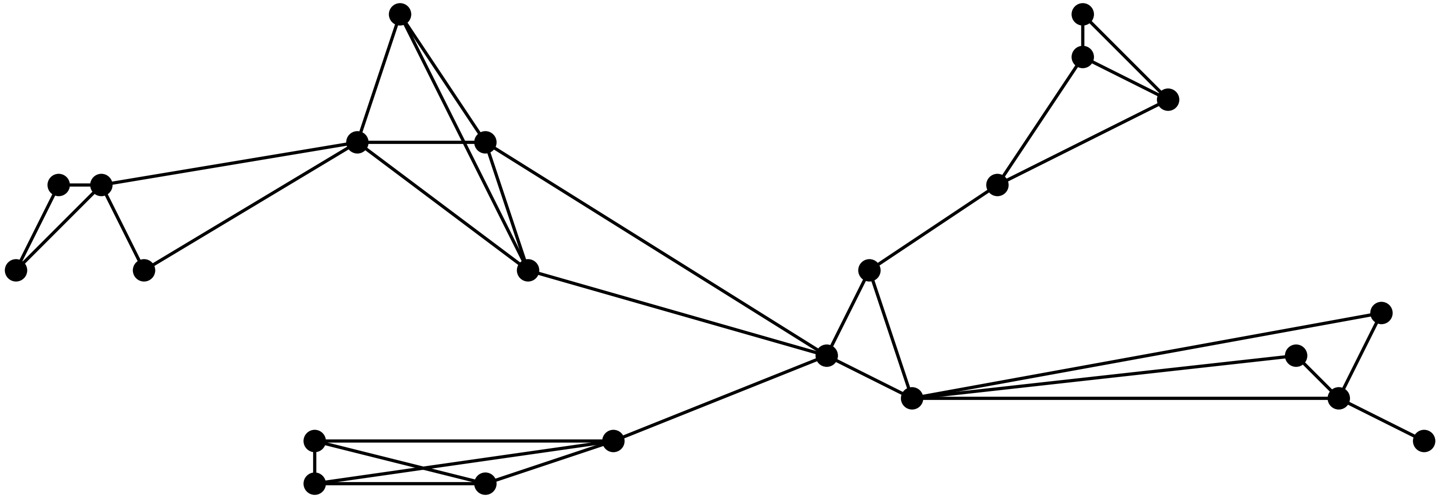
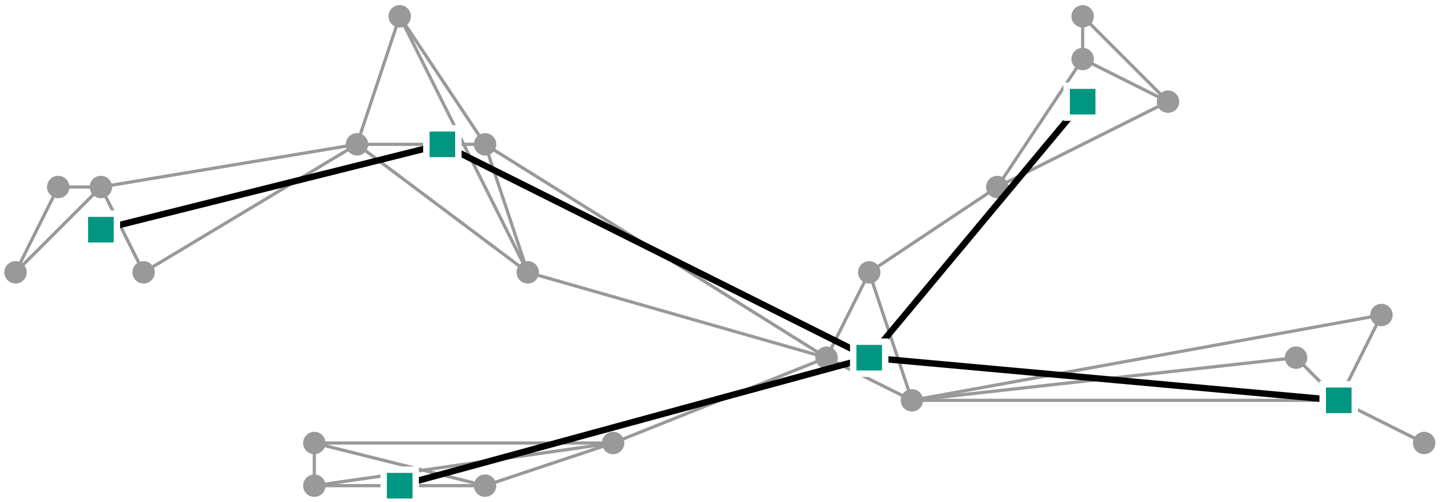
Tree Decompositions & Graph Decompositions

