On the Locality of Hall's Theorem

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Florian Schager · Jara Uitto

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Contributions

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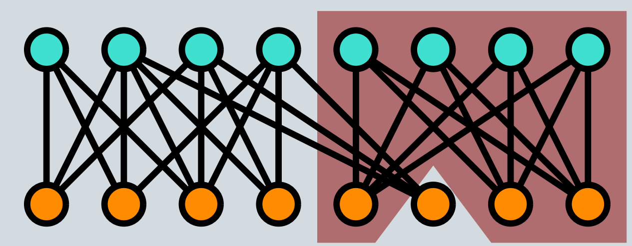
Algorithm design technique	Distributed Hall's Theorem	

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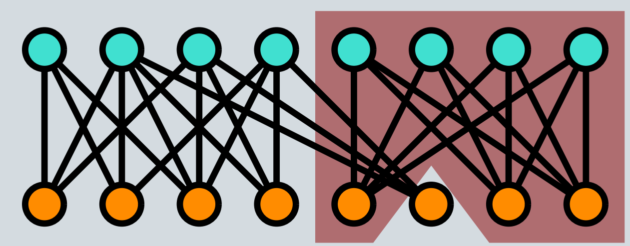
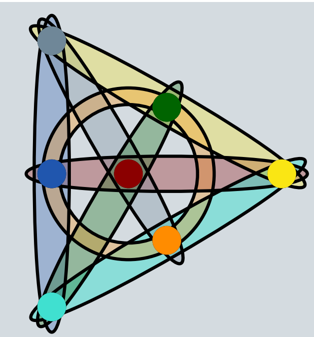
Algorithm design technique	Distributed Hall's Theorem	
Application	Hypergraph sinkless orientation	

Model of computation

Definition (LOCAL model) [Linial, 1992]

- communication network is abstracted as an undirected graph $G = (V, E)$
- nodes have unique IDs, edges serve as communication channels
- communication happens in synchronous rounds
- message size and local computation is unlimited
- each node outputs a (local part of a) solution
- time complexity is the number of communication rounds

Asymptotically optimal algorithms in LOCAL

Problem: Sinkless orientation [Brandt et al. STOC'16]

Input: A Δ-regular graph $G = (V, E)$.

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	Algorithms	**Lower bounds**
DetLOCAL		
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RandLOCAL	$\mathcal{O}(\log_\Delta \log n)$ [Ghaffari & Su, 2017]	$\Omega(\log_\Delta \log n)$ [Brandt et al. STOC'16]

Algorithm design technique

Hall's Theorem [Hall 1935]

A bipartite graph with node sets V and U has a U-saturating matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$.

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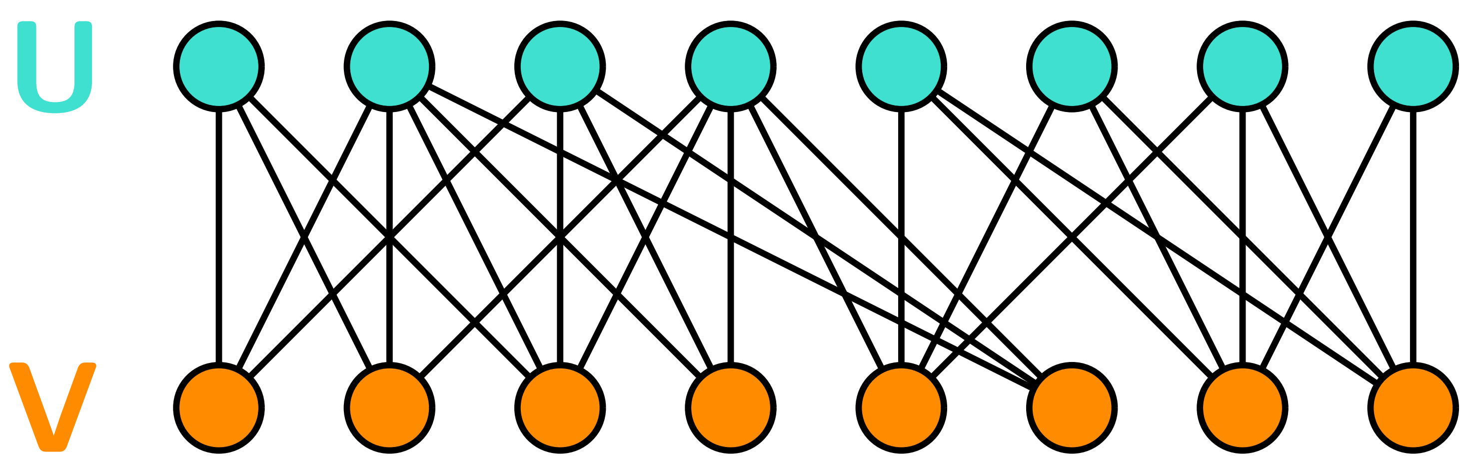

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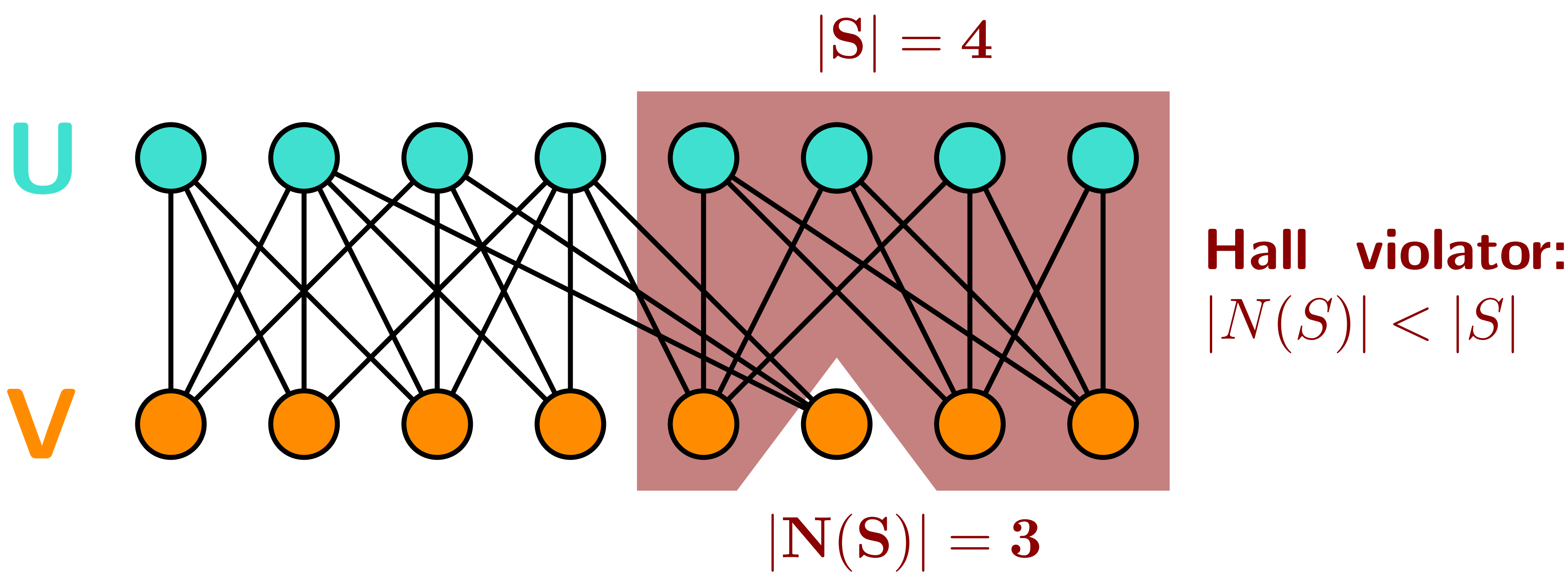

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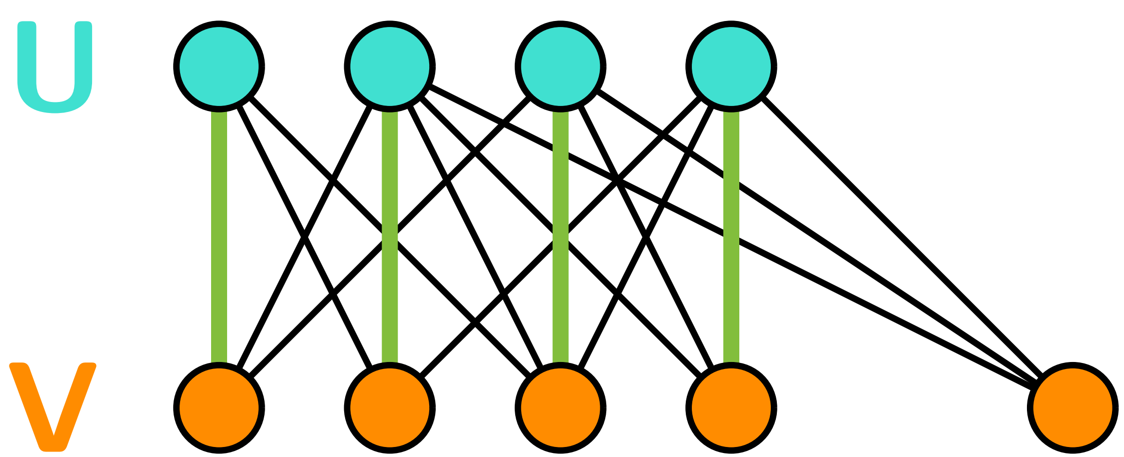

Hall graph

Definition (Hypergraph $H = (V, E)$)

- generalization of a graph, each edge may contain arbitrarily many vertices
- the *degree* $\deg(v)$ of a vertex $v \in V$ is the number of incident edges
- the *rank* $\mathrm{rank}(e)$ of an edge $e \in E$ is the number of vertices in the edge
- $\delta := \min_{v \in V} \deg(v)$ is the minimum degree of H
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Example ($\delta = 2, r = 4$)

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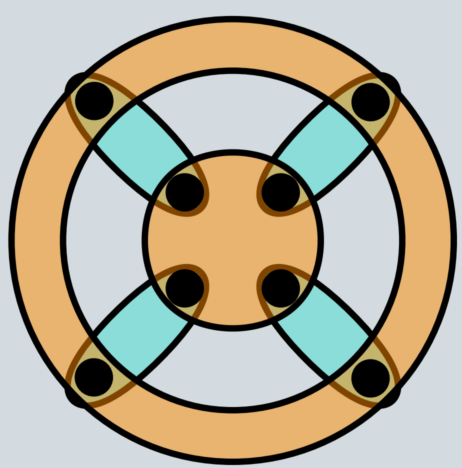

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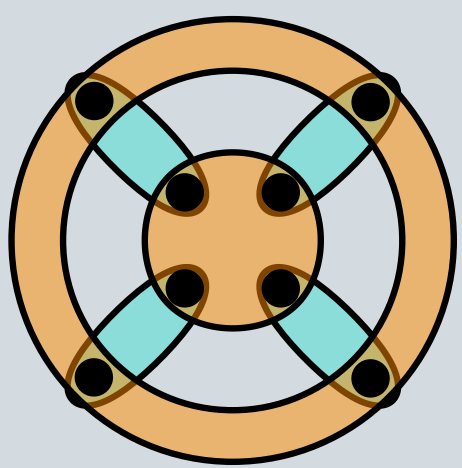


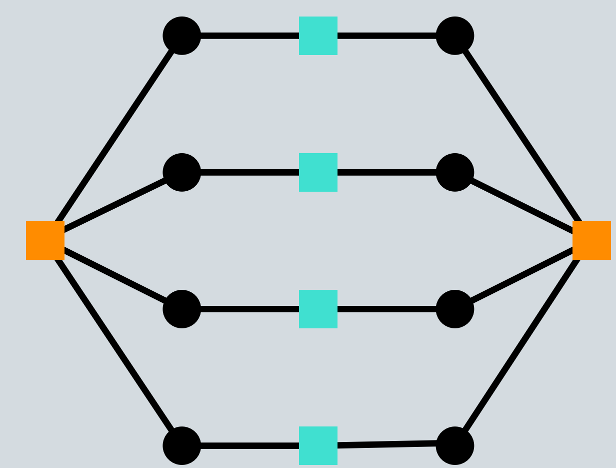
Bipartite representation

$\mathcal{B}_H = (V, E, F)$

$(v, e) \in F$

$\Longleftrightarrow$

$v \in e$



Every vertex is contained in a small-diameter Hall graph

Lemma 1: Many edges

Let $r < \delta$ and $x = \log_{(\delta-1)/(r-1)} n$. For any $v \in V$ there exists a subgraph $(V', E') \subseteq B_x^G(v)$ such that $v \in V'$ and $|E'| \geq |V'|$.

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Lemma 2: Non-empty Hall subgraph

Any non-empty hypergraph $G = (V, E)$ with $|V| \leq |E|$ contains a non-empty Hall subgraph.

