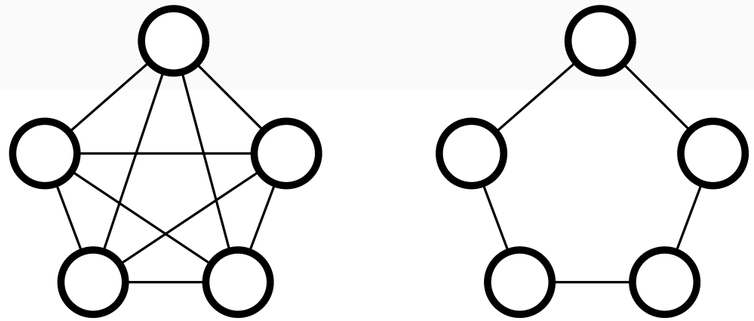

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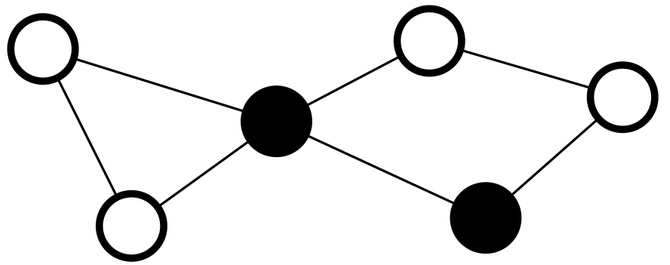
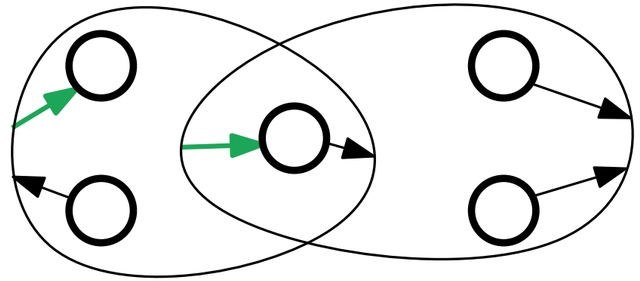


(Hyper-)Edge Sinkless Orientation (H)SO:

An orientation of (hyper-)edges where no vertex is a sink. $O(\log_{\delta/r} n)$ [Brandt, Maus, Narayanan, Schager '25]



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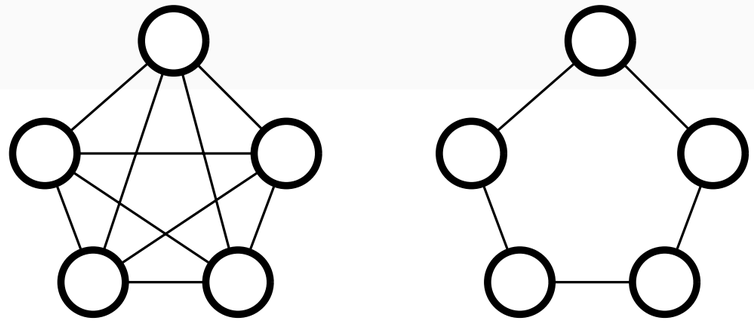

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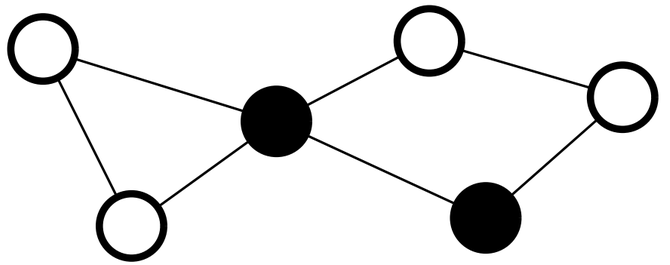

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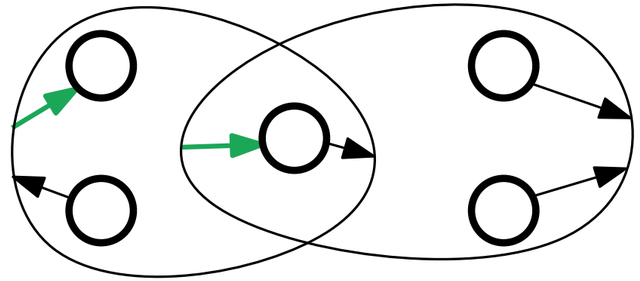

$(\Delta + 1)$ - Coloring / $deg + 1$ - Coloring

$O(\log^* n)$ [Linial '92]

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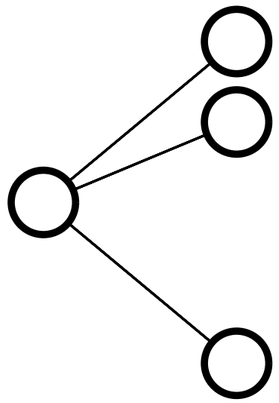

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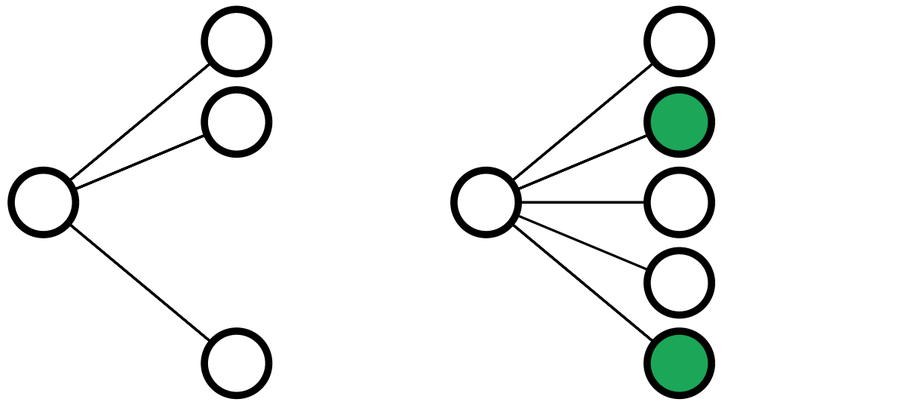

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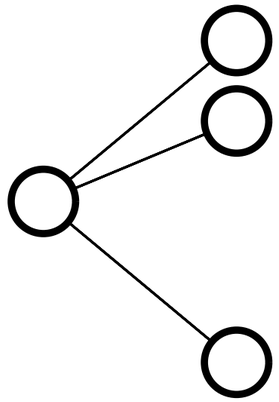
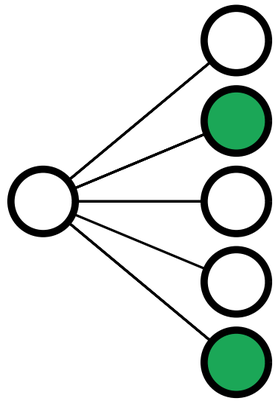
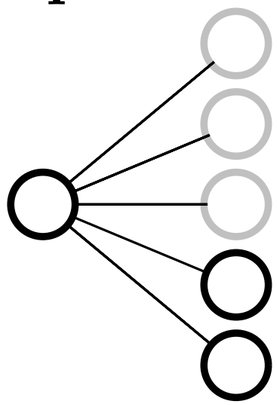




temporal slack



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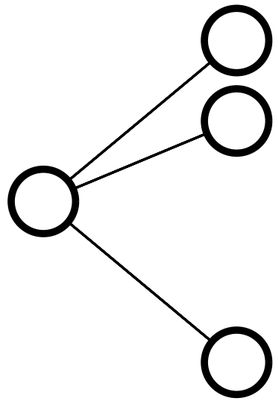

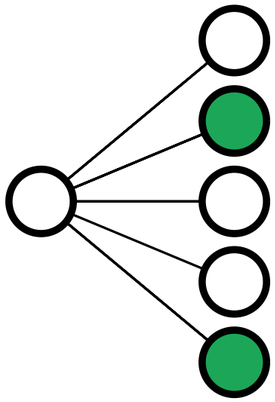

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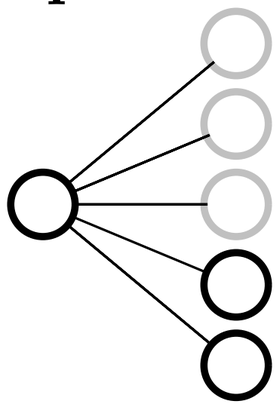

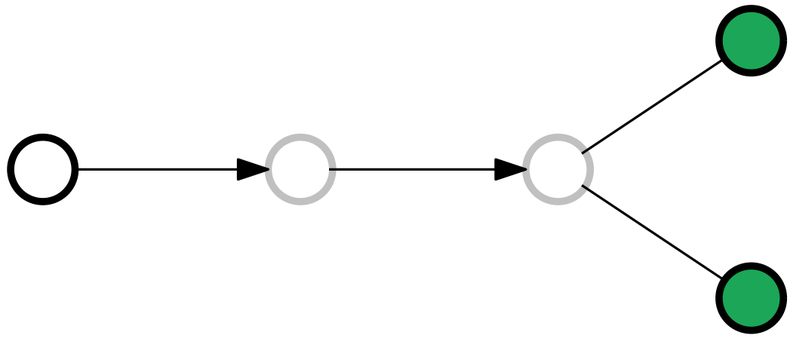

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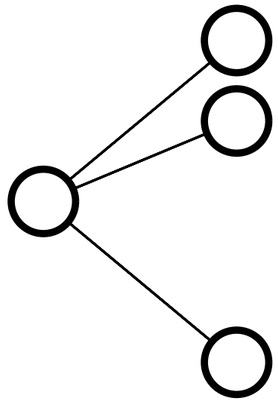