Tree decompositions

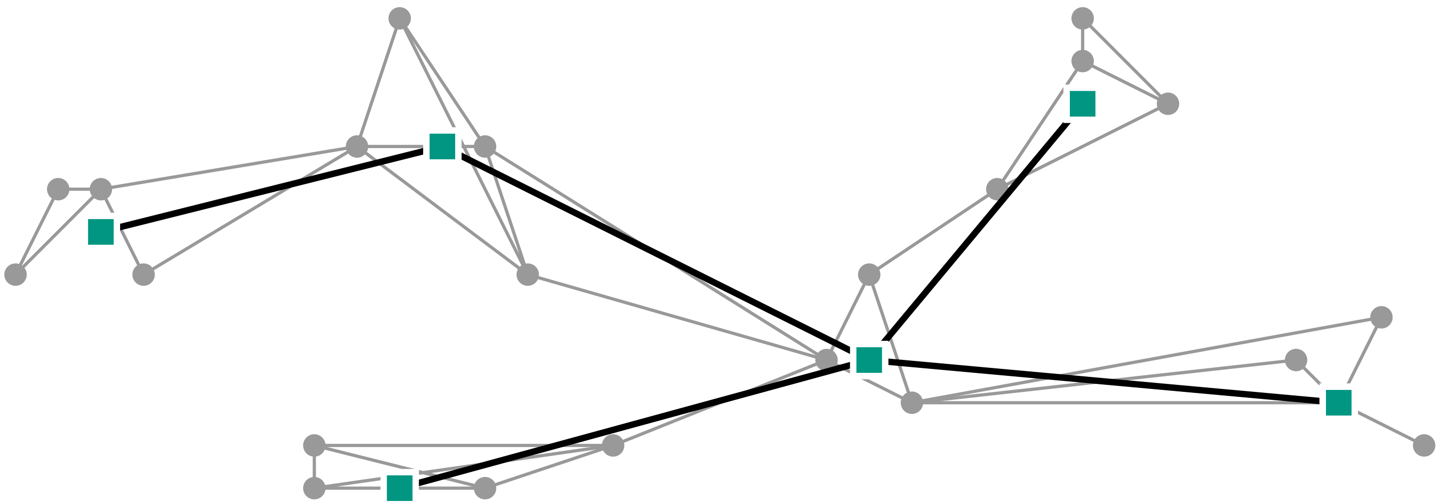


Tree decompositions



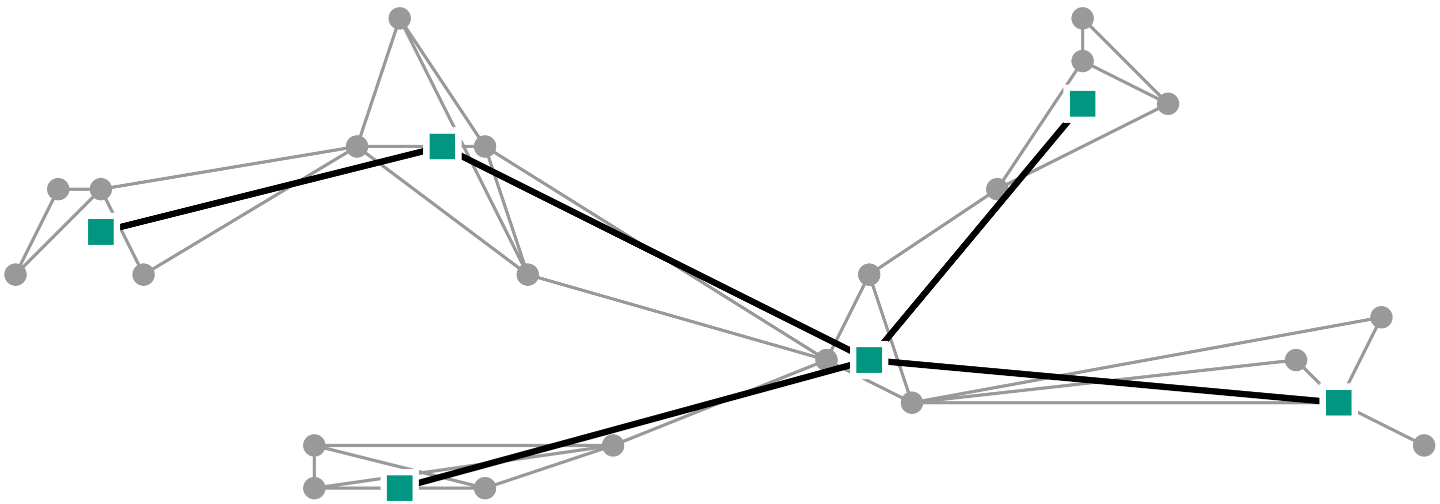
- How to capture the tree structure of a graph?