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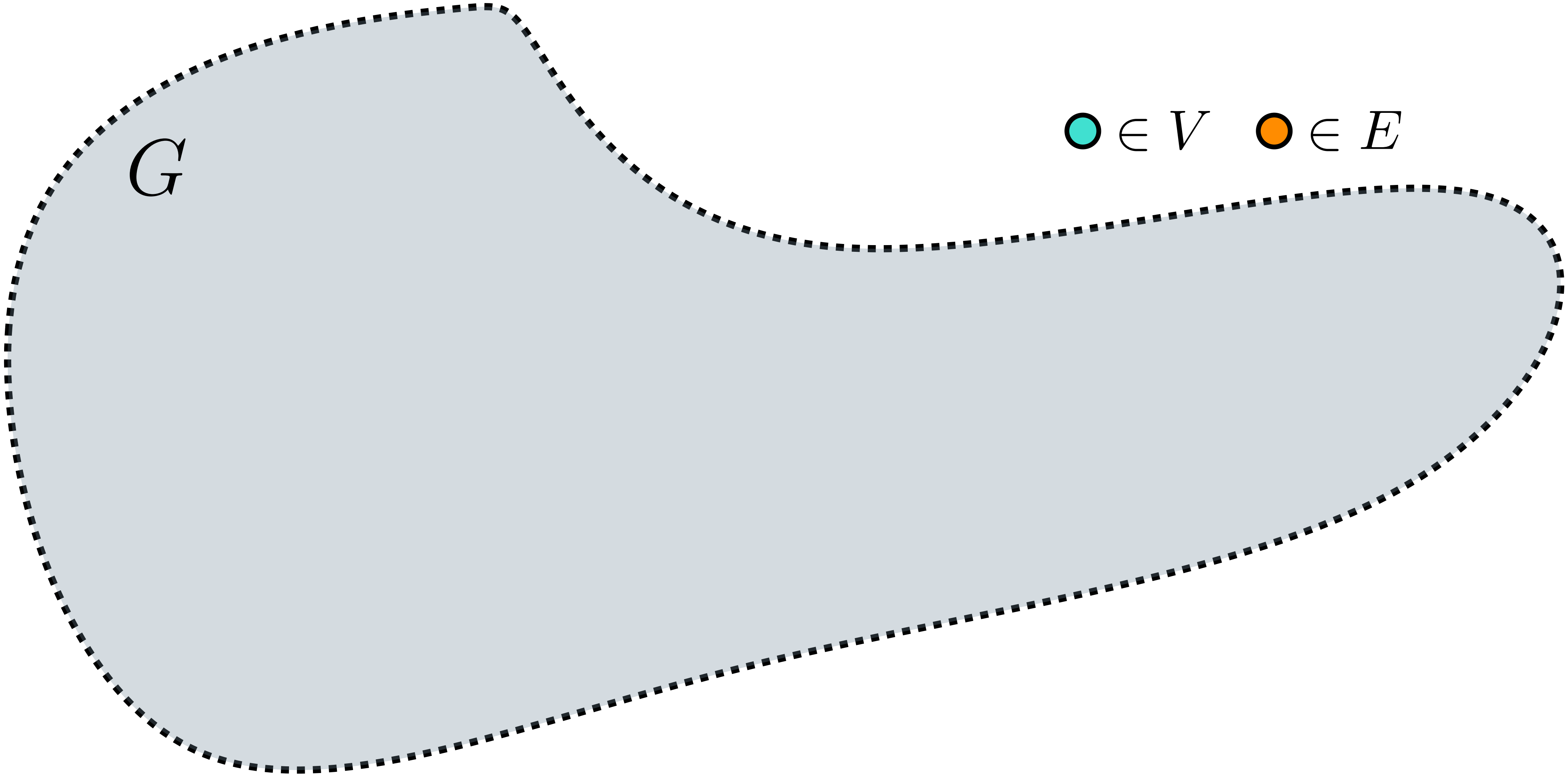
Distributed Hall's Theorem

Each node in any n-node hypergraph with minimum degree δ and maximum rank $r < \delta$ is contained in a Hall graph with diameter $\log_{\delta/r} n$.

Non-empty Hall subgraph



Let $G = (V, E)$ such that $|V| \leq |E|$.

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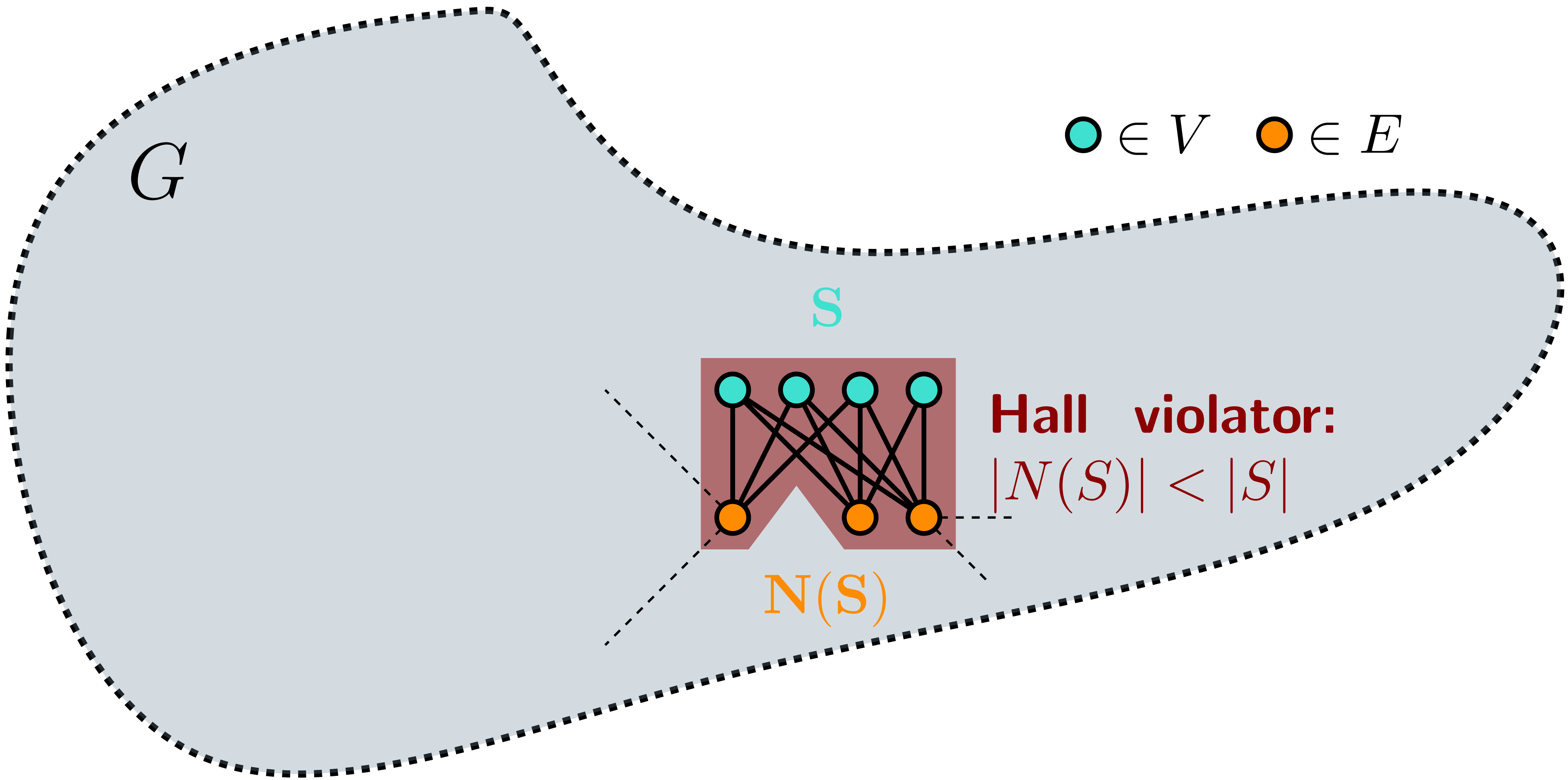
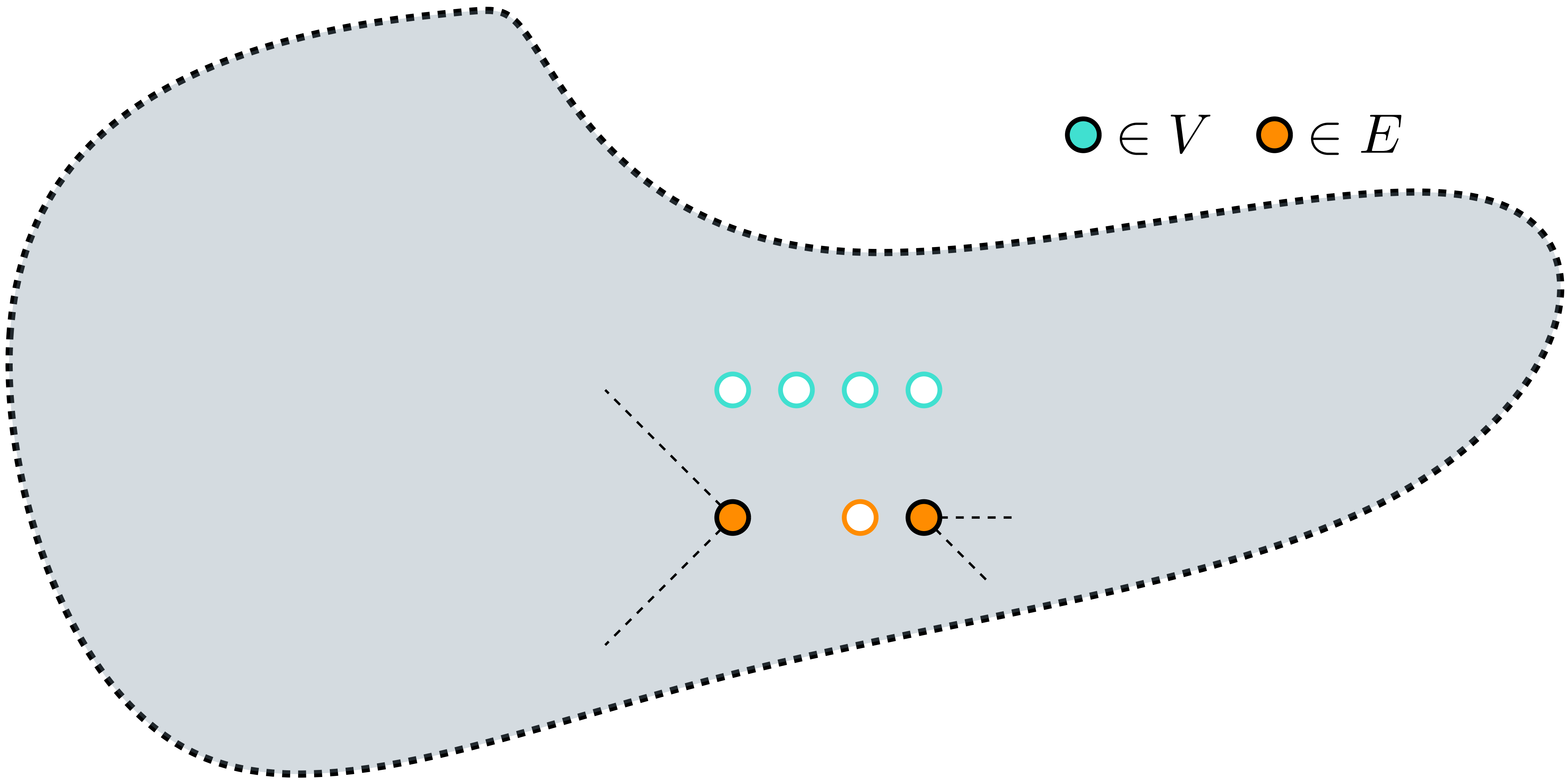


If G is not a Hall graph, there exists a Hall violator $S \subseteq V$.

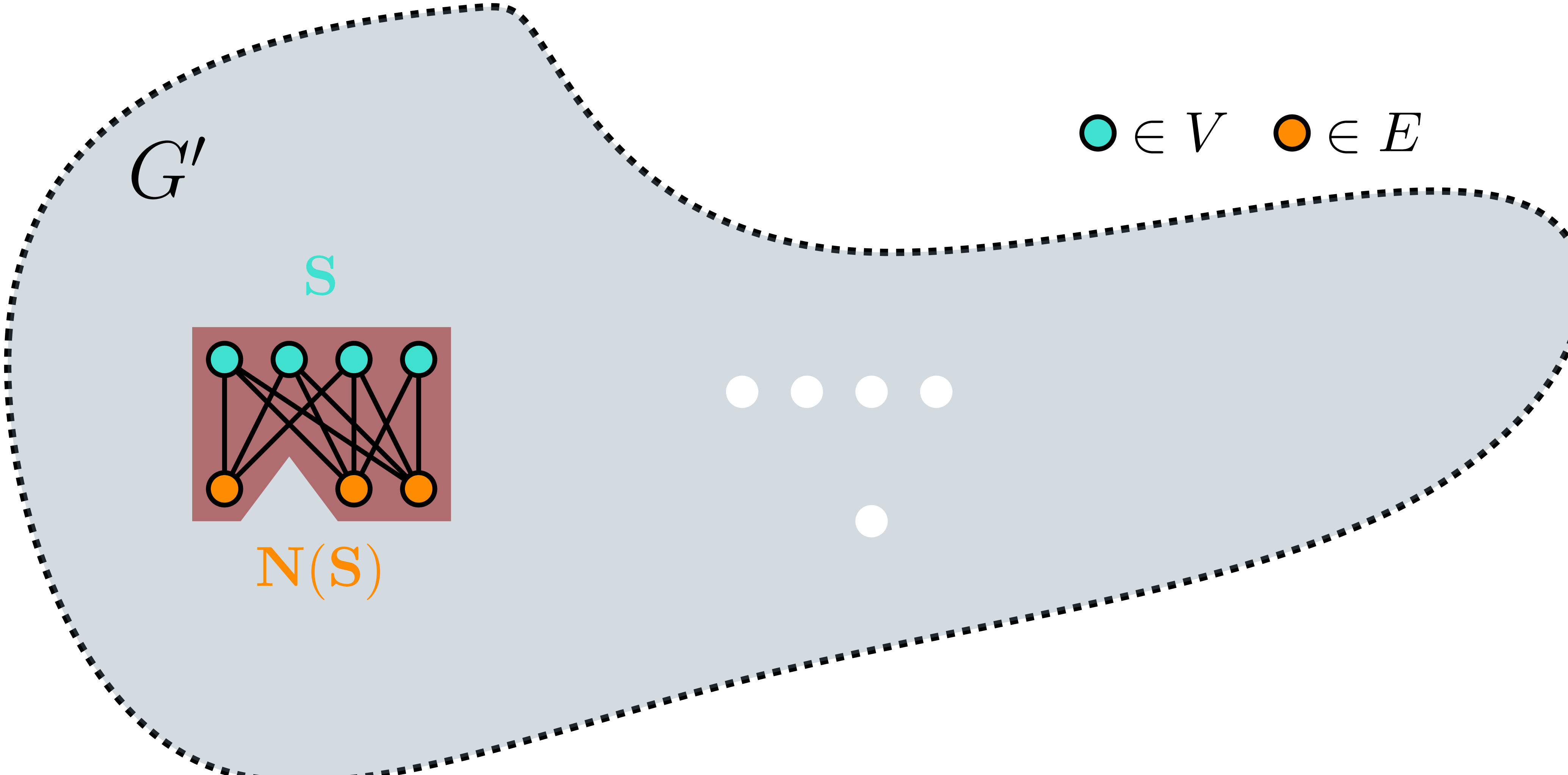
Non-empty Hall subgraph



Define $V' := V \setminus S$ and $E := E|_{V'}$.