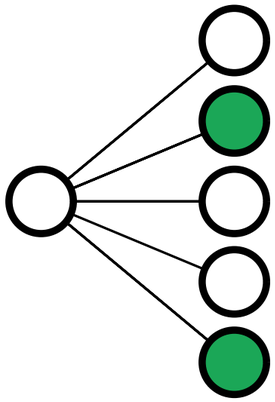

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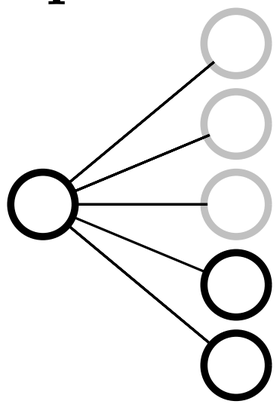

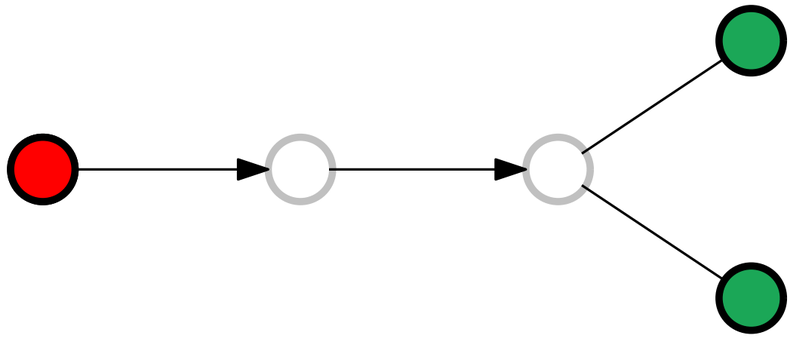

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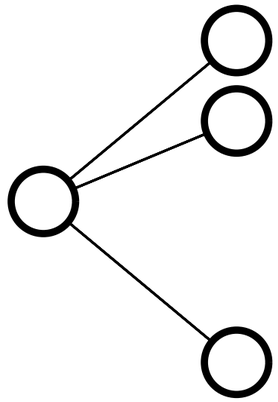

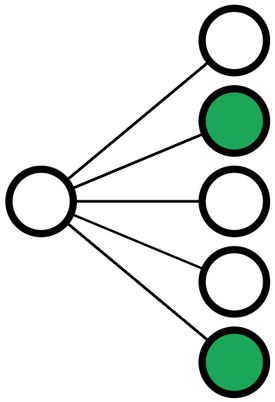

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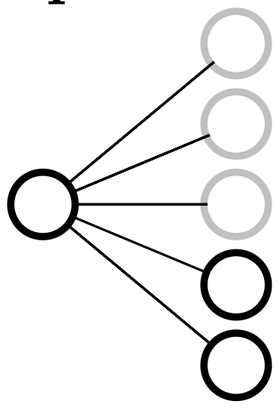

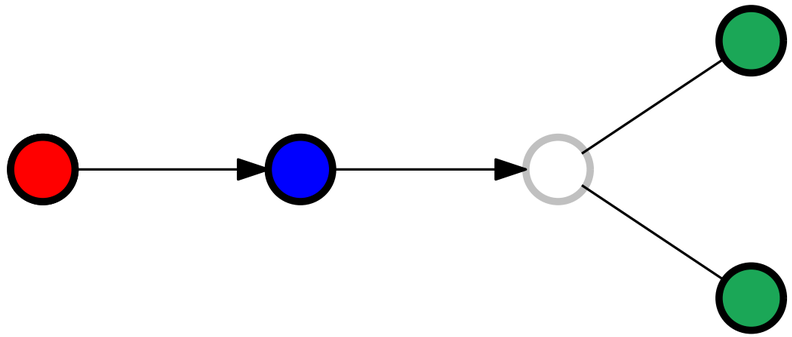

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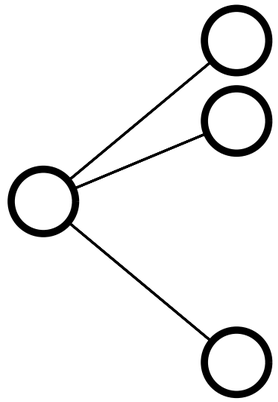

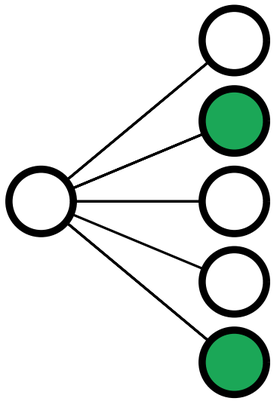

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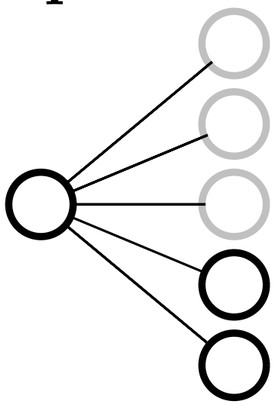

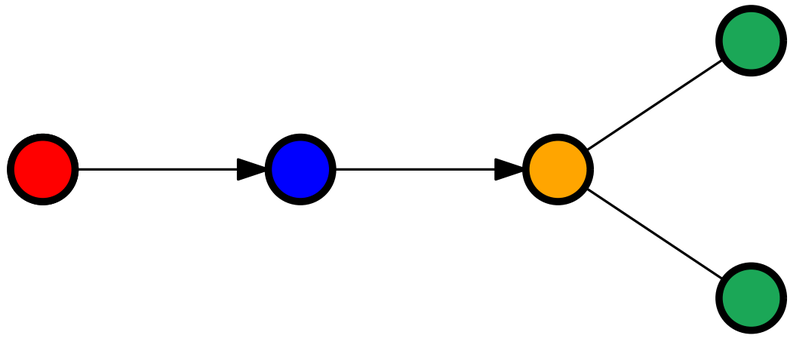

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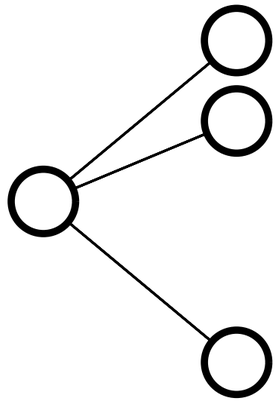

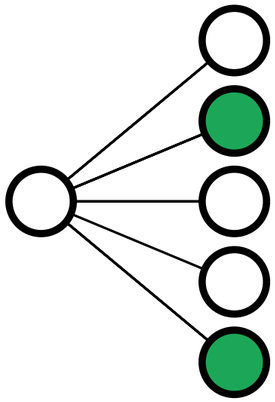

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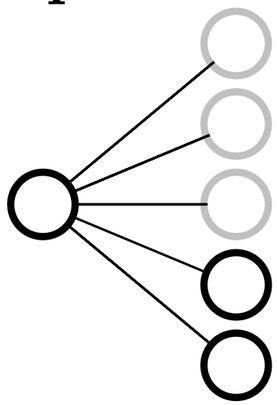
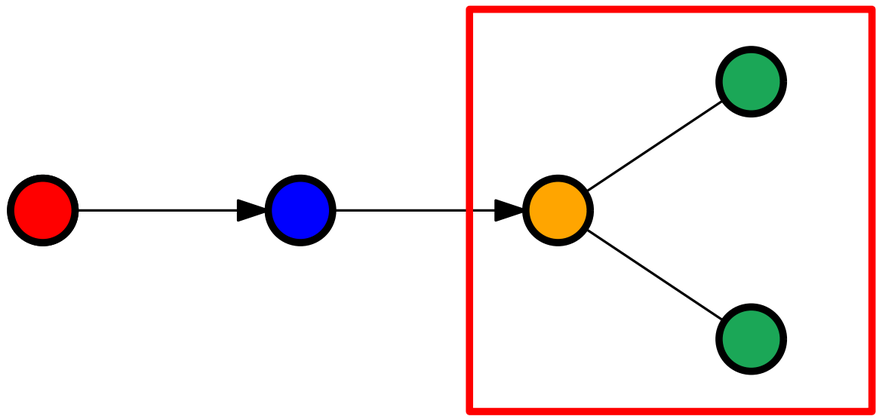


Slack Triad



propagation of slack

Δ-Coloring via Ruling Subgraphs

[Bourreau, Brandt, Nolin '25]

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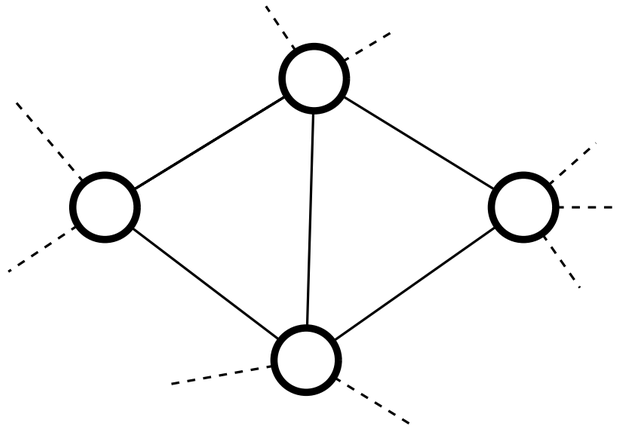
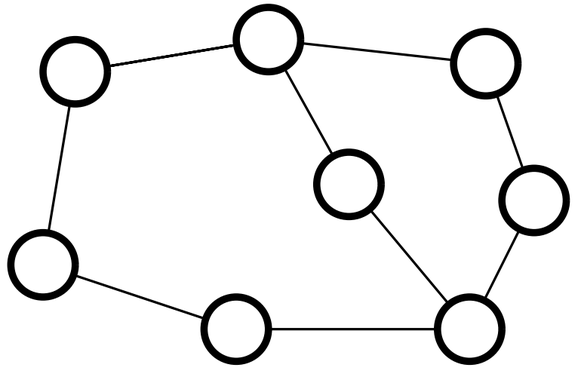
Definition: Degree Choosable Component (DCC)