Tree decompositions



- How to capture the tree structure of a graph?
- A *tree decomposition* of a graph G is a tree T and a collection of *bags* $(B_t)_{t \in V(T)}$, $B_t \subseteq V(G)$ such that
    1. $V(G) = \cup_{t \in V(T)} B_t$
    2. For every edge $uv \in E(G)$, there exists a bag $V_t \supseteq \{u, v\}$.
    3. For any v, the bags containing v form a subtree in T.

Tree decompositions



- How to capture the tree structure of a graph?
- A *tree decomposition* of a graph G is a tree T and a collection of *bags* $(B_t)_{t \in V(T)}$, $B_t \subseteq V(G)$ such that
  1. $V(G) = \cup_{t \in V(T)} B_t$
  2. For every edge $uv \in E(G)$, there exists a bag $V_t \supseteq \{u, v\}$.
  3. For any v, the bags containing v form a subtree in T.

How does this differ from "T is induced minor of G"?

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

- The treewidth of a graph measures its *tree-likeness*

Treewidth

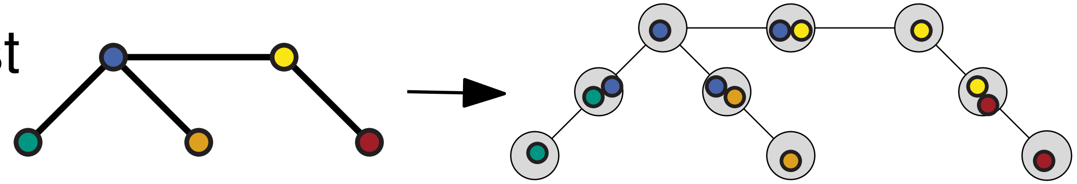
$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

- The treewidth of a graph measures its *tree-likeness*
  - $\text{tw}(G) = 1 \Leftrightarrow G$ is a forest
  - $\text{tw}(K_n) = n - 1$

Treewidth

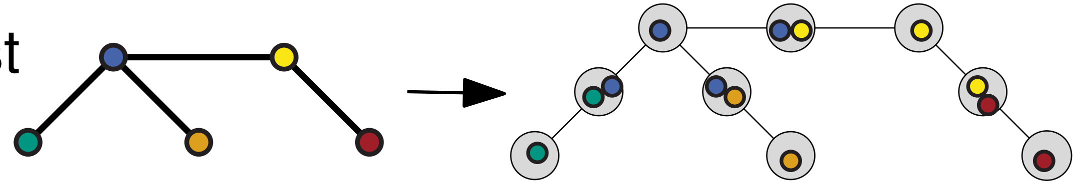
$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

- The treewidth of a graph measures its *tree-likeness*
  - $\text{tw}(G) = 1 \Leftrightarrow G$ is a forest
  - $\text{tw}(K_n) = n - 1$



$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices } \} - 1$$

- The treewidth of a graph measures its *tree-likeness*
  - $\text{tw}(G) = 1 \Leftrightarrow G$ is a forest
  - $\text{tw}(K_n) = n - 1$



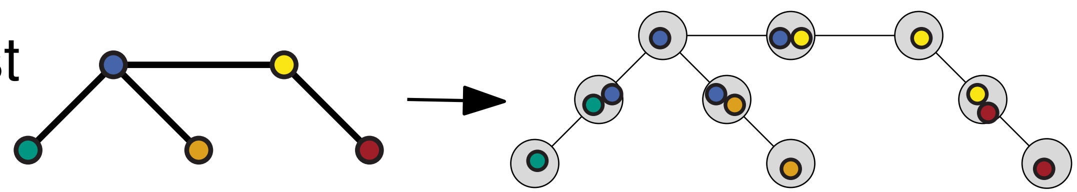
Alternative definitions

- $\text{tw}(G) = \min\{\omega(H) \mid H \text{ is chordal}, G \subseteq H\} - 1$