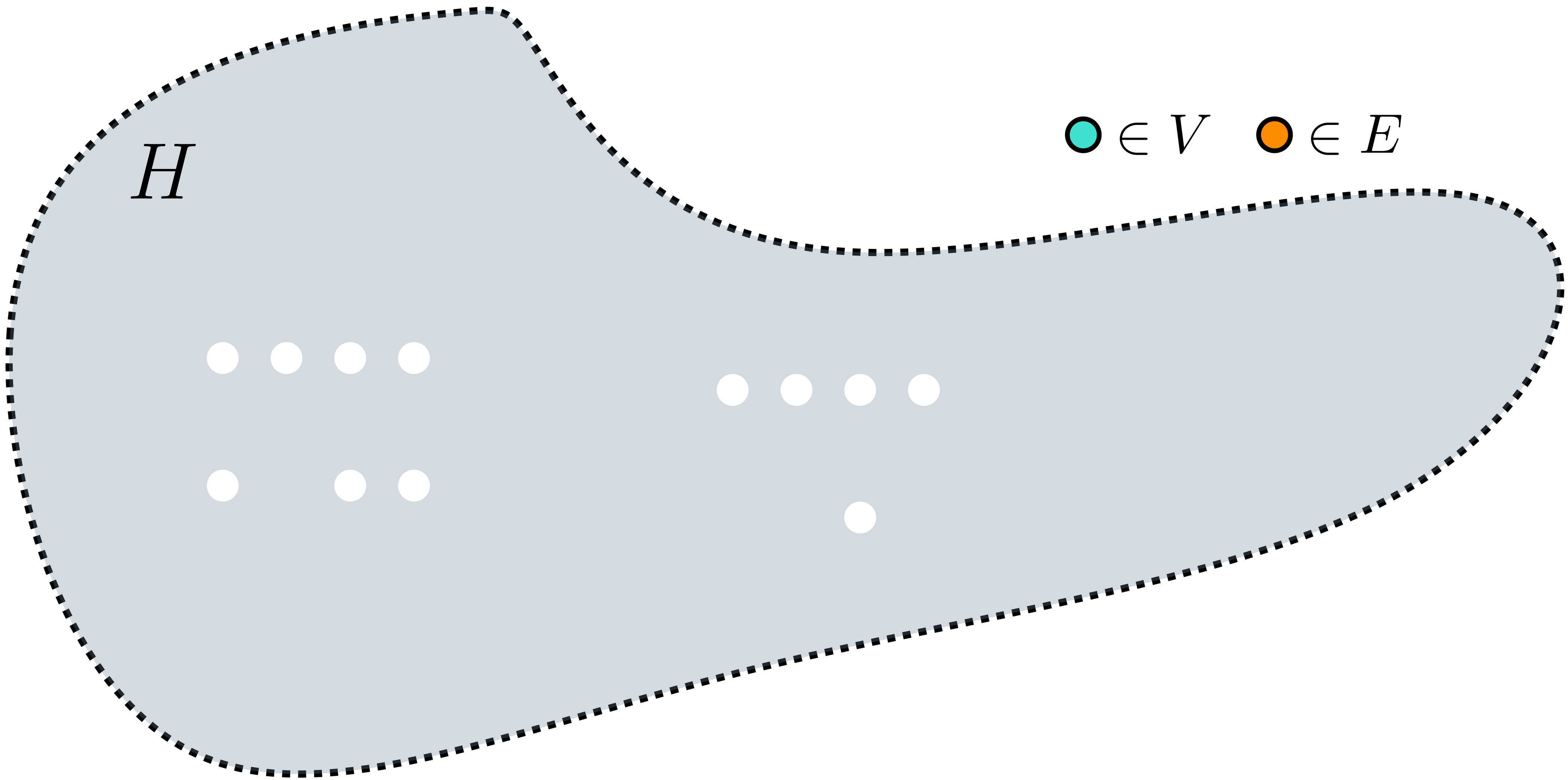
$|V'| = |V| - |S| < |V| - |N(S)| \leq |E| - |N(S)| \leq |E'|$.

Non-empty Hall subgraph



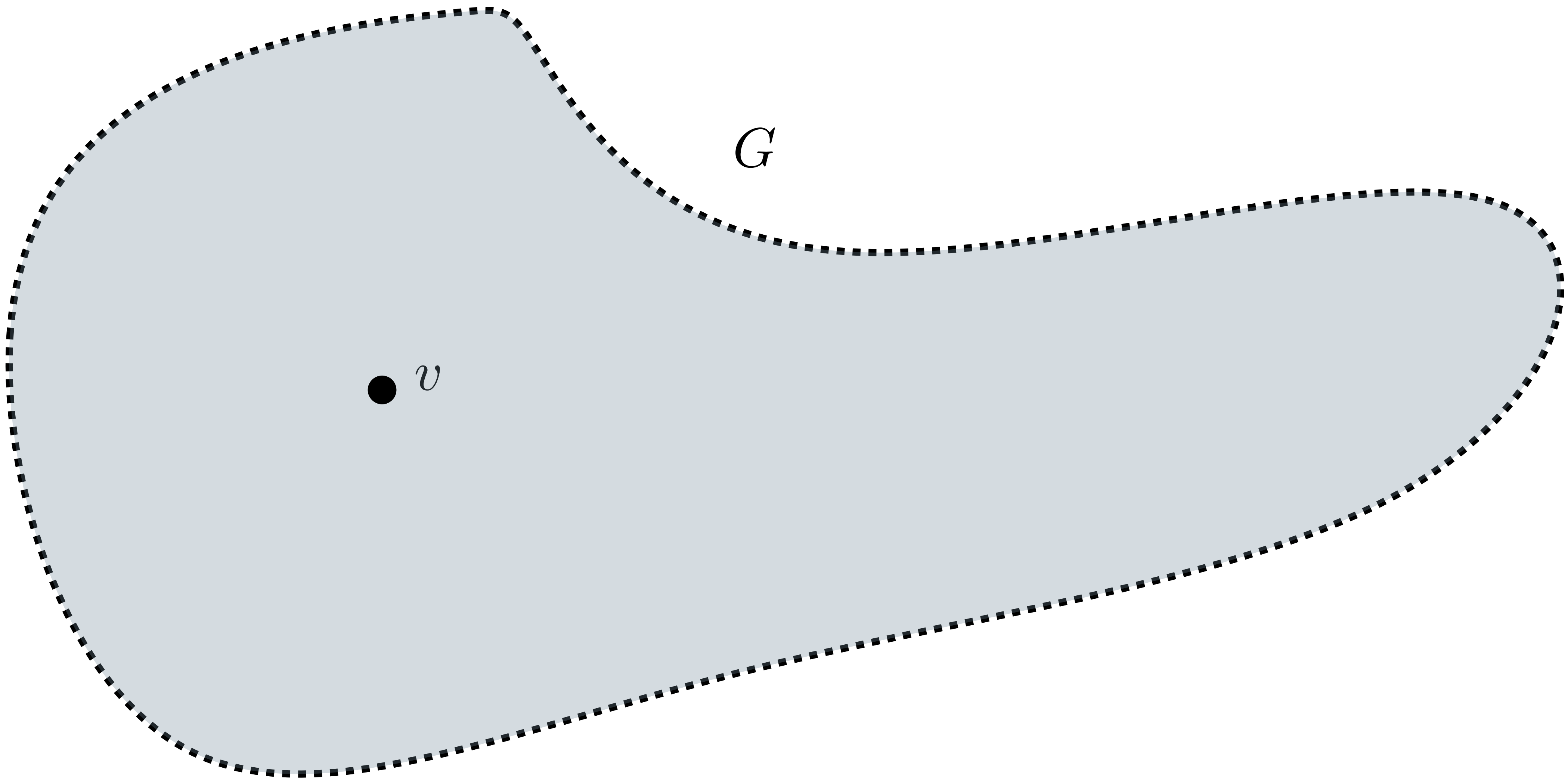
Repeat for $G' = (V', E')$.

Non-empty Hall subgraph



Since $|V'| < |E'|$, this process cannot end with an empty graph.

Distributed Hall's algorithm



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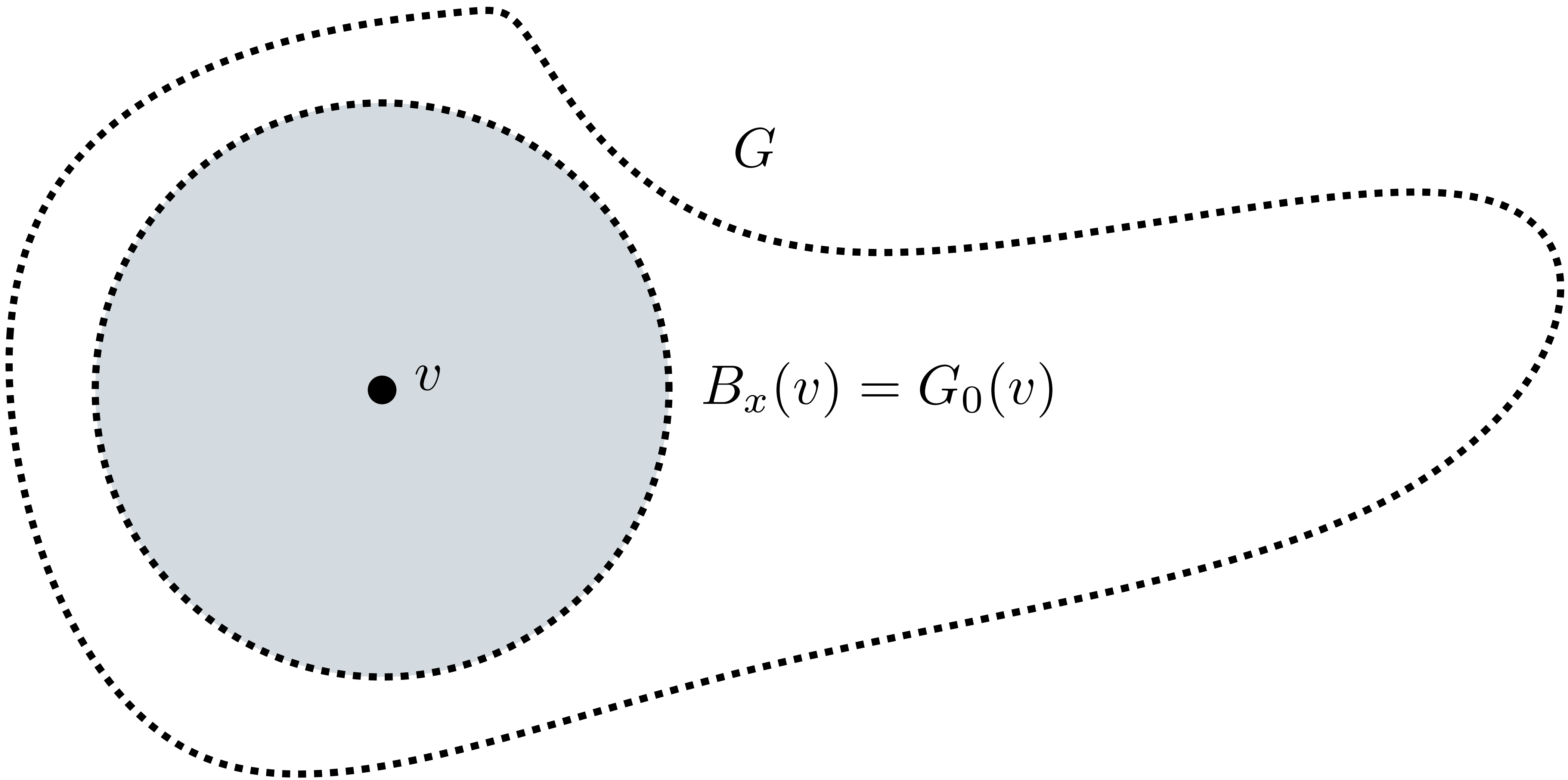