is a graph for which it is always possible to complete a proper Δ-coloring





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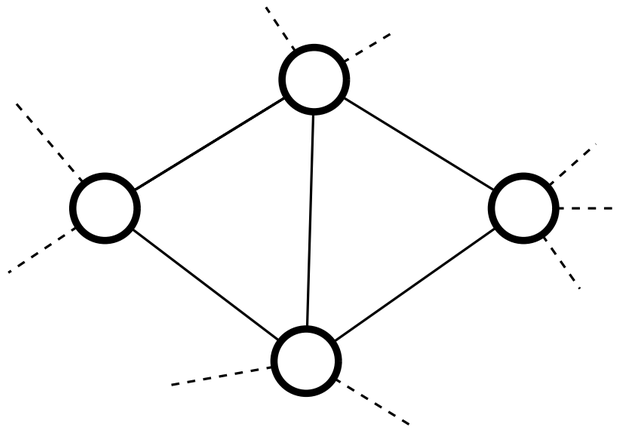
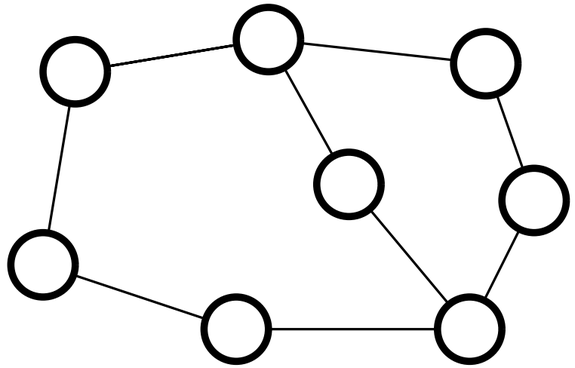
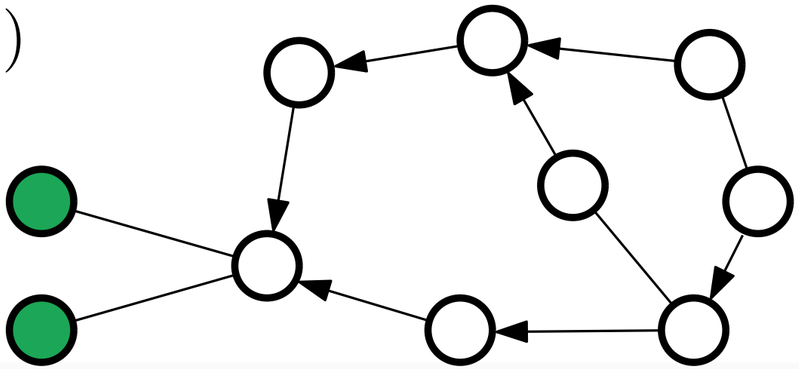


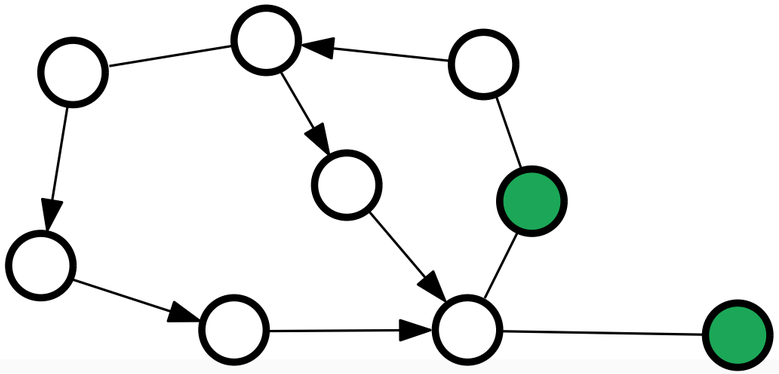


Proof:

(1)



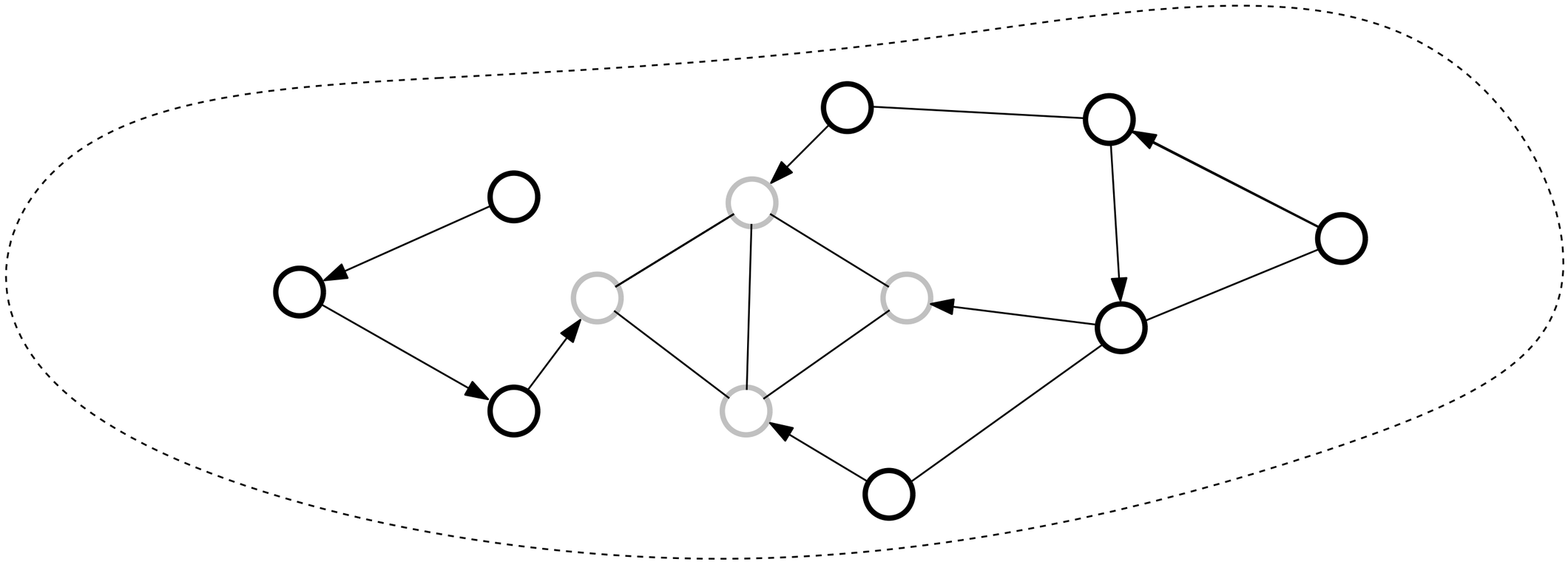
(2)



Δ-Coloring via Ruling Subgraphs

[Bourreau, Brandt, Nolin '25]

Approach:



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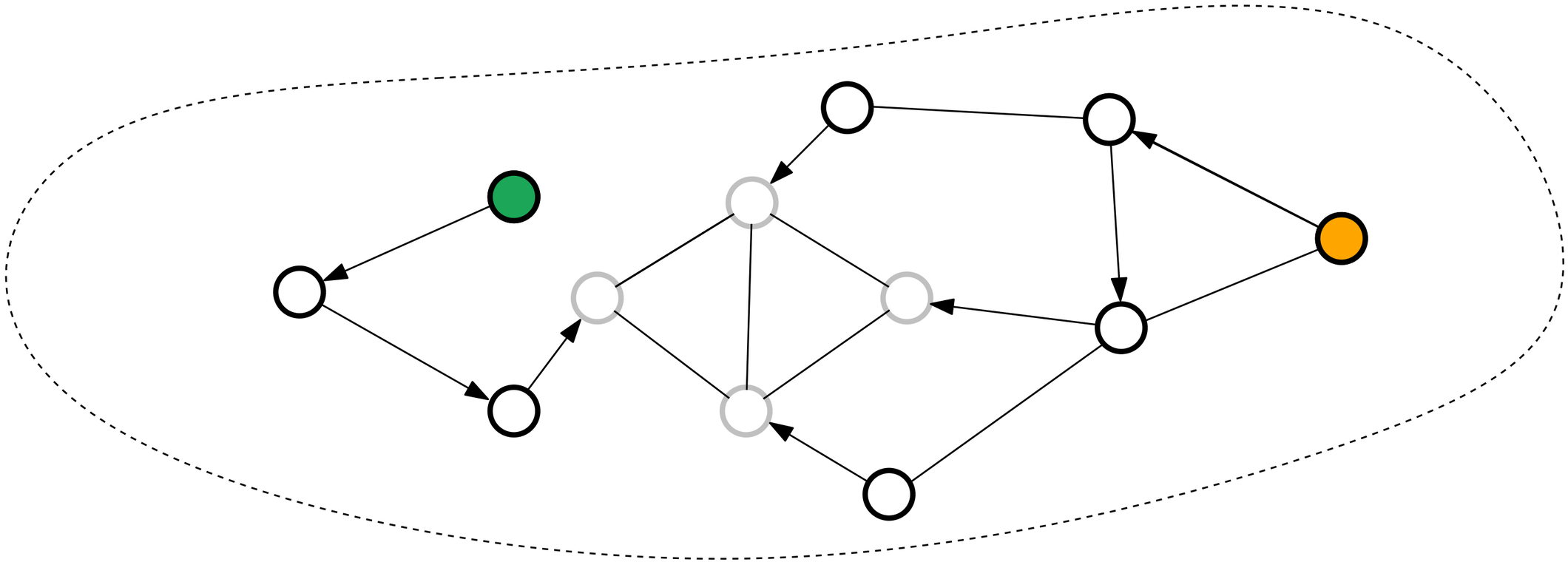