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

- The treewidth of a graph measures its *tree-likeness*
  - $\text{tw}(G) = 1 \Leftrightarrow G$ is a forest
  - $\text{tw}(K_n) = n - 1$



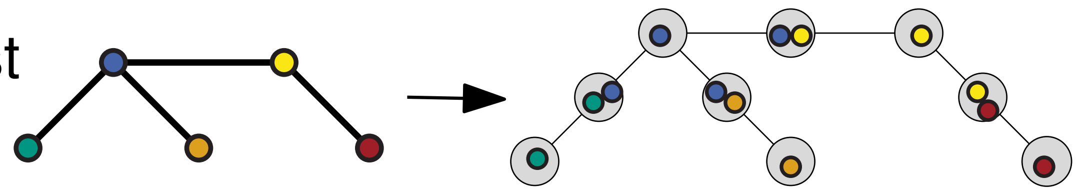
Alternative definitions

- $\text{tw}(G) = \min\{\omega(H) \mid H \text{ is chordal}, G \subseteq H\} - 1$
- $\text{tw}(G) =$ cop-number of G in a variant of “cops and robbers”.

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

- The treewidth of a graph measures its *tree-likeness*
  - $\text{tw}(G) = 1 \Leftrightarrow G$ is a forest
  - $\text{tw}(K_n) = n - 1$



Alternative definitions

- $\text{tw}(G) = \min\{\omega(H) \mid H \text{ is chordal}, G \subseteq H\} - 1$
- $\text{tw}(G) =$ cop-number of G in a variant of “cops and robbers”.
- First choose starting positions (first cops, then robber). Then, each turn
  - One cop announces a vertex he will fly towards with a helicopter and is removed from the graph
  - The robber may move to any vertex reachable via a path that isn’t blocked by the cops
  - The cop in the helicopter lands on his chosen vertex.

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Computational interest

- Many NP-hard problems on graphs can be solved in running time $\leq f(\text{tw}(G)) \cdot n^{\mathcal{O}(1)}$. (Fixed parameter tractable)

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Computational interest

- Many NP-hard problems on graphs can be solved in running time $\leq f(\text{tw}(G)) \cdot n^{\mathcal{O}(1)}$. (Fixed parameter tractable)

VERTEX COVER MAXCUT STEINER TREE HAMILTON CYCLE LONGEST CYCLE CYCLE PACKING
DOMINATING SET ODD CYCLE TRANSVERSAL LONGEST PATH CONNECTED DOMINATING SET
INDEPENDENT SET FEEDBACK VERTEX SET HAMILTON PATH CONNECTED VERTEX COVER
CHROMATIC NUMBER CONNECTED FEEDBACK VERTEX SET

- Idea: Run a dynamic program on the tree-decomposition of G

What could this look like for 3-COLOR?

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Computational interest

- Many NP-hard problems on graphs can be solved in running time $\leq f(\text{tw}(G)) \cdot n^{\mathcal{O}(1)}$. (Fixed parameter tractable)

VERTEX COVER MAXCUT STEINER TREE HAMILTON CYCLE LONGEST CYCLE CYCLE PACKING
DOMINATING SET ODD CYCLE TRANSVERSAL LONGEST PATH CONNECTED DOMINATING SET
INDEPENDENT SET FEEDBACK VERTEX SET HAMILTON PATH CONNECTED VERTEX COVER
CHROMATIC NUMBER CONNECTED FEEDBACK VERTEX SET

- Idea: Run a dynamic program on the tree-decomposition of G

What could this look like for 3-COLOR?

- Problem: Finding $\text{tw}(G)$ and a tree-decomposition is NP-hard

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Computational interest

- Many NP-hard problems on graphs can be solved in running time $\leq f(\text{tw}(G)) \cdot n^{\mathcal{O}(1)}$. (Fixed parameter tractable)

VERTEX COVER, MAXCUT, STEINER TREE, HAMILTON CYCLE, LONGEST CYCLE, CYCLE PACKING, DOMINATING SET, ODD CYCLE TRANSVERSAL, LONGEST PATH, CONNECTED DOMINATING SET, INDEPENDENT SET, FEEDBACK VERTEX SET, HAMILTON PATH, CONNECTED VERTEX COVER, CHROMATIC NUMBER, CONNECTED FEEDBACK VERTEX SET

- Idea: Run a dynamic program on the tree-decomposition of G
- Problem: Finding tw(G) and a tree-decomposition is NP-hard
- Solution: There is an $k \cdot 2^{\mathcal{O}(k^2)} \cdot n^4$ algorithm for both

What could this look like for 3-COLOR?