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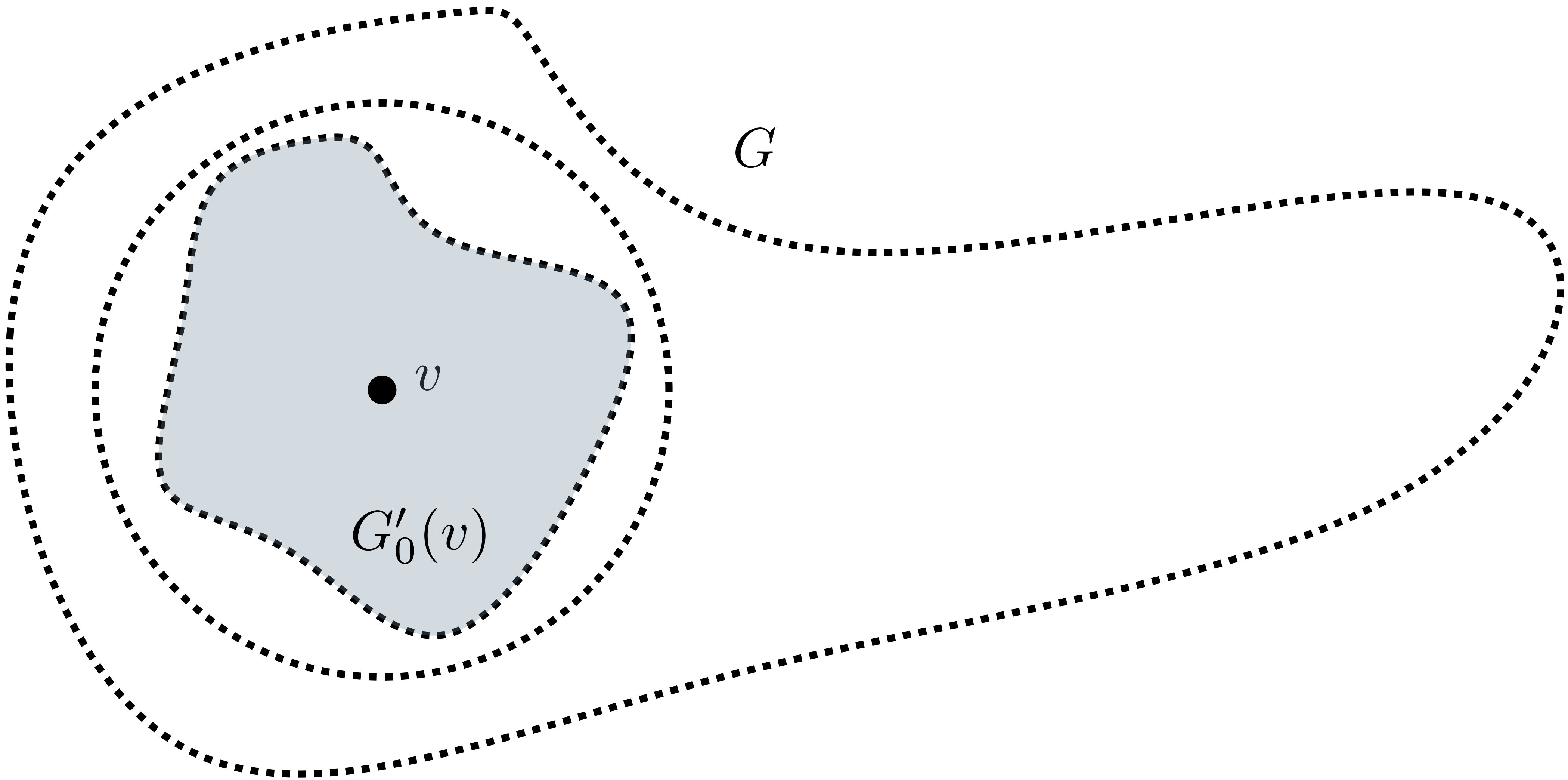

Lemma 1: Find a subgraph $v \in G'_0(v) \subseteq G_0(v)$ with $|V'| \geq |E'|$.

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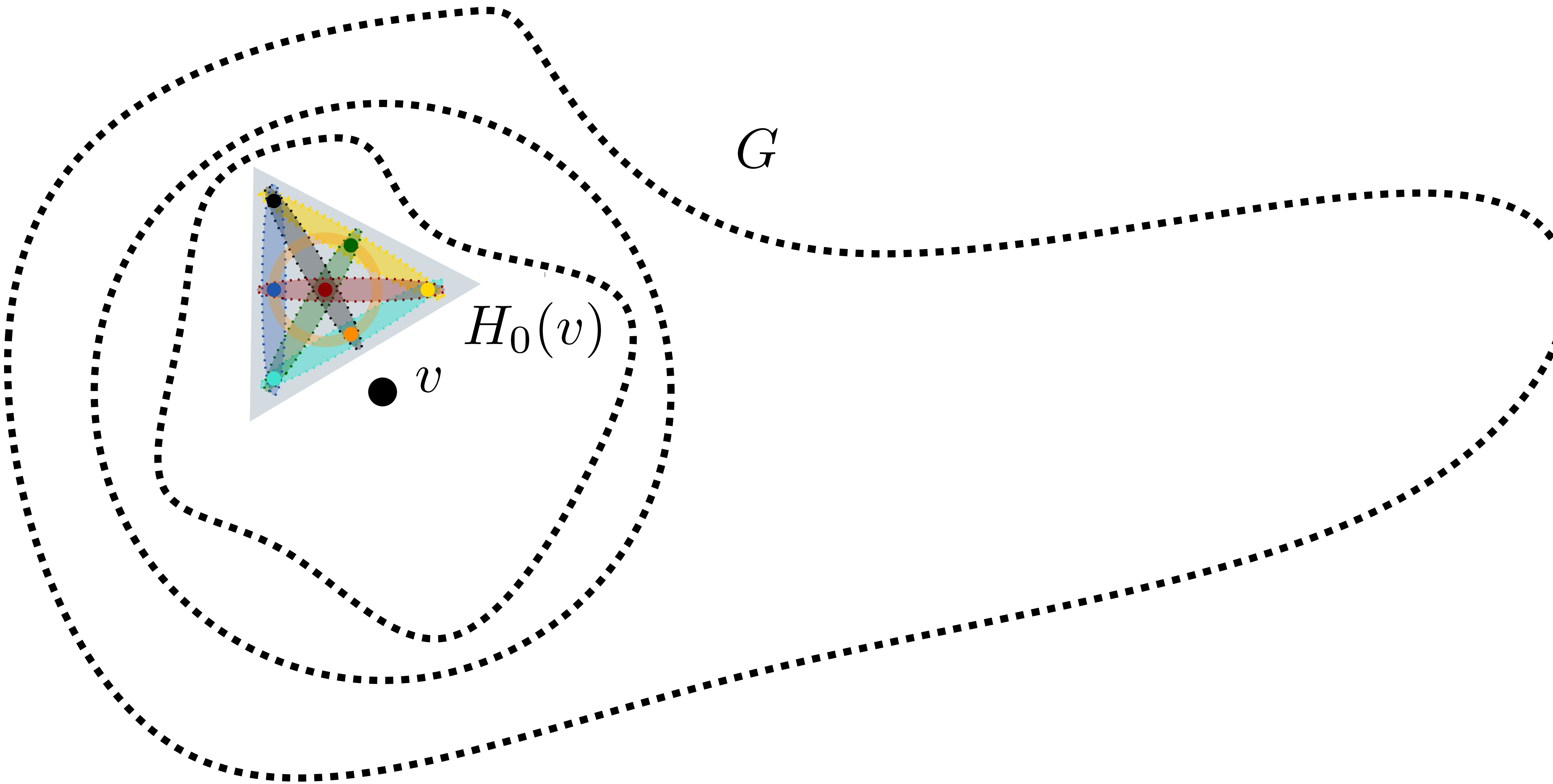

Lemma 2: Find Hall subgraph $H_0 \subseteq G'_0(v)$.

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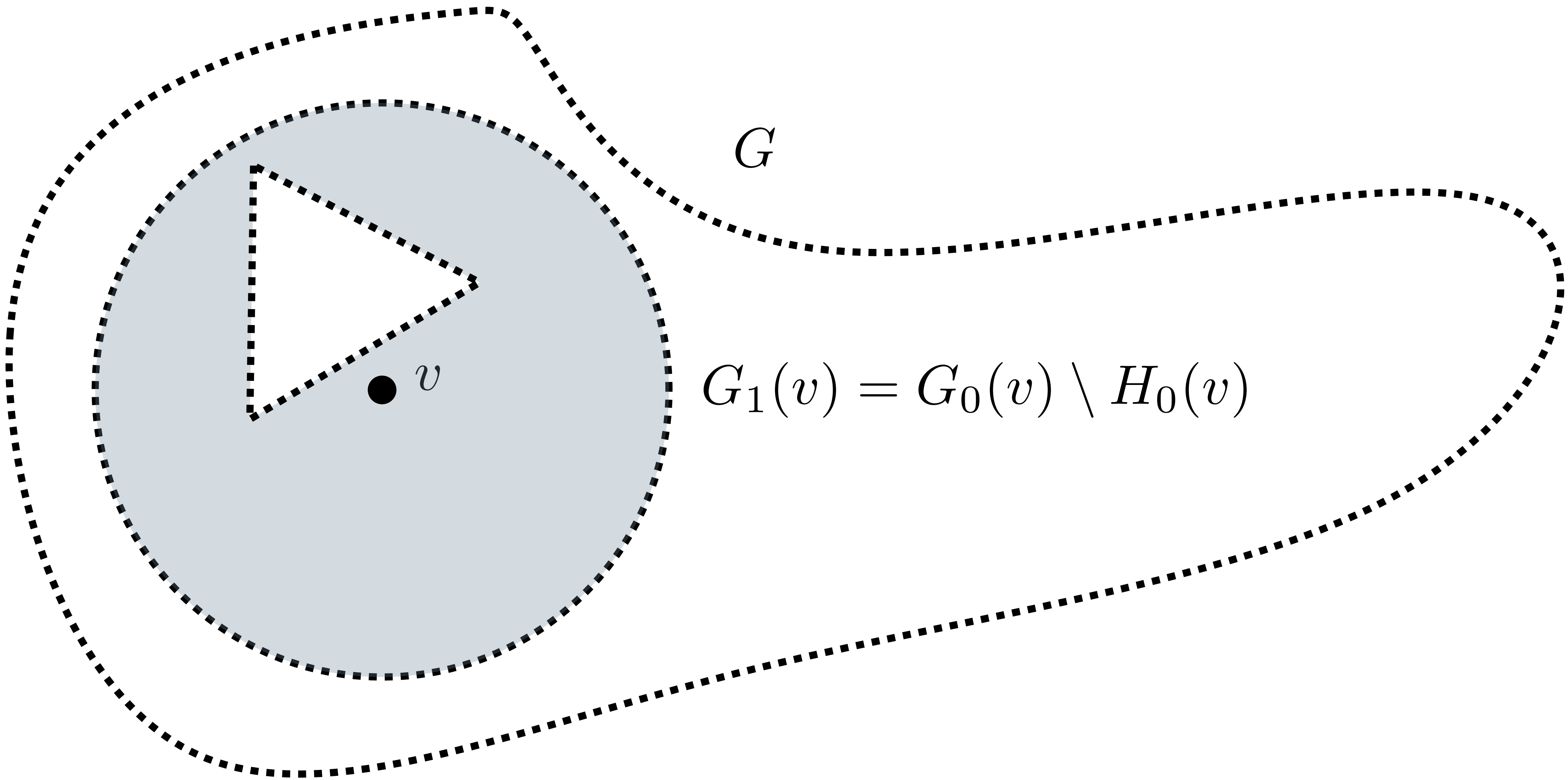

Invariant: $\delta(G_i(v)) > r(G_i(v))$

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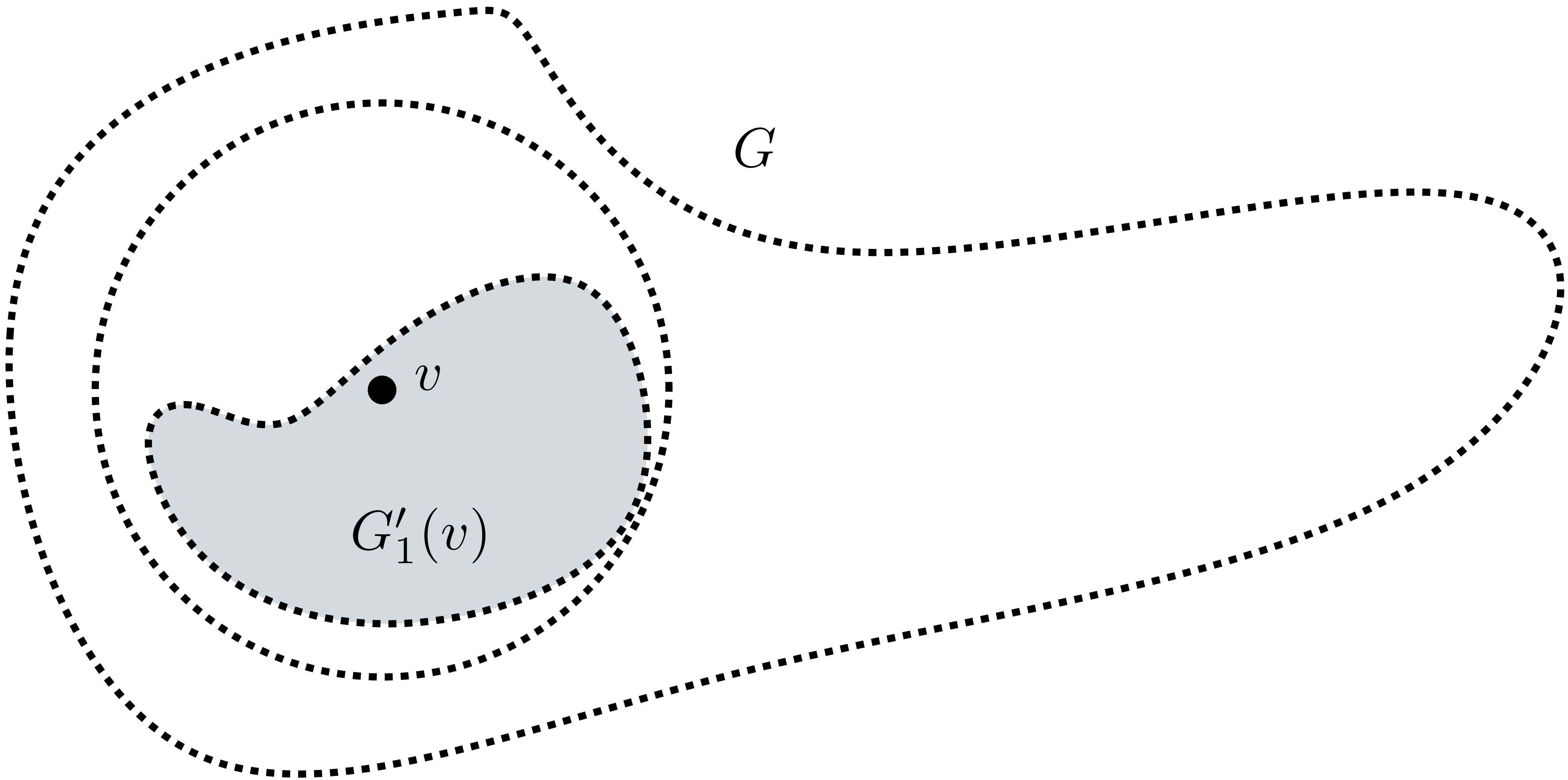