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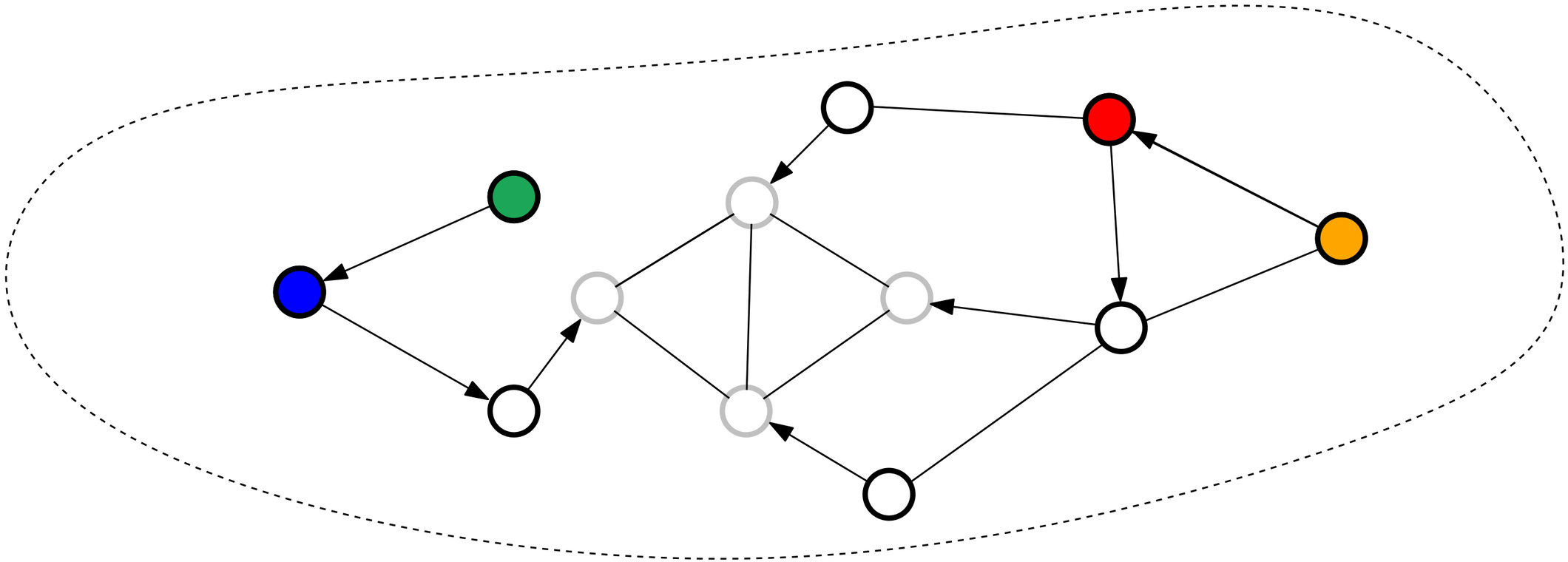

Δ-Coloring via Ruling Subgraphs

[Bourreau, Brandt, Nolin '25]

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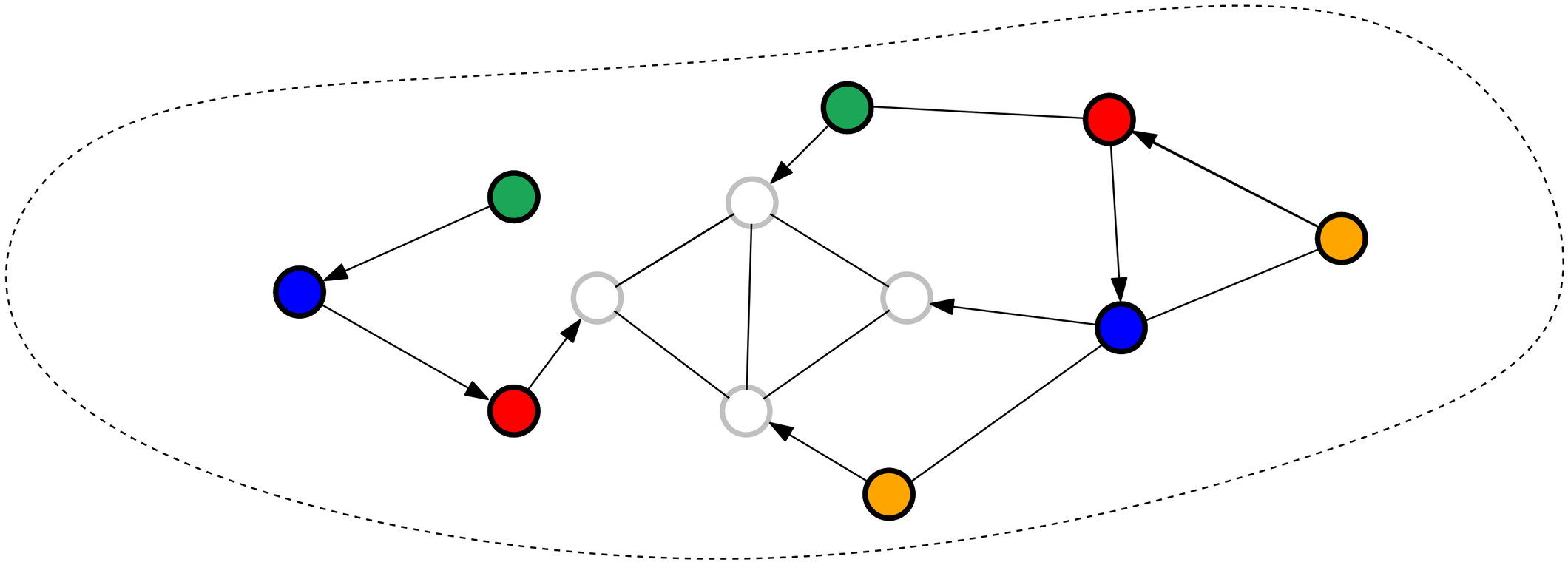
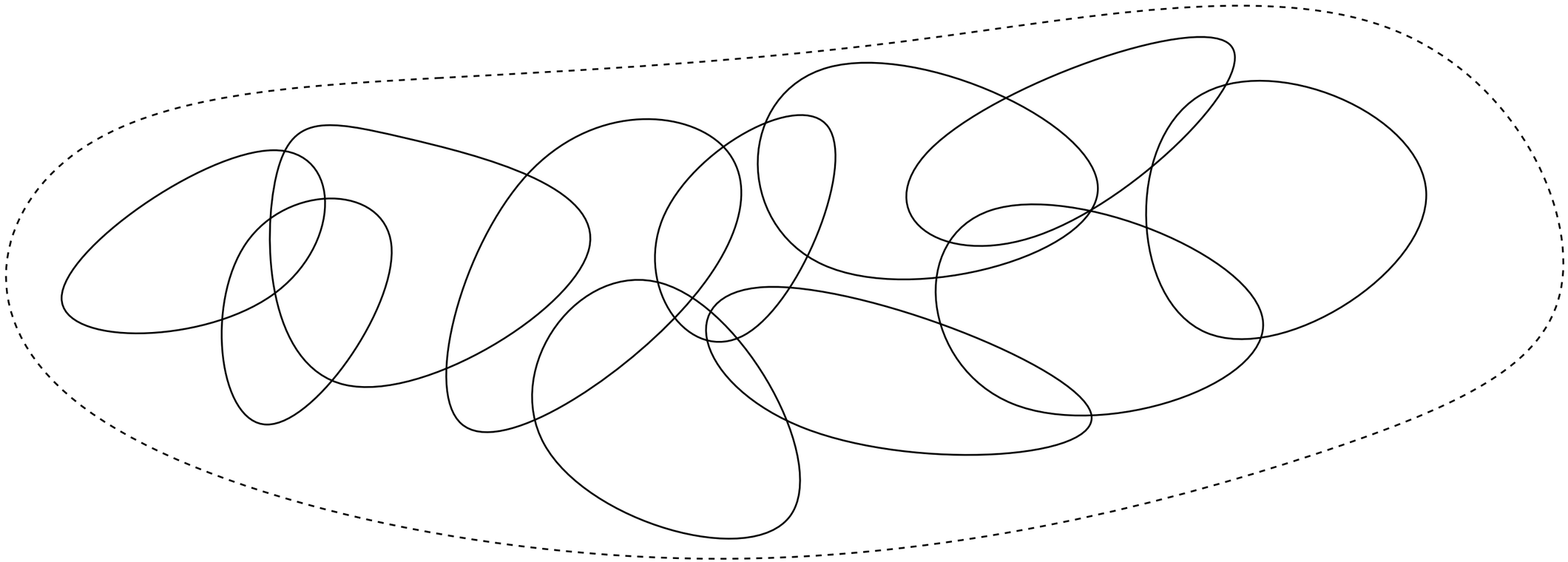


Δ-Coloring via Ruling Subgraphs

[Bourreau, Brandt, Nolin '25]

Problem: DCCs can overlap...



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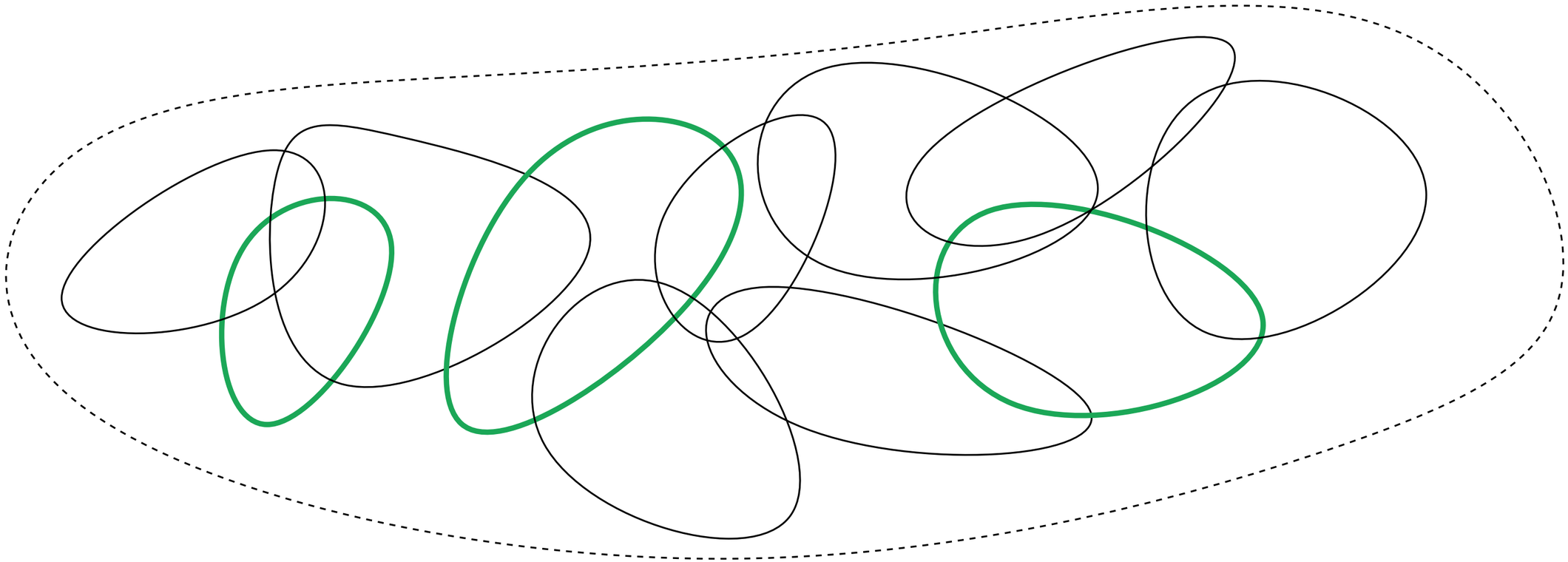