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices }\} - 1$$

Computational interest

- Many NP-hard problems on graphs can be solved in running time $\leq f(\text{tw}(G)) \cdot n^{\mathcal{O}(1)}$. (Fixed parameter tractable)

VERTEX COVER MAXCUT STEINER TREE HAMILTON CYCLE LONGEST CYCLE CYCLE PACKING
DOMINATING SET ODD CYCLE TRANSVERSAL LONGEST PATH CONNECTED DOMINATING SET
INDEPENDENT SET FEEDBACK VERTEX SET HAMILTON PATH CONNECTED VERTEX COVER
CHROMATIC NUMBER CONNECTED FEEDBACK VERTEX SET

- Idea: Run a dynamic program on the tree-decomposition of G

What could this look like for 3-COLOR?

- Problem: Finding tw(G) and a tree-decomposition is NP-hard
- Solution: There is an $k \cdot 2^{\mathcal{O}(k^2)} \cdot n^4$ algorithm for both

Courcelle's theorem (1990)

Let φ be an MSO_2-formula and G some graph. Then, there is an $f(|\varphi|, \text{tw}(G)) \cdot n$ algorithm for testing whether G satisfies φ (for some computable function f)

Treewidth

$$\mathrm{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.

$$\mathrm{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices } \} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
    - $H_{\text{planar graphs}} = \{K_3, K_{5,5}\}$

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
    - $H_{\text{planar graphs}} = \{K_3, K_{5,5}\}$
    - $H_{\text{graphs of tw} \leq 1} = \{K_3\}$
    - $H_{\text{graphs of tw} \leq 2} = \{K_4\}$

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices }\} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
    - $H_{\text{planar graphs}} = \{K_3, K_{5,5}\}$
    - $H_{\text{graphs of tw} \leq 1} = \{K_3\}$
    - $H_{\text{graphs of tw} \leq 2} = \{K_4\}$
    - $|H_{\text{graphs of tw} \leq 3}| = 4$

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices }\} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
    - $H_{\text{planar graphs}} = \{K_3, K_{5,5}\}$
    - $H_{\text{graphs of tw} \leq 1} = \{K_3\}$
    - $H_{\text{graphs of tw} \leq 2} = \{K_4\}$
    - $|H_{\text{graphs of tw} \leq 3}| = 4$
    - $|H_{\text{toroidal graphs}}| \geq 17523$

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices }\} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
- G has a K_n-minor $\Rightarrow \text{tw}(G) \geq n - 1$. On the other hand, $\not\Leftarrow$.

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices}\} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
- G has a K_n-minor $\Rightarrow$ $\text{tw}(G) \geq n-1$. On the other hand, $\not\Leftarrow$.
- However, there is a function $f(k) = \mathcal{O}(k^{98+o(1)})$ such that every graph with treewidth $f(k)$ contains a $k \times k$-grid as a minor.

Treewidth

$$\text{tw}(G) := \min\{k \mid \text{There is a tree-decomp. of } G \text{ in which all bags contain at most } k \text{ vertices } \} - 1$$

Structural interest

- Treewidth was a central notion in the 23-paper spanning Graph Minor program of Robertson and Seymour (1983-2010).
  - One main result: Any minor-closed graph family $\mathcal{G}$ is characterized by a finite set of forbidden minors $H_{\mathcal{G}}$.
- G has a K_n-minor $\Rightarrow \text{tw}(G) \geq n - 1$. On the other hand, $\not\Leftarrow$.
- However, there is a function $f(k) = \mathcal{O}(k^{98+o(1)})$ such that every graph with treewidth $f(k)$ contains a $k \times k$-grid as a minor.
- Getting further into modern structural graph theory (of minors), one arrives at brambles, blocks and especially *tangles*

Graph decompositions

- What if the global structure of G is better explained by some graph that is not a tree?