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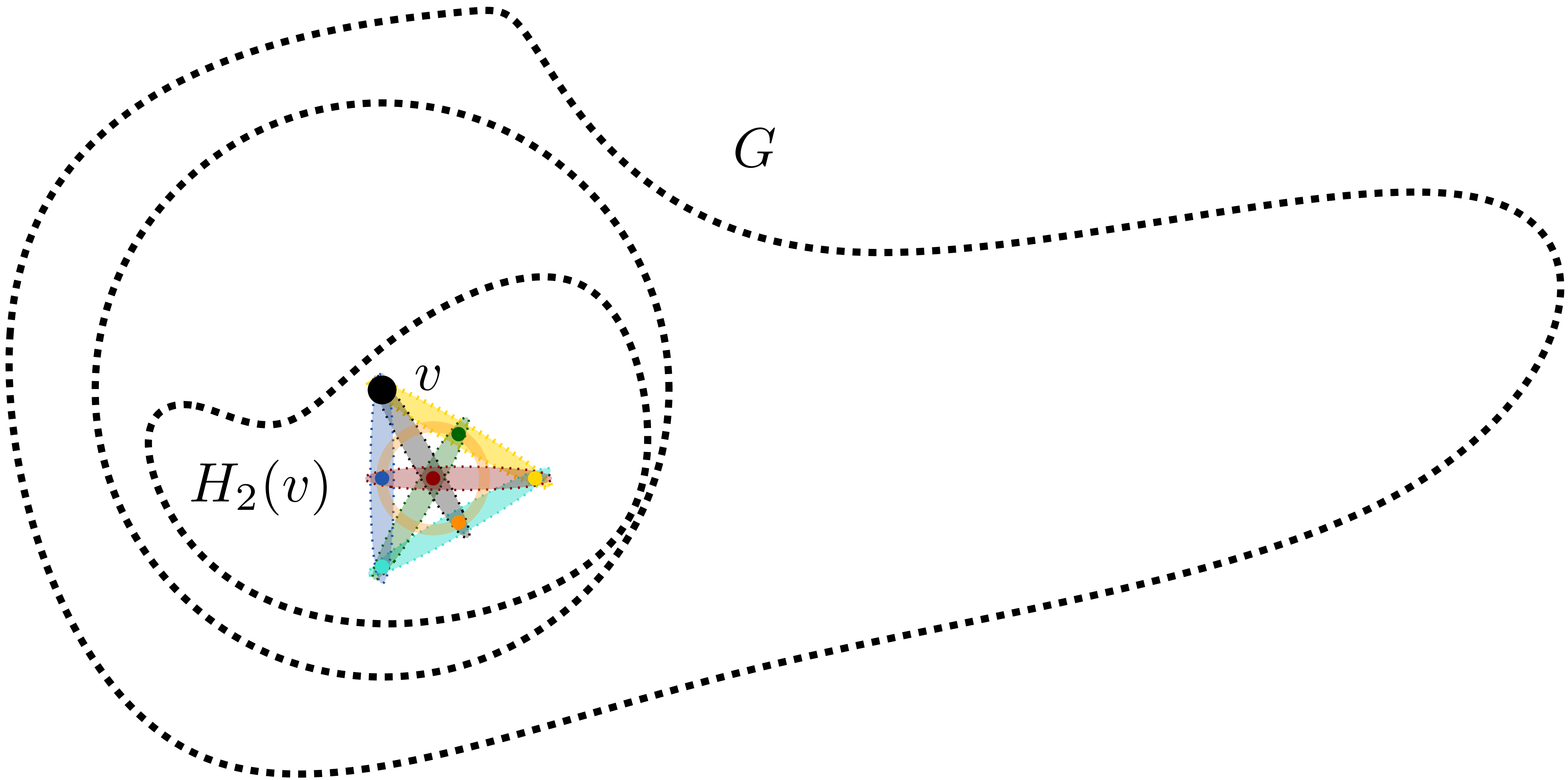

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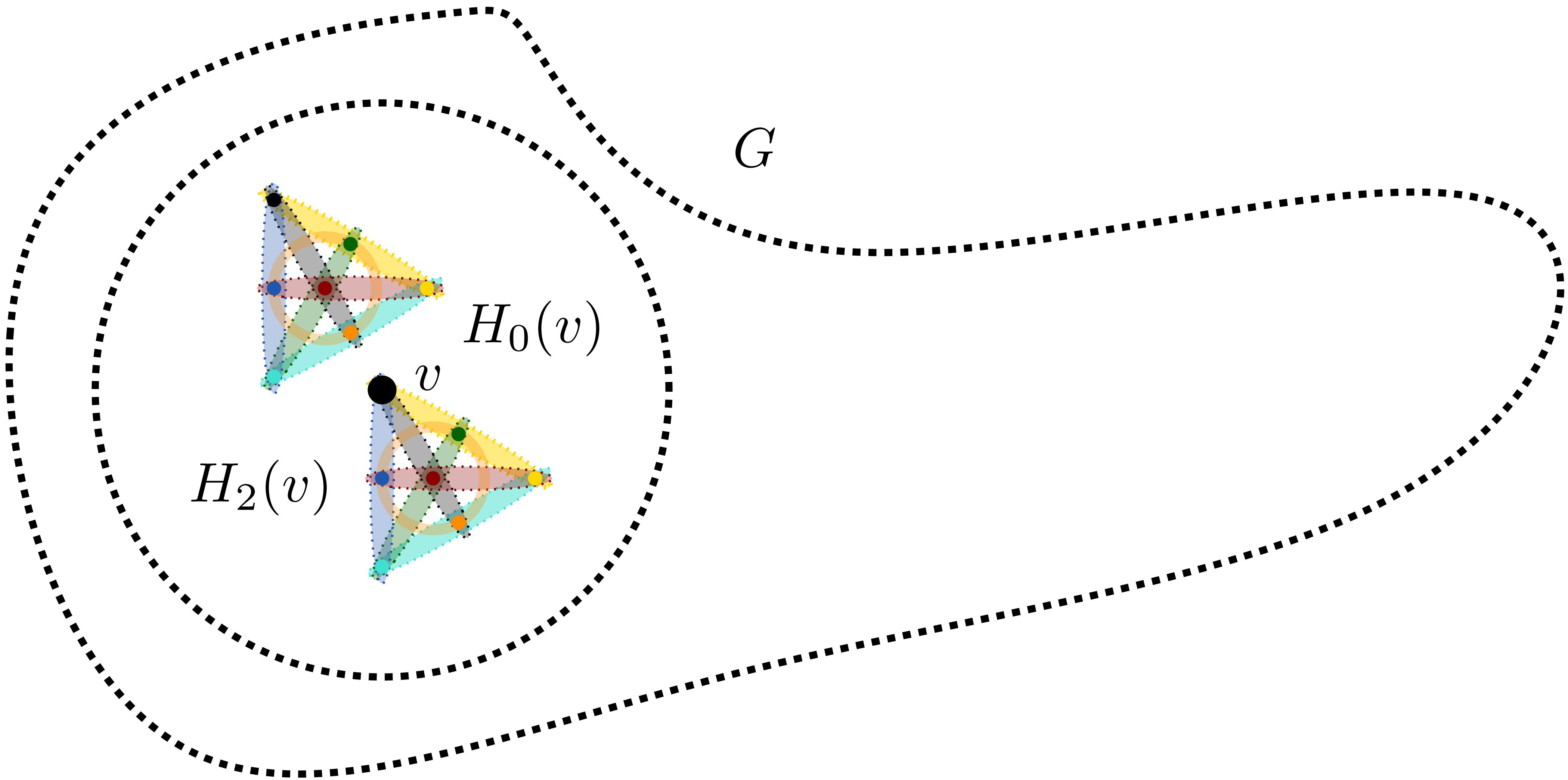

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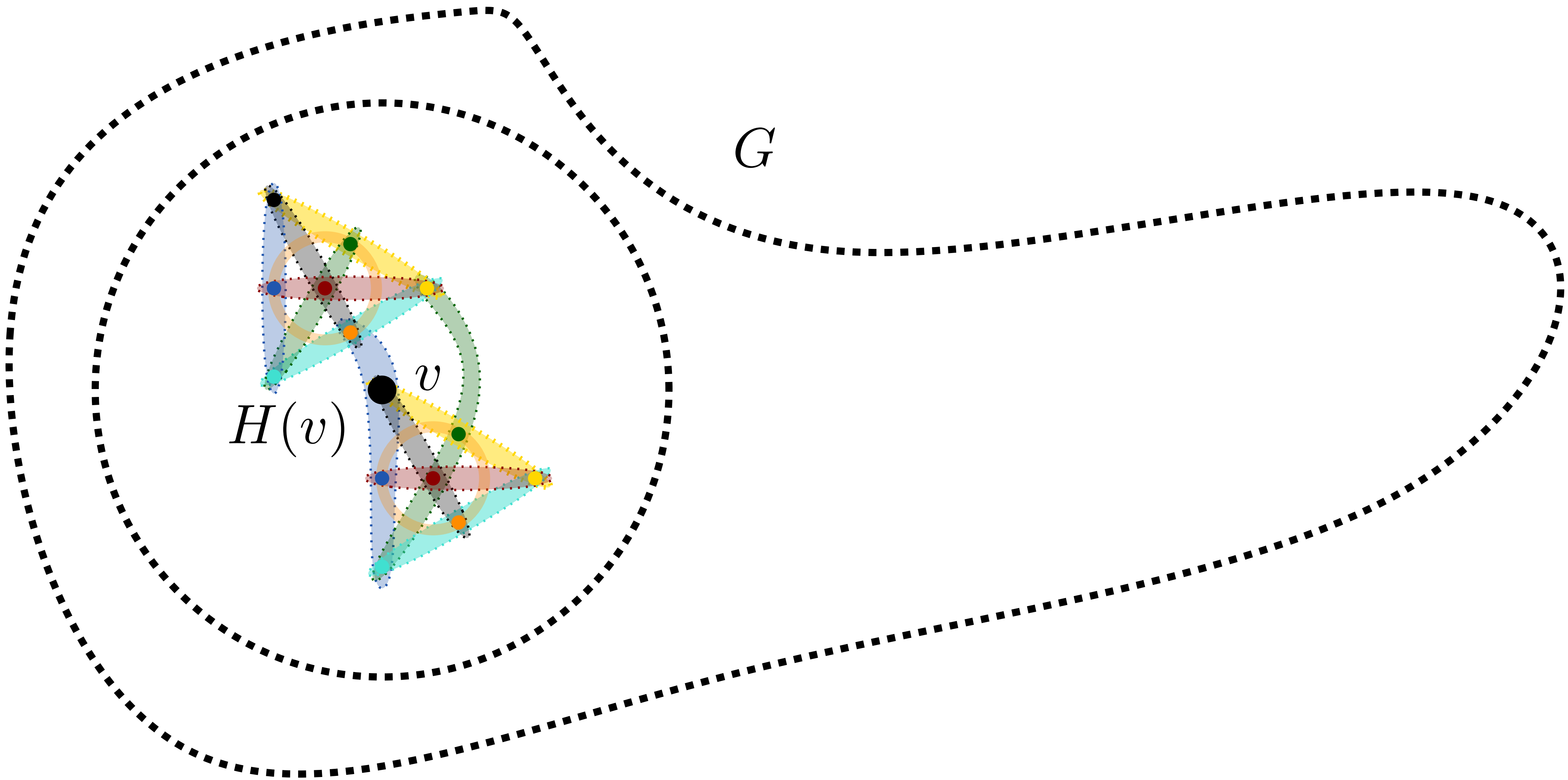

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Input: A hypergraph H with min. degree δ and max. rank r.

Output: An orientation of the hyperedges such that every vertex has at least one outgoing hyperedge.