Δ-Coloring via Ruling Subgraphs

[Bourreau, Brandt, Nolin '25]

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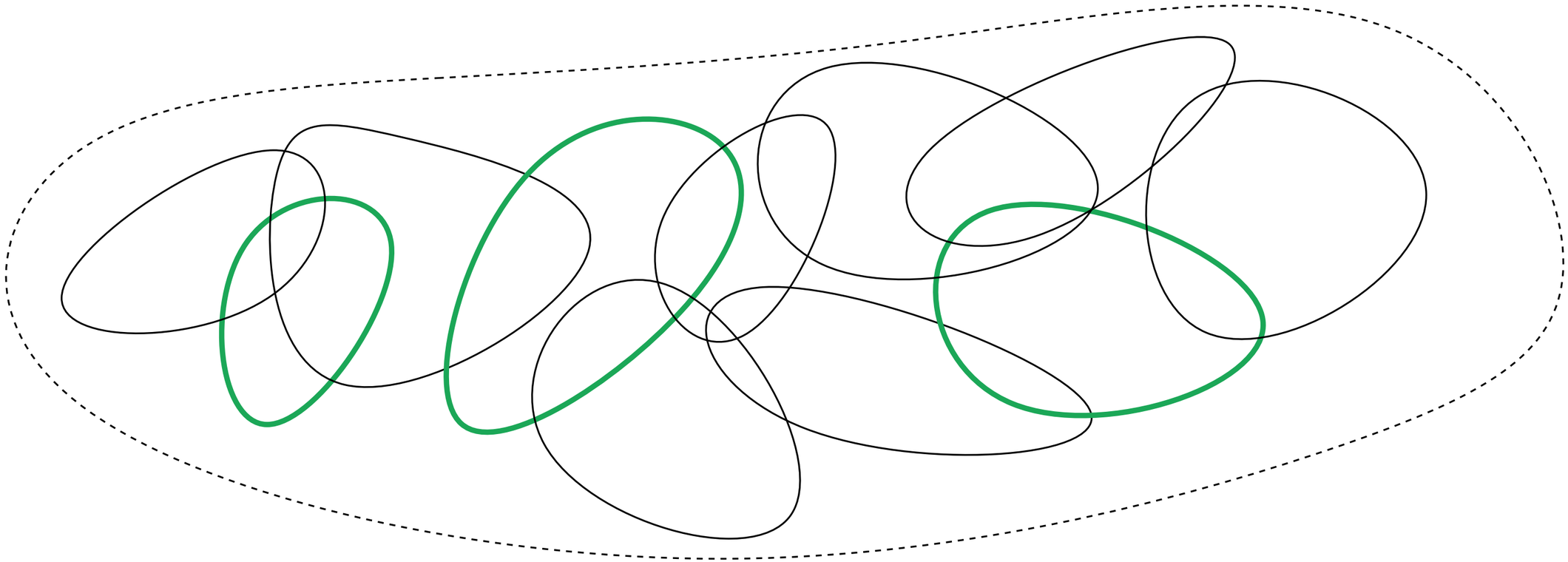

Algorithm (simplified):

1. each vertex finds all DCCs in a $\log n$ neighborhood and selects one arbitrary
2. compute an independent set of DCCs
3. color vertices in decreasing distance to the closest DCC ($(\Delta+1)$-coloring)

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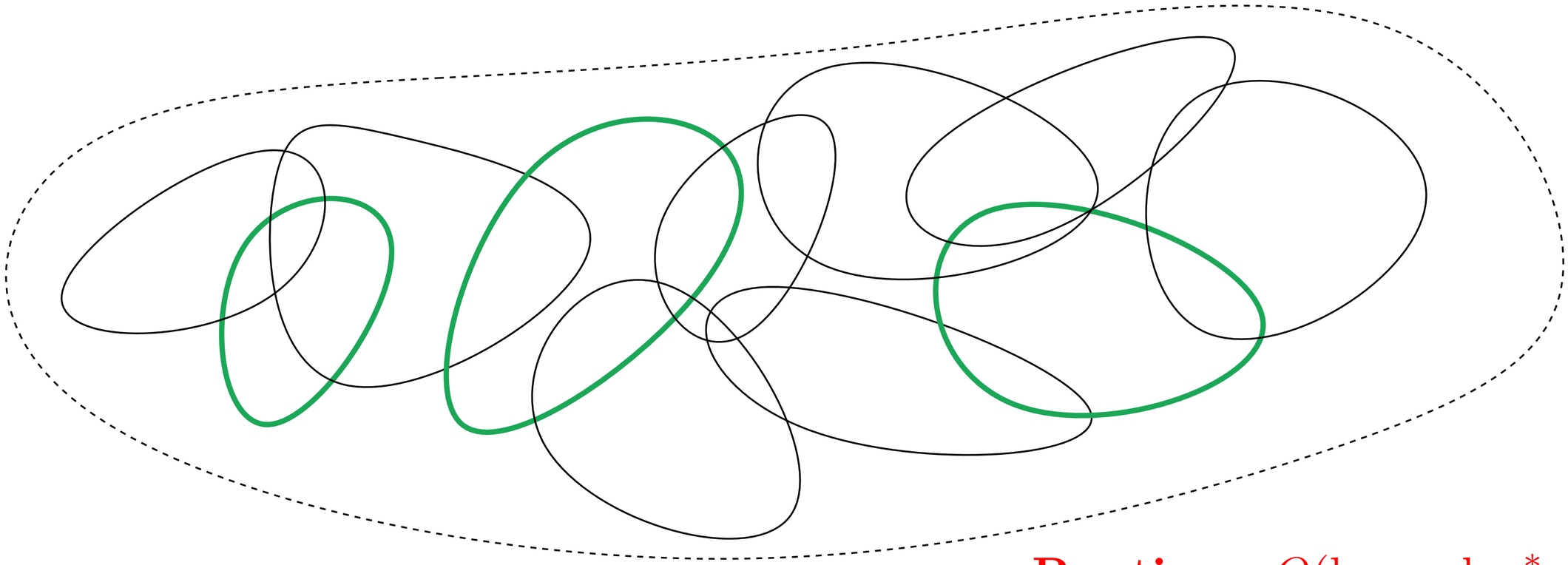

Runtime: $O(\log n \cdot \log^* n)$

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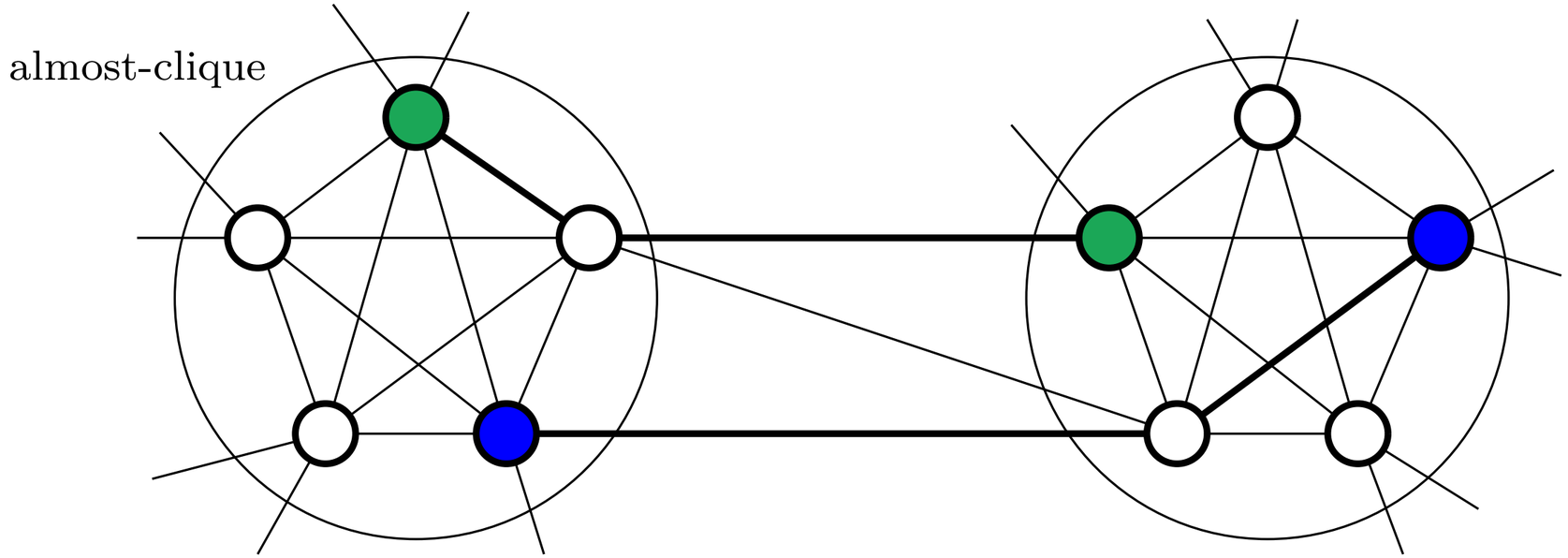
Towards an optimal algorithm