Graph decompositions

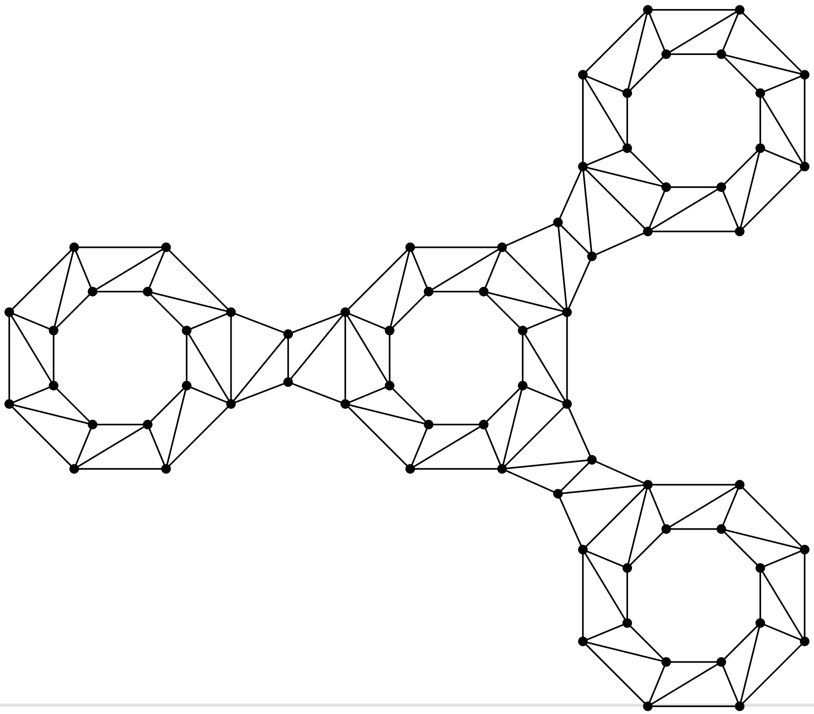
- What if the global structure of G is better explained by some graph that is not a tree?
- Let G, H be two graphs and $(G_h)_{h\in V(H)}$ a family of subgraphs of $G_h \subseteq G$. The pair $(H, (G_h)_{h\in V(H)})$ is called a *H-decomposition* of G if
  1. $G = \cup_{h\in V(H)} G_h$
  2. For any v, the bags containing v form a connected subgraph in H.

Graph decompositions

- What if the global structure of G is better explained by some graph that is not a tree?
- Let G, H be two graphs and $(G_h)_{h \in V(H)}$ a family of subgraphs of $G_h \subseteq G$. The pair $(H, (G_h)_{h \in V(H)})$ is called a *H-decomposition* of G if
  1. $G = \cup_{h \in V(H)} G_h$
  2. For any v, the bags containing v form a connected subgraph in H.
- Given a graph G, is there a canonical choice of H which witnesses the global structure of G?

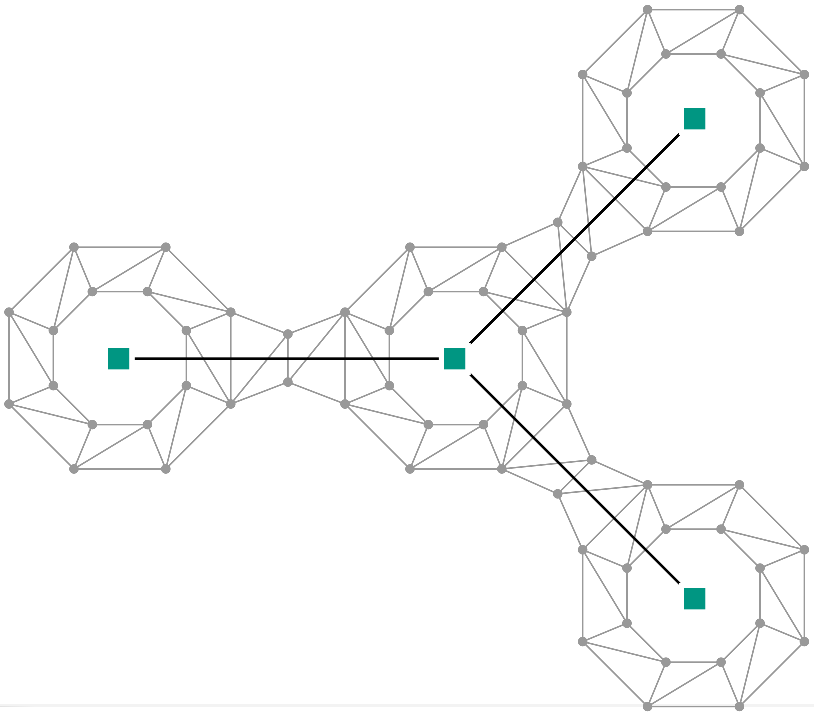
Graph decompositions

- What if the global structure of G is better explained by some graph that is not a tree?
- Let G, H be two graphs and $(G_h)_{h \in V(H)}$ a family of subgraphs of $G_h \subseteq G$. The pair $(H, (G_h)_{h \in V(H)})$ is called a *H-decomposition* of G if
  1. $G = \cup_{h \in V(H)} G_h$
  2. For any v, the bags containing v form a connected subgraph in H.
- Given a graph G, is there a canonical choice of H which witnesses the global structure of G?
- *How* global?