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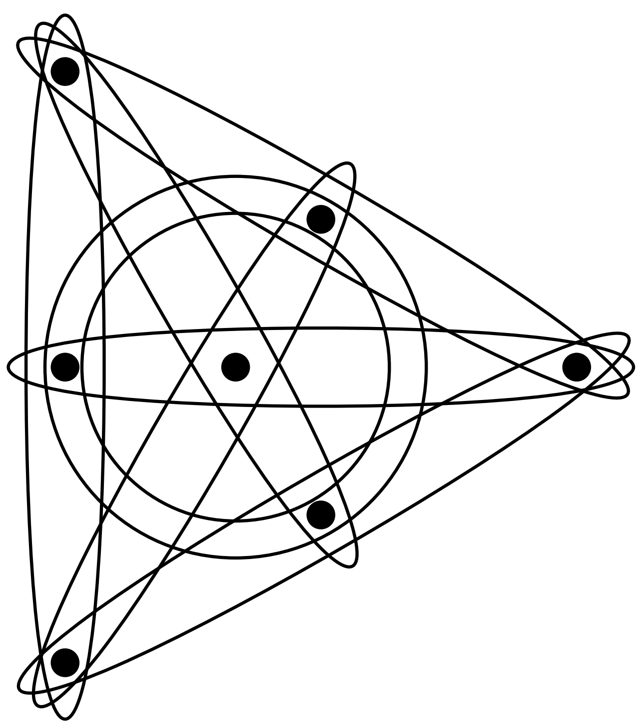

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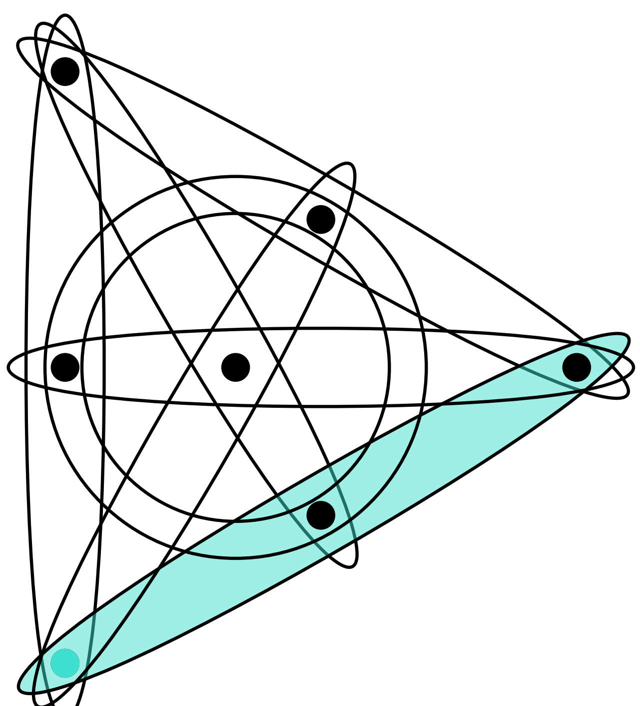

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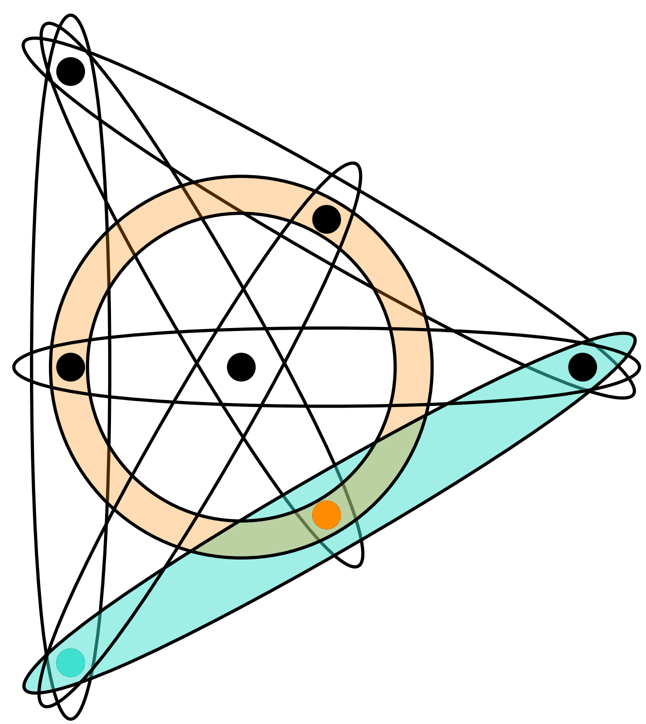

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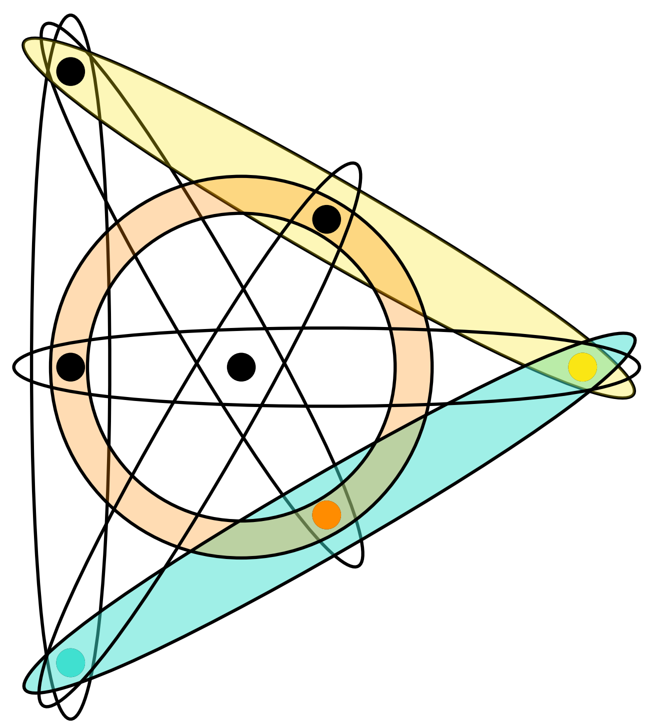

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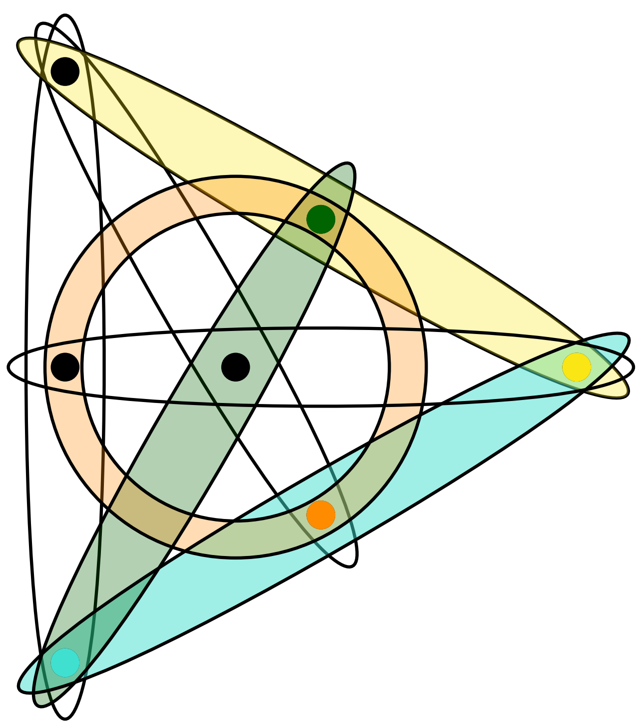

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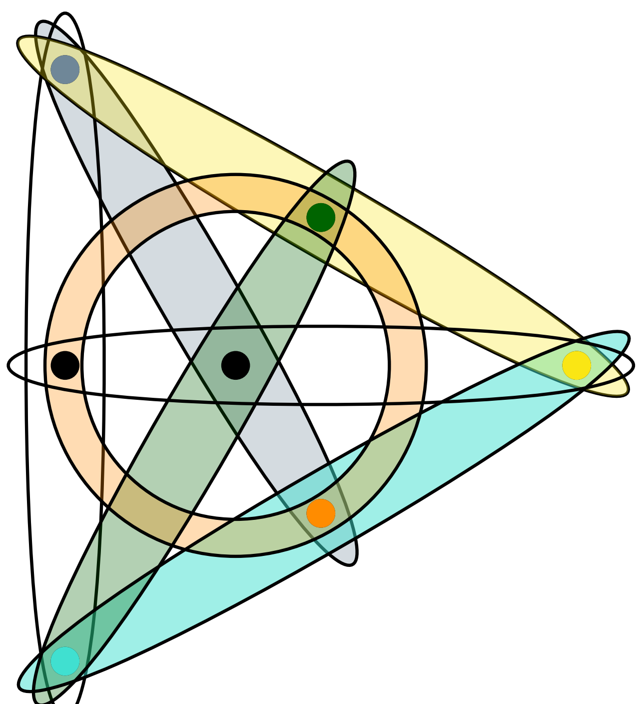

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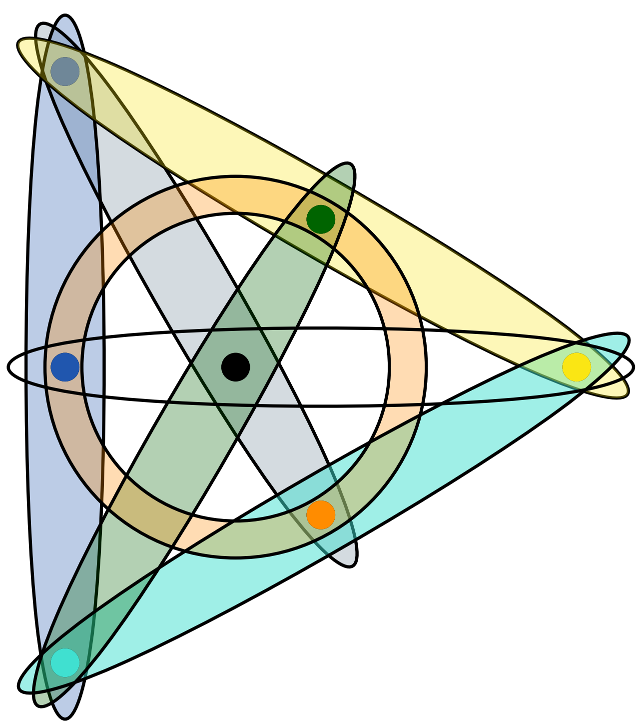

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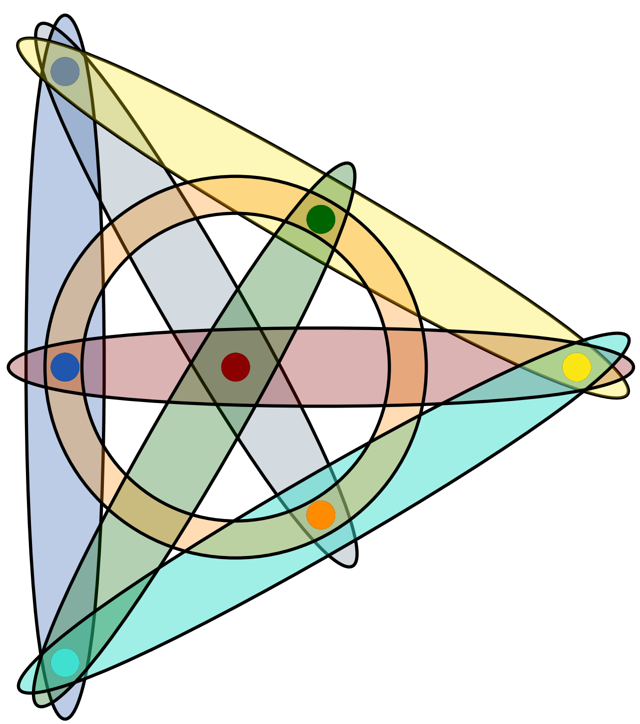

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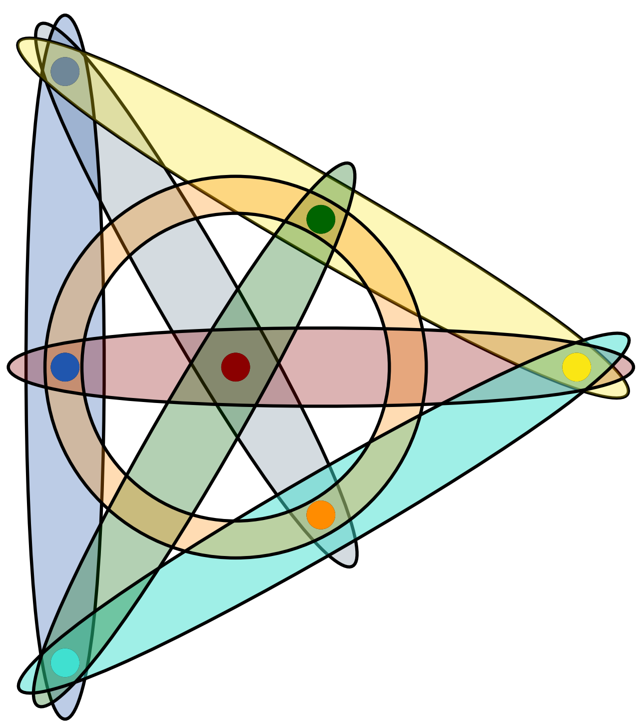

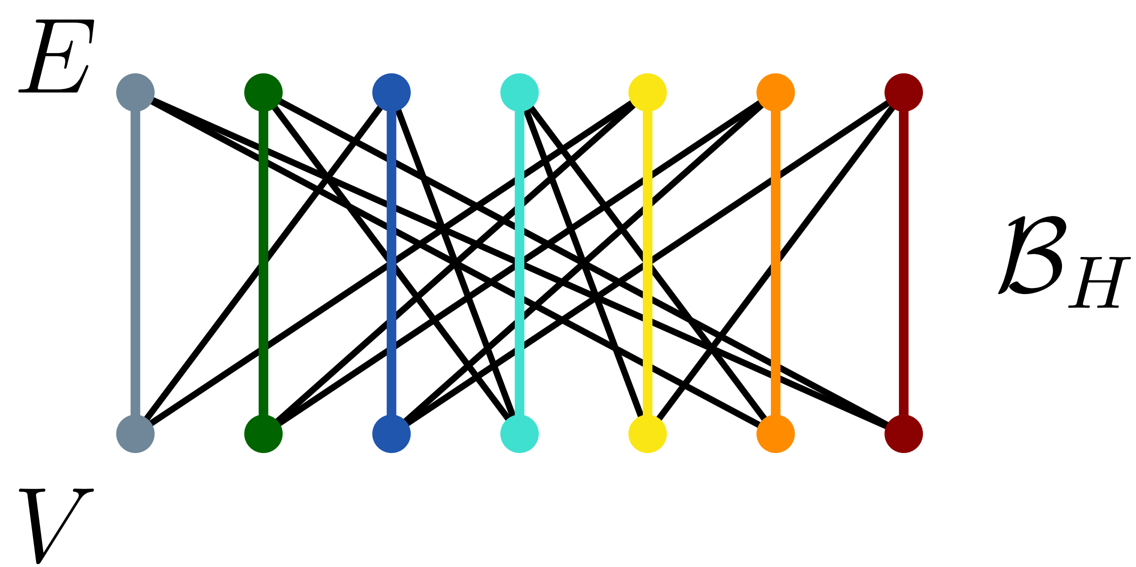

Deterministic HSO algorithm

Sequential algorithm

for $v \in V$:

  Find a local Hall graph $H(v) \ni v$.

  Compute a HSO of $H(v)$.

  Orient the edges in $H(v)$ accordingly.

end for

Orient all remaining hyperedges arbitrarily

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What happens if we reorient edges?