[**J**, Maus '25]

locally-dense graph:



Towards an optimal algorithm

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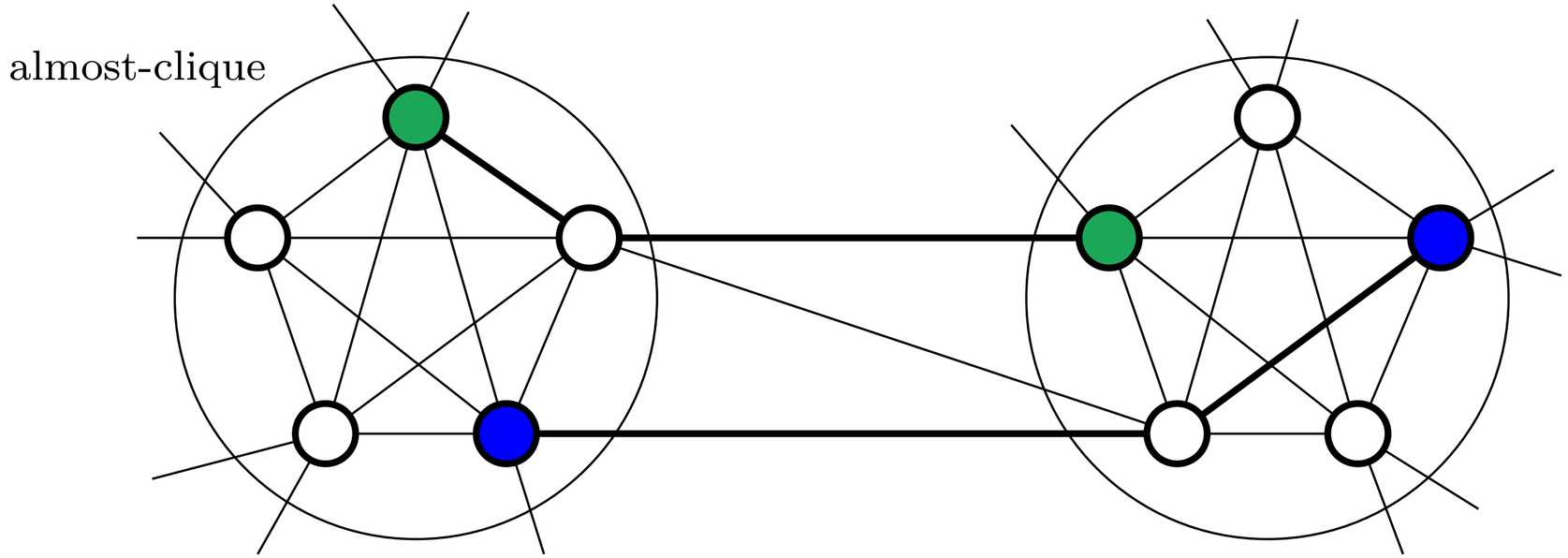

Algorithm (simplified):

1. compute a Maximal Matching of the edges between clusters
2. choose one edge for each cluster (HSO)
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Towards an optimal algorithm

[**J**, Maus '25]

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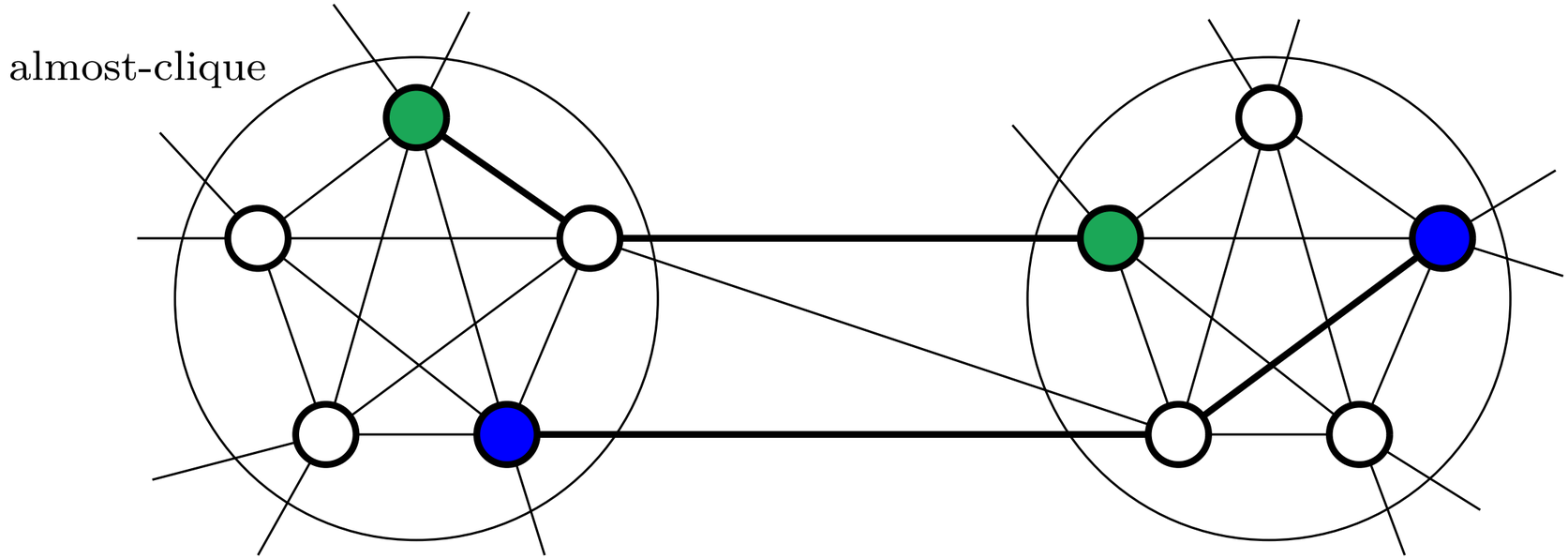

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Runtime: $O(\log n)$

Towards an optimal algorithm

[**J**, Maus '25]

Why is this not applicable to all graphs?

Towards an optimal algorithm

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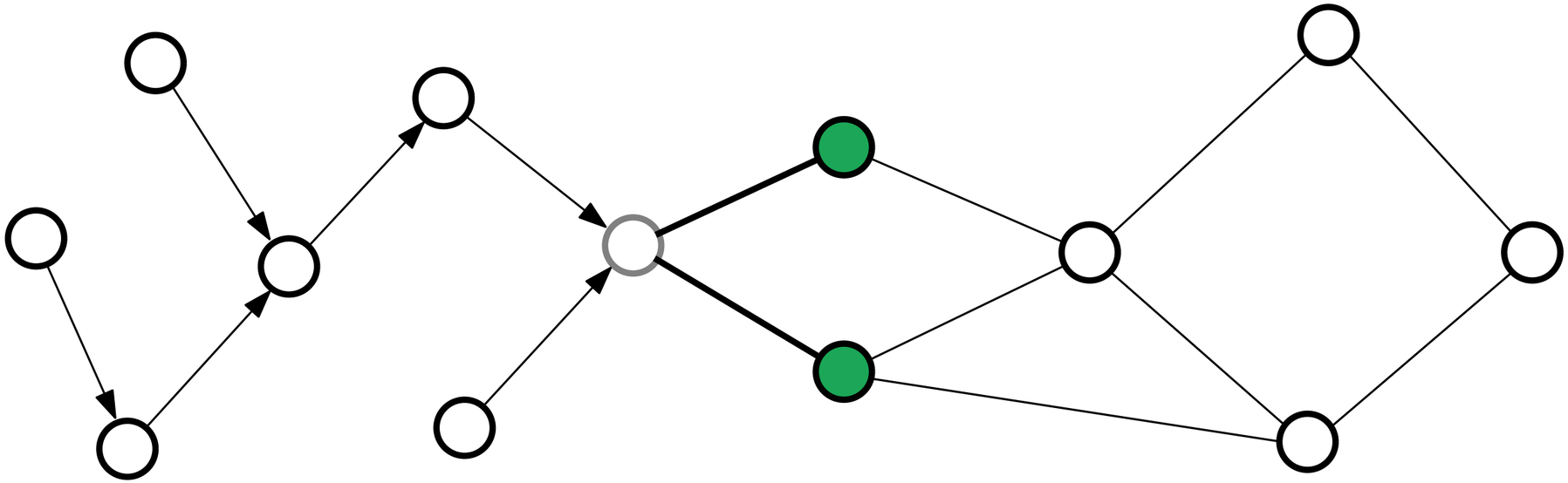
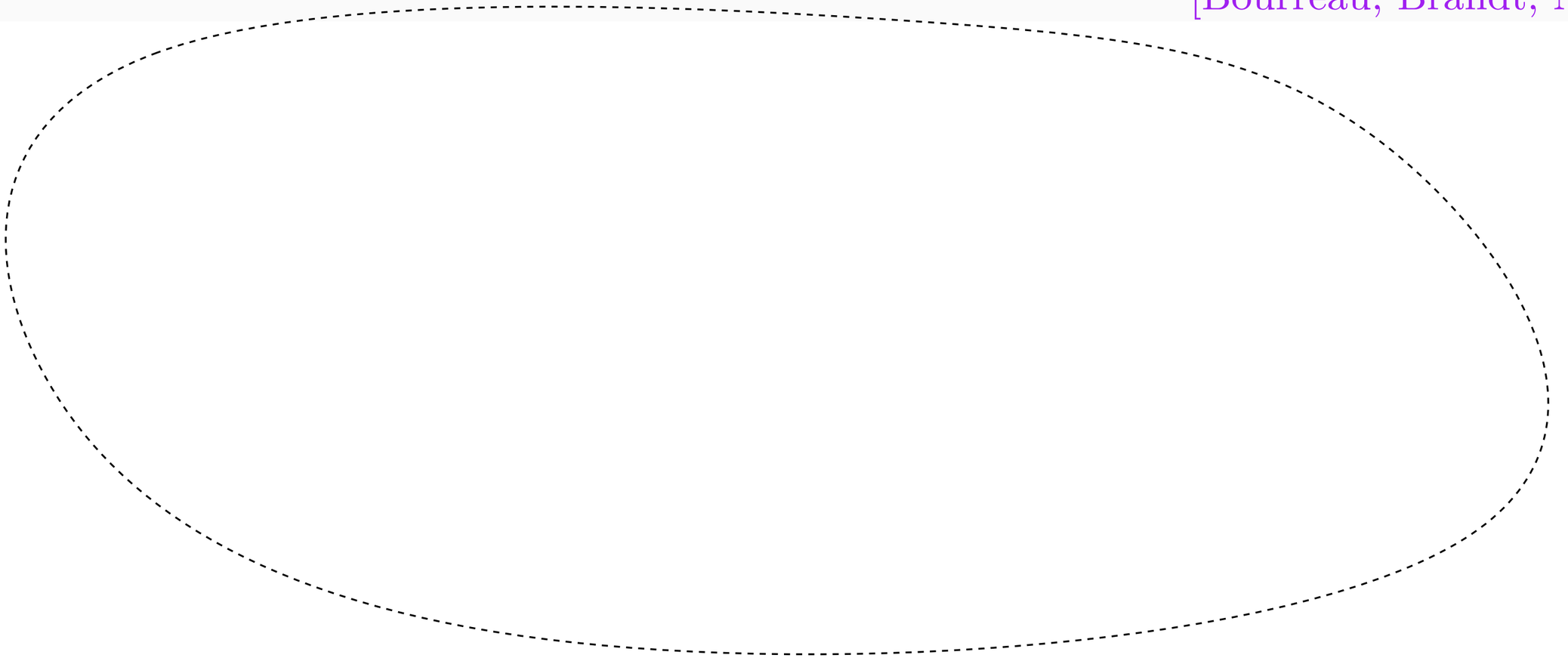