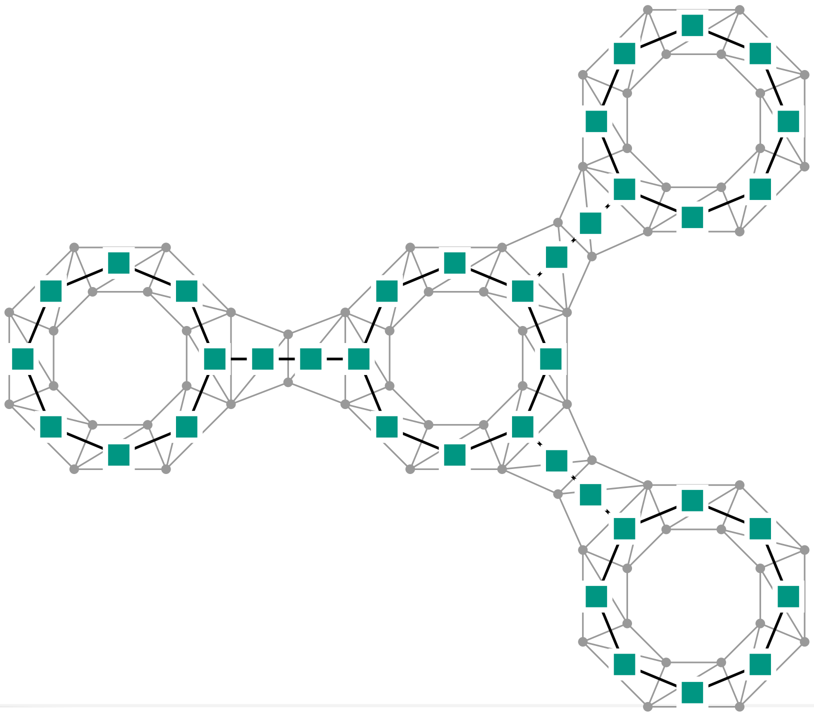
Graph decompositions

- What if the global structure of G is better explained by some graph that is not a tree?
- Let G, H be two graphs and $(G_h)_{h \in V(H)}$ a family of subgraphs of $G_h \subseteq G$. The pair $(H, (G_h)_{h \in V(H)})$ is called a *H-decomposition* of G if
  1. $G = \cup_{h \in V(H)} G_h$
  2. For any v, the bags containing v form a connected subgraph in H.
- Given a graph G, is there a canonical choice of H which witnesses the global structure of G?
- *How* global?