Orienting edges in a HSO cannot create a sink!

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Runtime: $2x = \mathcal{O}(\log_{\delta/r} n)$

HSO results

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$\delta = r$	$\Omega(n)$ [**new result**]	

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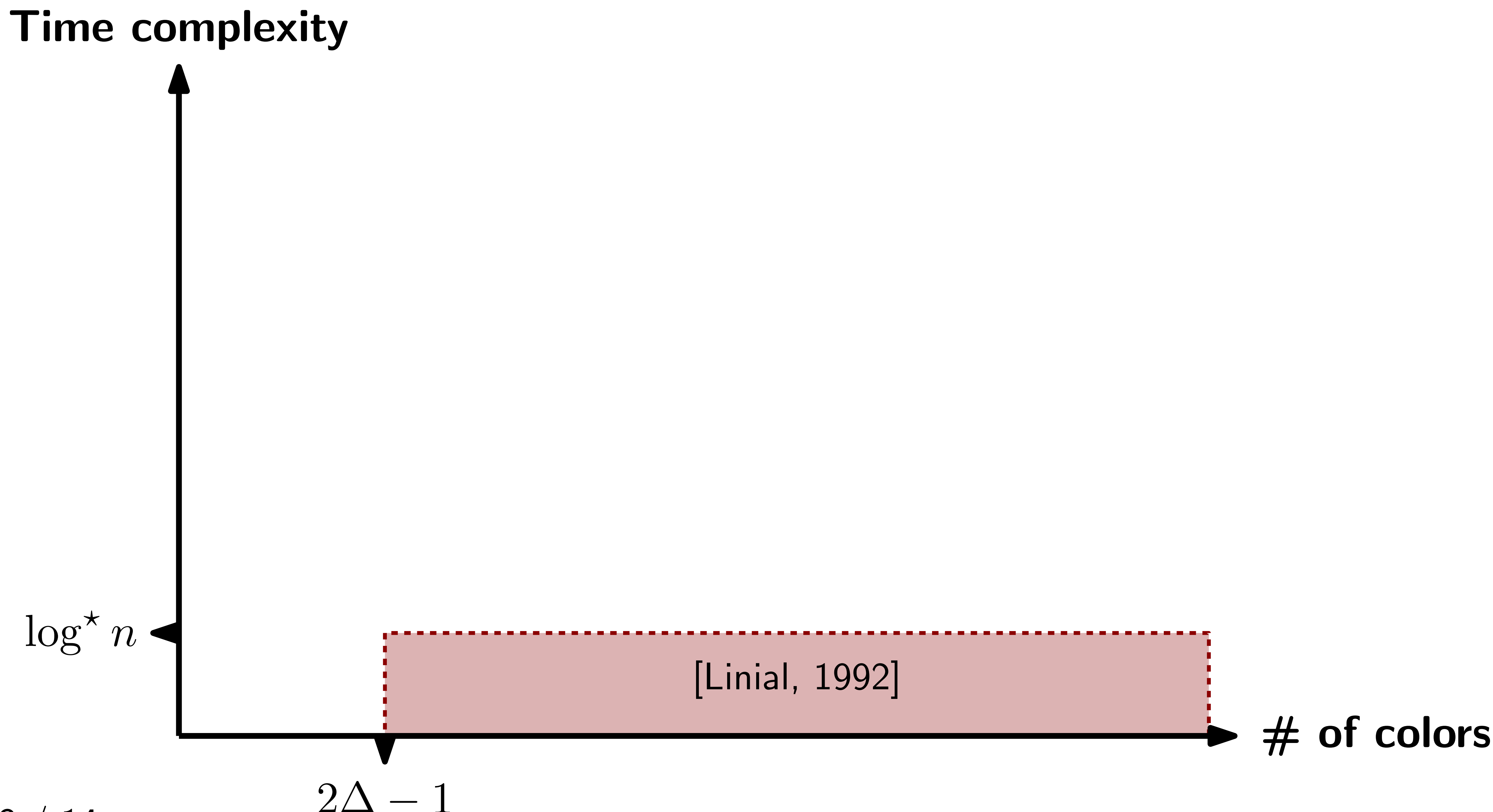

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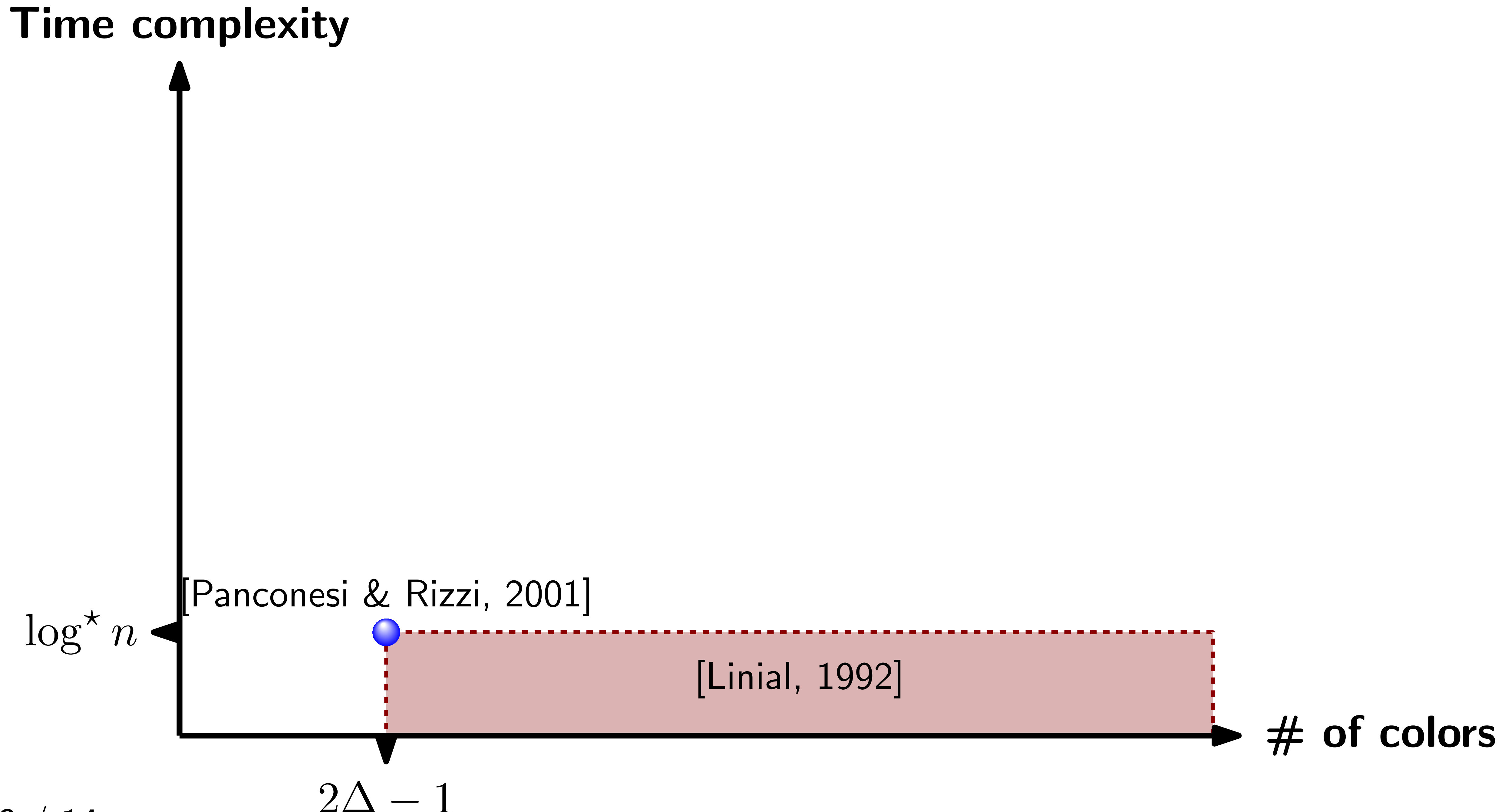

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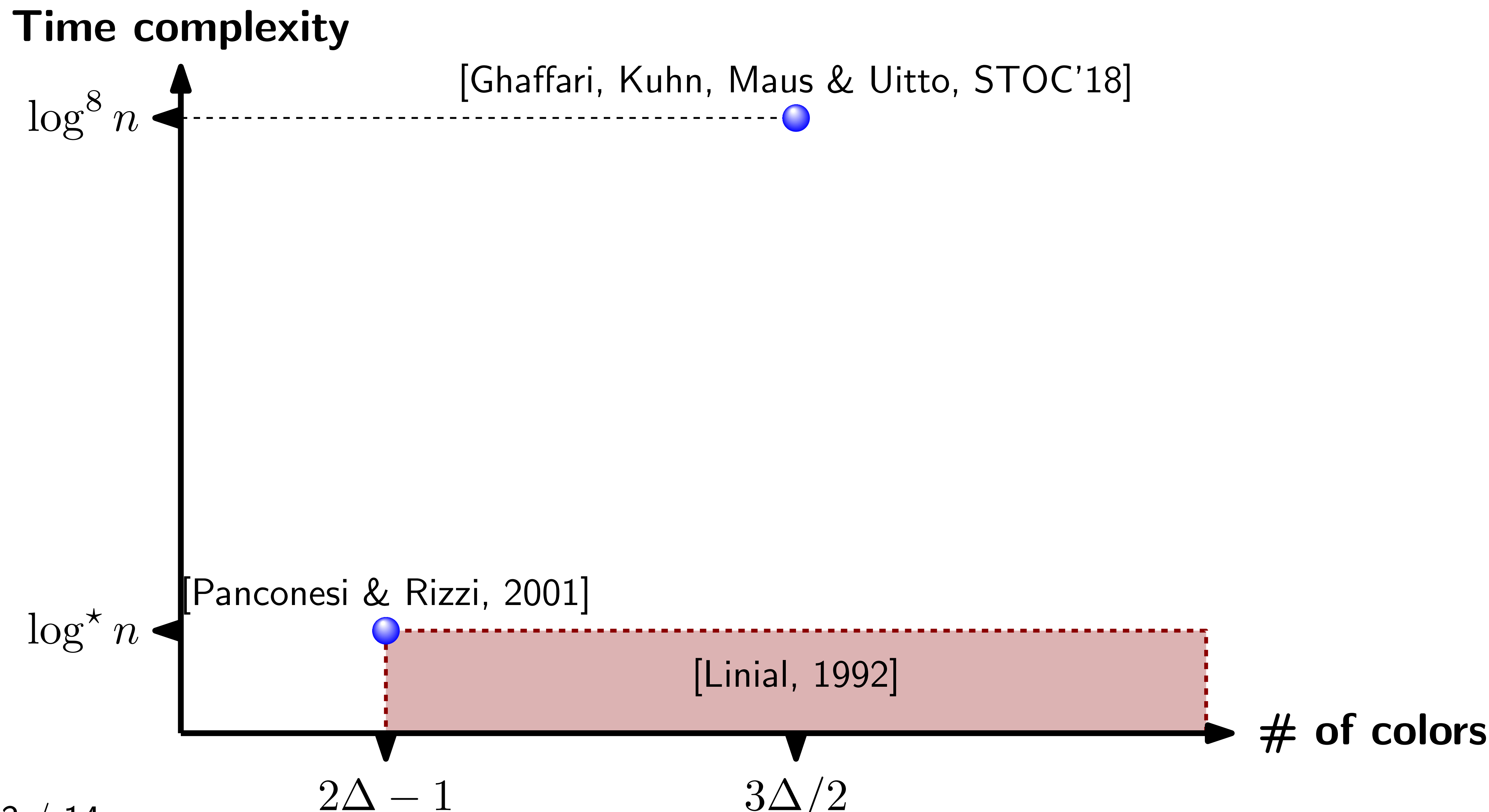

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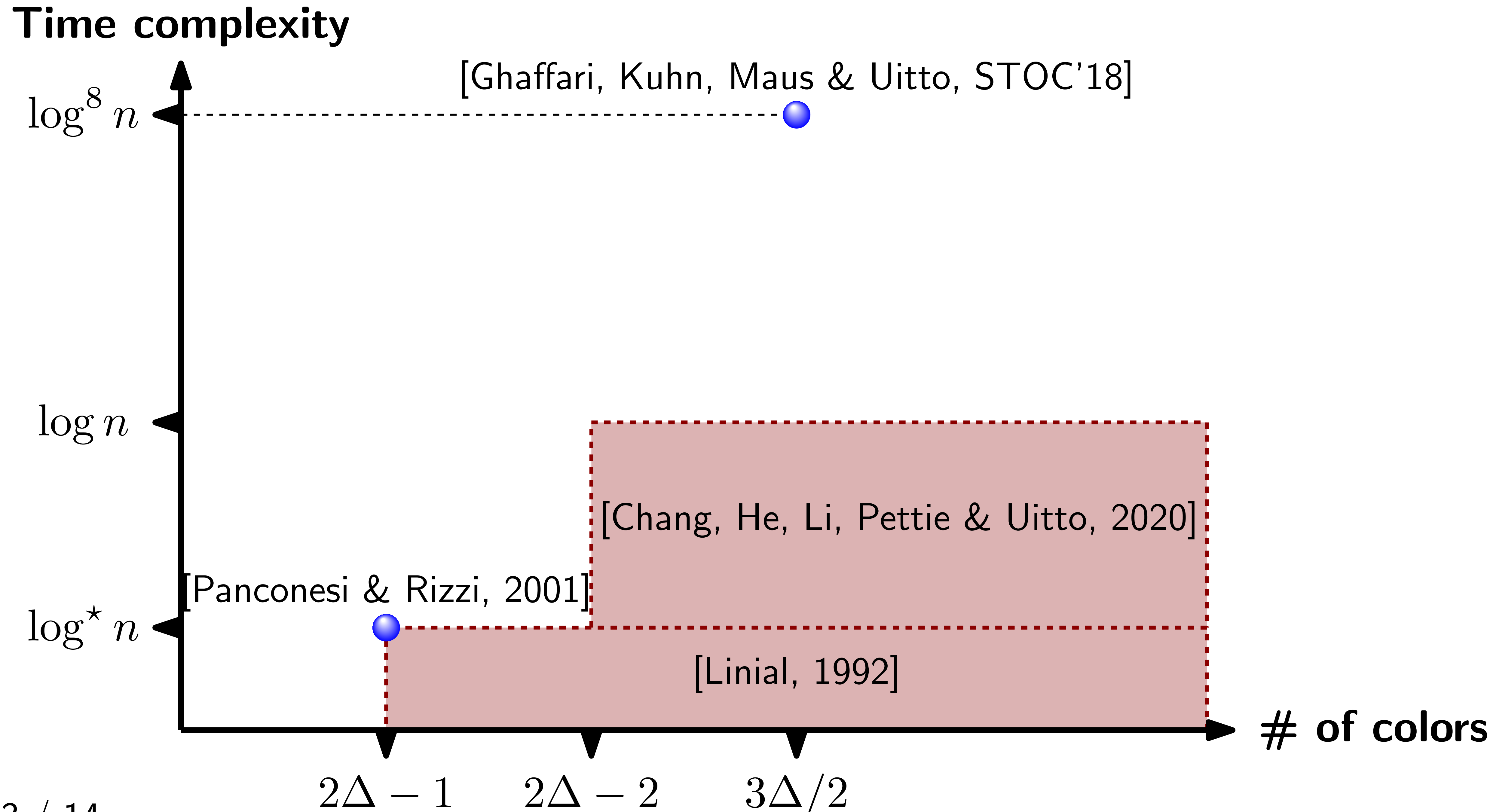