Reduction to MIS

[Bourreau, Brandt, Nolin '26]



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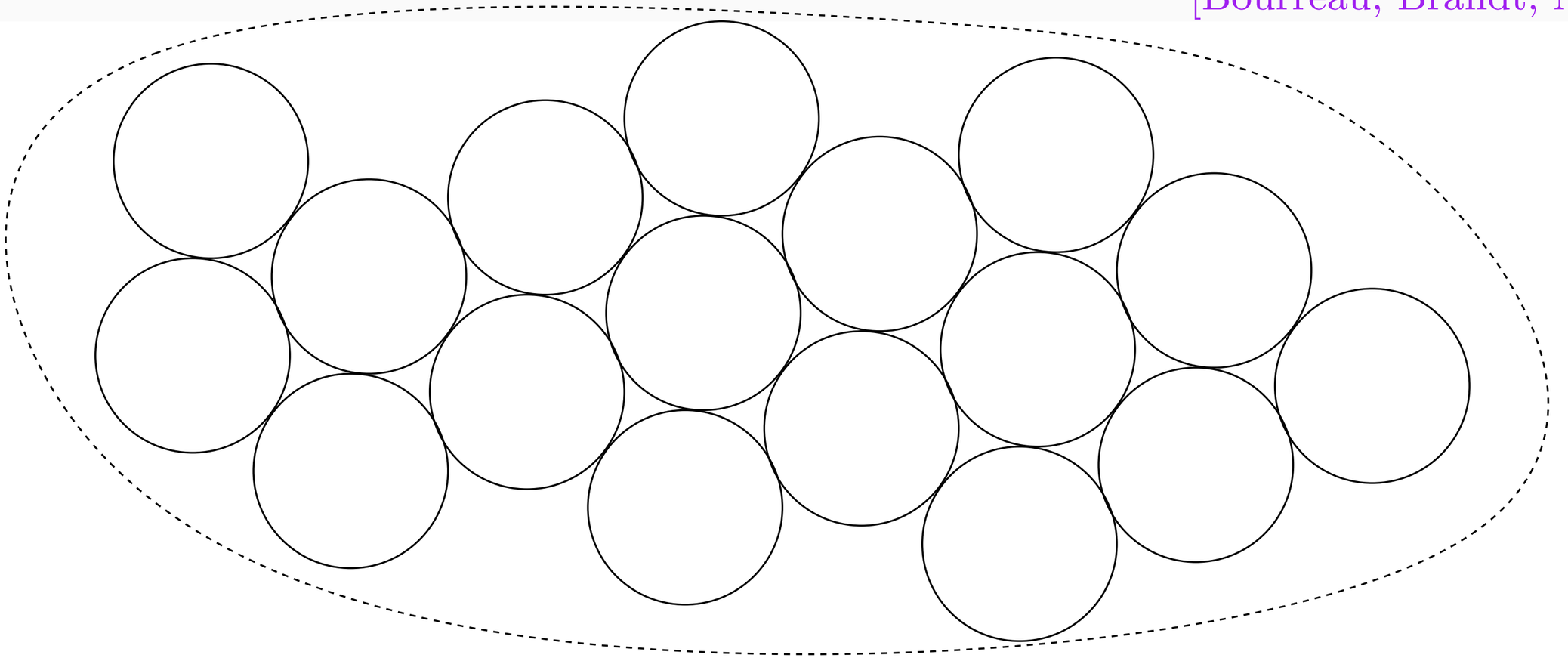
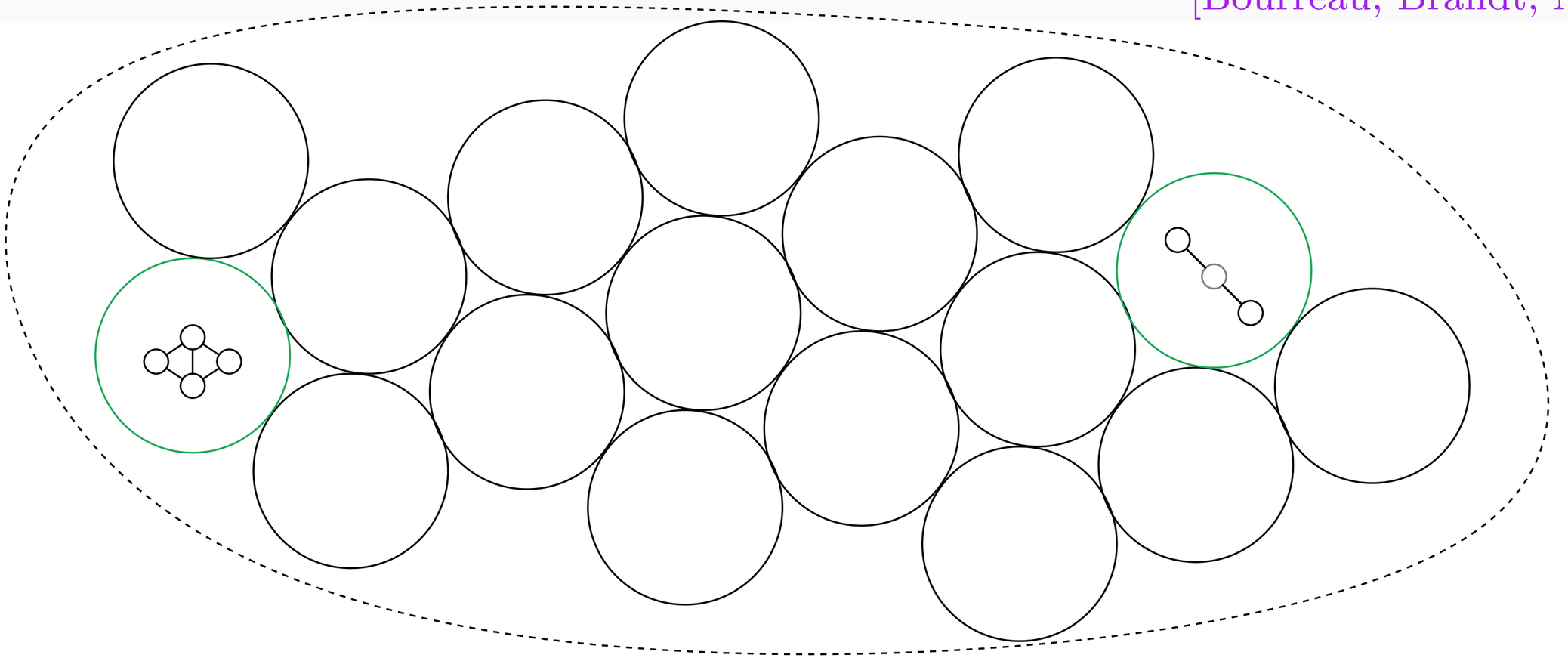


Algorithm (simplified):

1. compute a MIS on G^c

Reduction to MIS

[Bourreau, Brandt, Nolin '26]