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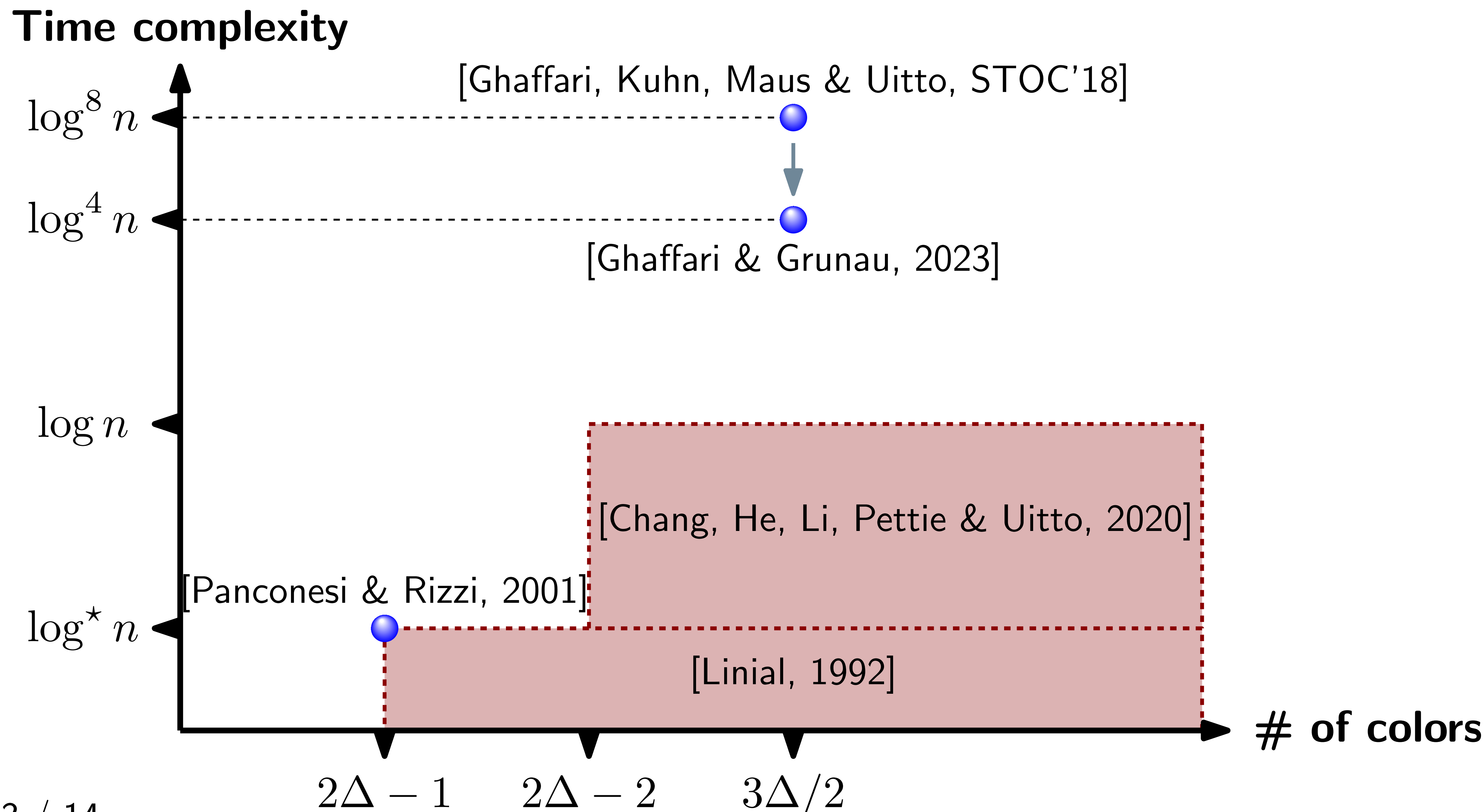

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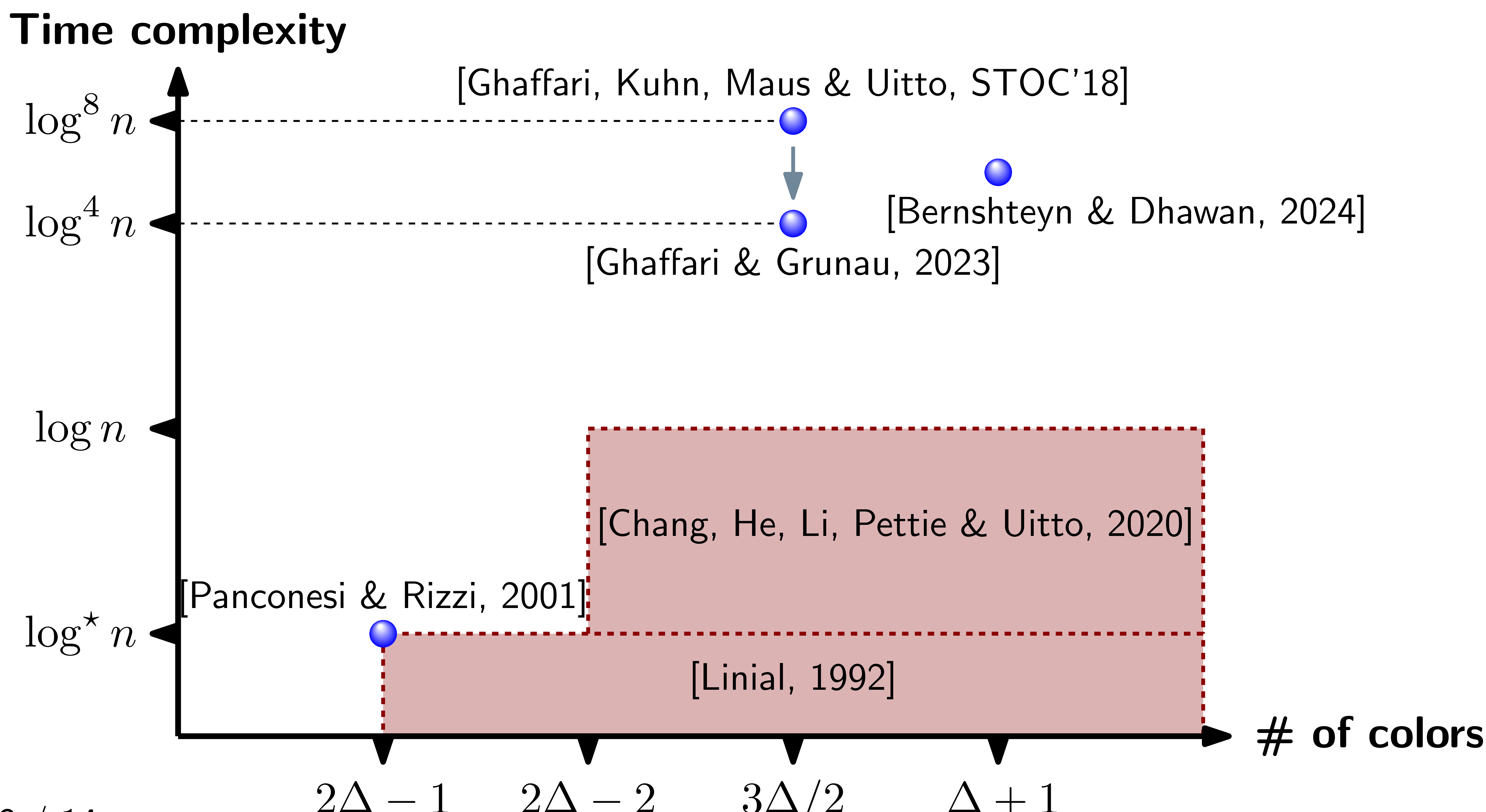
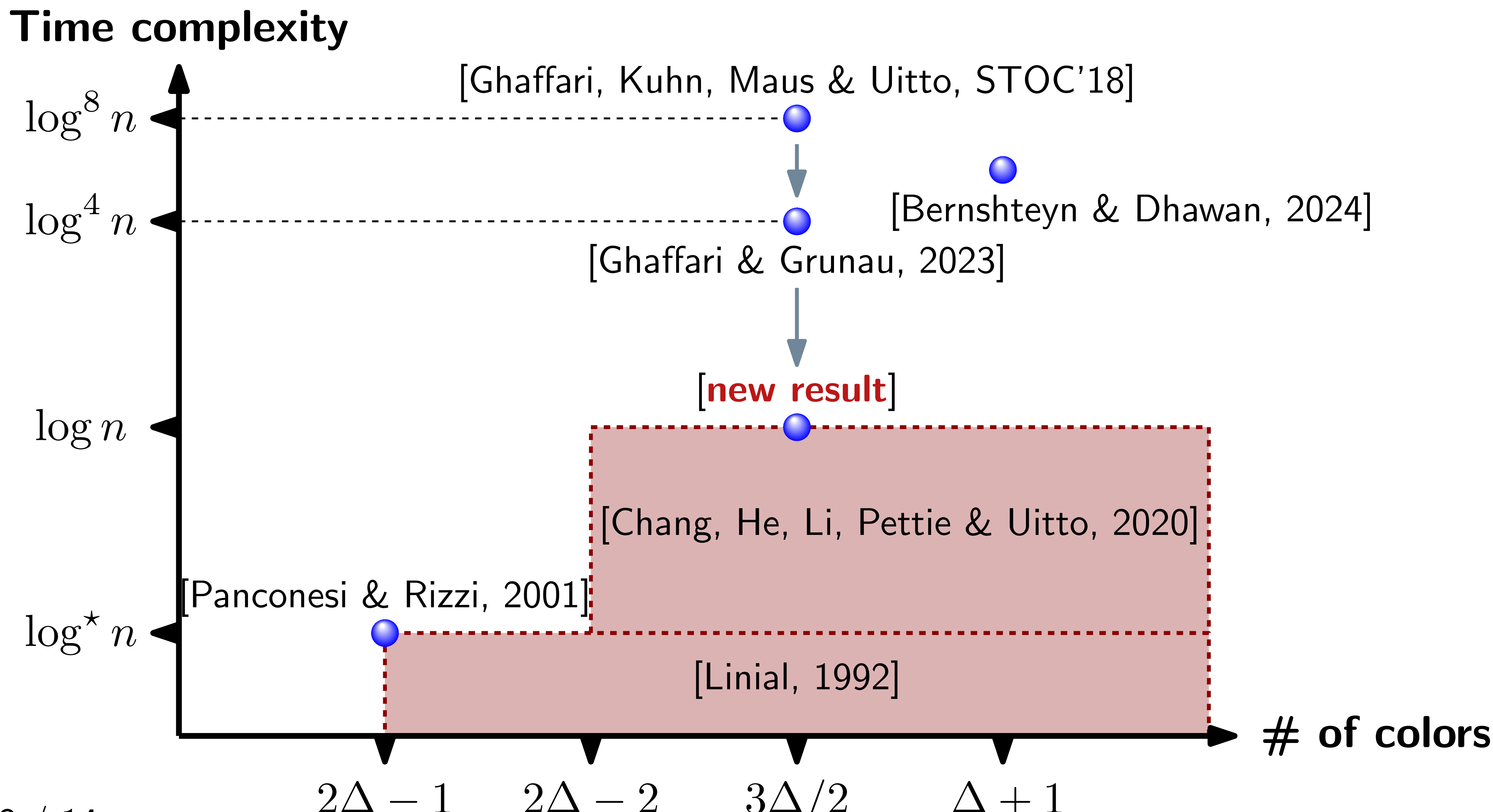
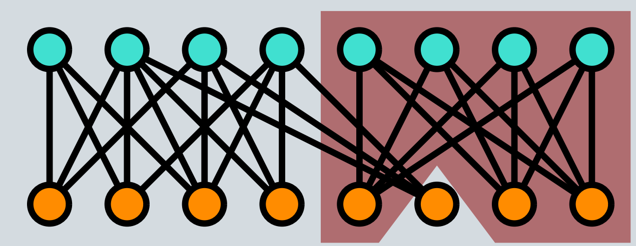


State-of-the-art edge coloring in DetLOCAL for const. Δ



Final slide

Motivation	Design asymptotically optimal algorithms in the LOCAL model	$\Theta(\log n)$
Algorithm design technique	Distributed Hall's Theorem	
Application	$3\Delta/2$-edge coloring algorithm