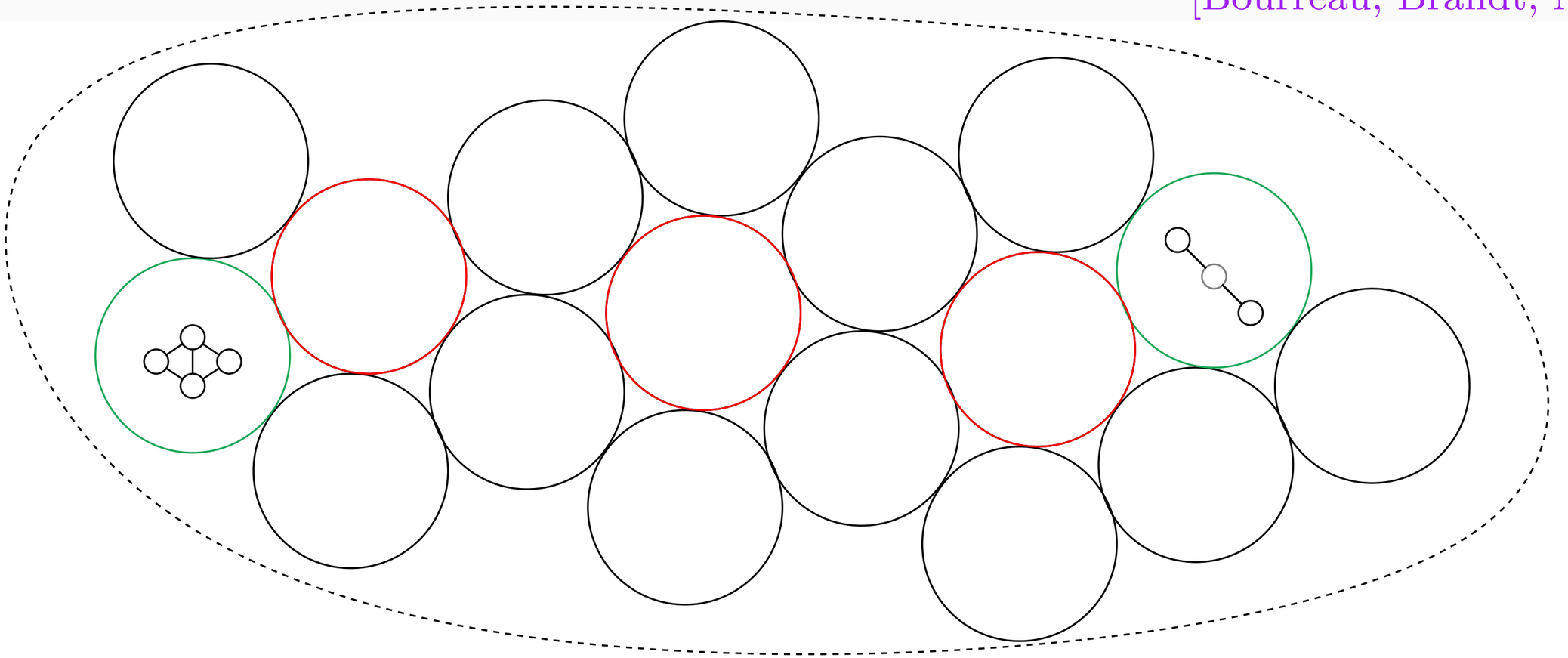


Algorithm (simplified):

1. compute a MIS on G^c
2. define C_{DCC} clusters

Reduction to MIS

[Bourreau, Brandt, Nolin '26]



Algorithm (simplified):

1. compute a MIS on G^c
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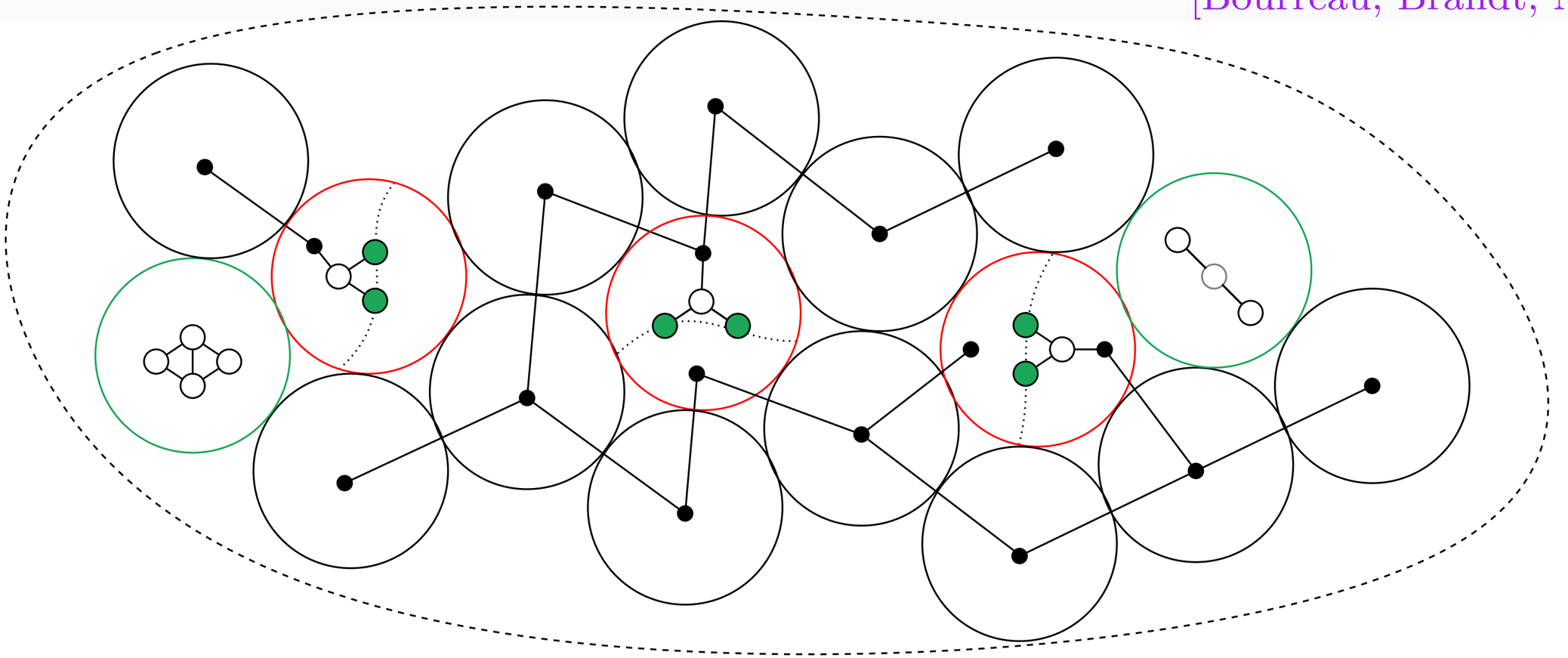
3. compute MIS to get C_{flex} and C_{link}

Reduction to MIS

[Bourreau, Brandt, Nolin '26]



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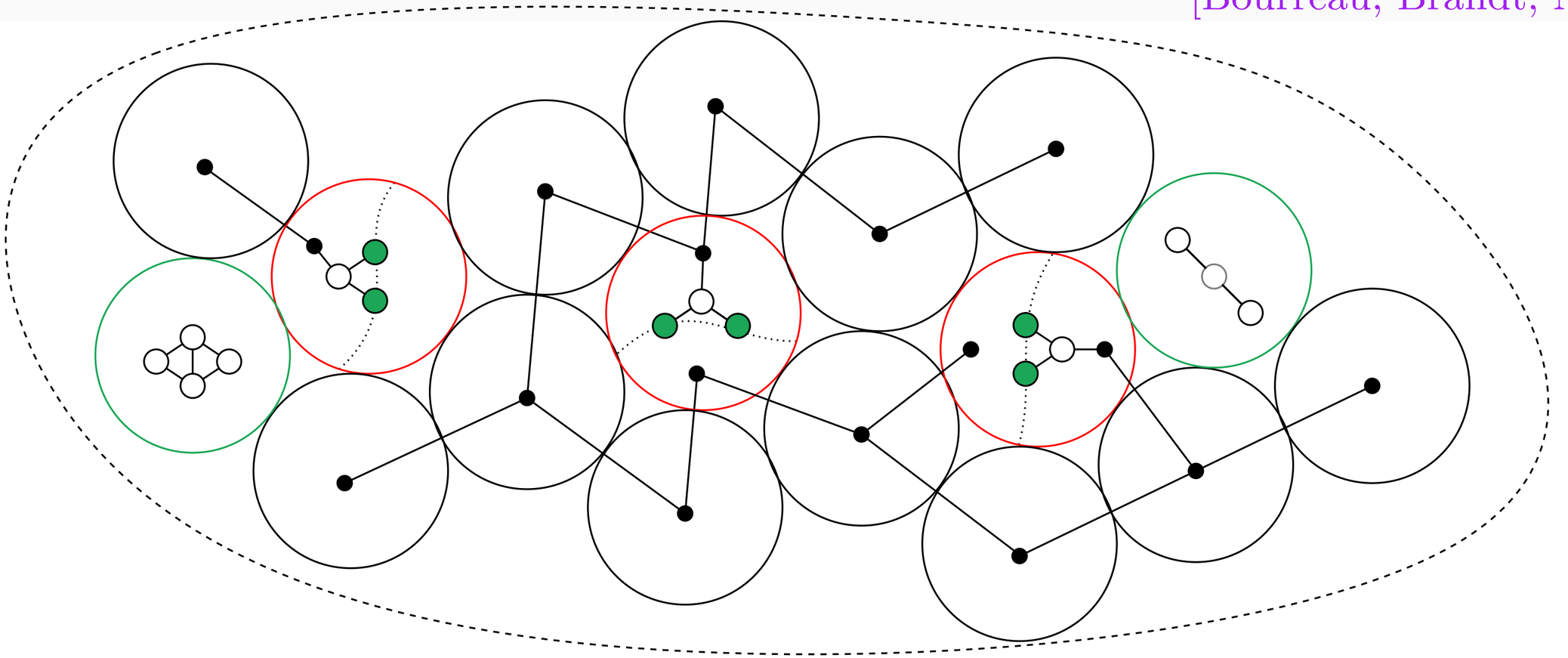
5. $(\Delta+1)$-coloring

Reduction to MIS

[Bourreau, Brandt, Nolin '26]



Algorithm (simplified):

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5. $(\Delta+1)$-coloring

Runtime: $O(\log n)$

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Conclusion

Δ-coloring is tight $\Theta(\log n)$ rounds for const. Δ

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What was missing?

Δ-coloring for $\Delta = f(n)$

randomized Δ-coloring

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Thanks!