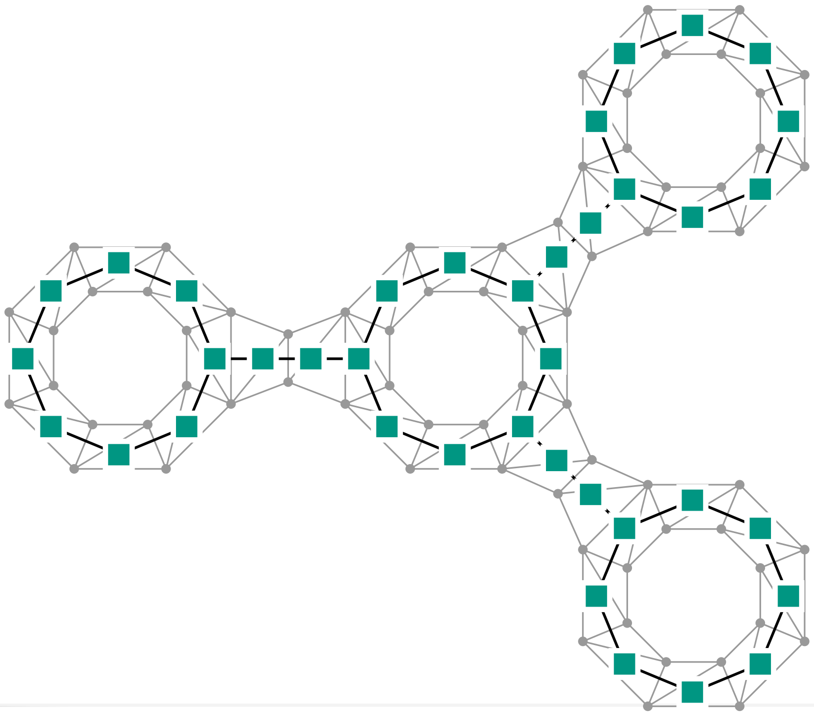
Graph decompositions

- What if the global structure of G is better explained by some graph that is not a tree?
- Let G, H be two graphs and $(G_h)_{h \in V(H)}$ a family of subgraphs of $G_h \subseteq G$. The pair $(H, (G_h)_{h \in V(H)})$ is called a *H-decomposition* of G if
  1. $G = \cup_{h \in V(H)} G_h$
  2. For any v, the bags containing v form a connected subgraph in H.
- Given a graph G, is there a canonical choice of H which witnesses the global structure of G?
- *How* global?



Graph decompositions

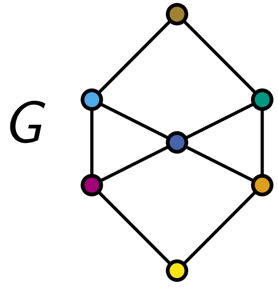
- What if the global structure of G is better explained by some graph that is not a tree?
- Let G, H be two graphs and $(G_h)_{h \in V(H)}$ a family of subgraphs of $G_h \subseteq G$. The pair $(H, (G_h)_{h \in V(H)})$ is called a *H-decomposition* of G if
  1. $G = \cup_{h \in V(H)} G_h$
  2. For any v, the bags containing v form a connected subgraph in H.
- Given a graph G, is there a canonical choice of H which witnesses the global structure of G?
- *How* global?
- Idea:
  Consider all cycles of length $\leq r$ and their "compositions" as local.



Canonical graph decompositions via coverings

REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*



Canonical graph decompositions via coverings

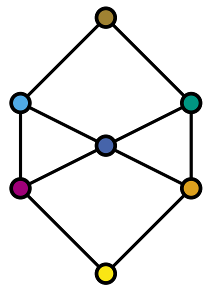
REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*

Construction idea

- Use the fundamental group $\pi_1(G, x_0)$: The group of closed walks from x_0 up to local reductions.

G



$$\pi_1(G) = F_4$$

Canonical graph decompositions via coverings

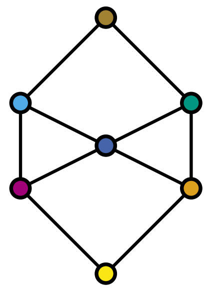
REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*

Construction idea

- Use the fundamental group $\pi_1(G, x_0)$: The group of closed walks from x_0 up to local reductions.
- Let $\pi_1^r(G, x_0) \leq \pi_1(G, x_0)$ be generated by all closed walks of length $\leq r$.

G



$$\pi_1(G) = F_4$$
$$\pi_1^r(G) = F_2$$

Canonical graph decompositions via coverings

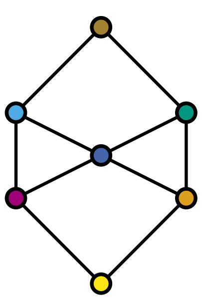
REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*

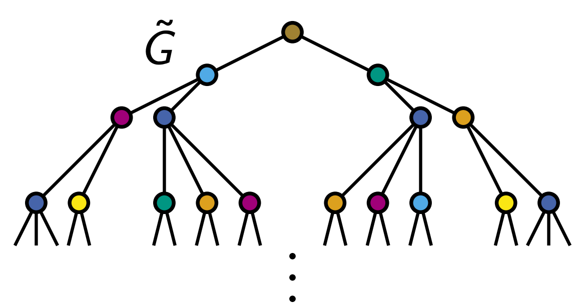
Construction idea

- Use the fundamental group $\pi_1(G, x_0)$: The group of closed walks from x_0 up to local reductions.
- Let $\pi_1^r(G, x_0) \leq \pi_1(G, x_0)$ be generated by all closed walks of length $\leq r$.
- Consider the universal covering $\tilde{G}$. Its isometry group (of Deck transformations) is exactly $\pi_1(G, x_0)$.

G



$$\pi_1(G) = F_4$$
$$\pi_1^r(G) = F_2$$



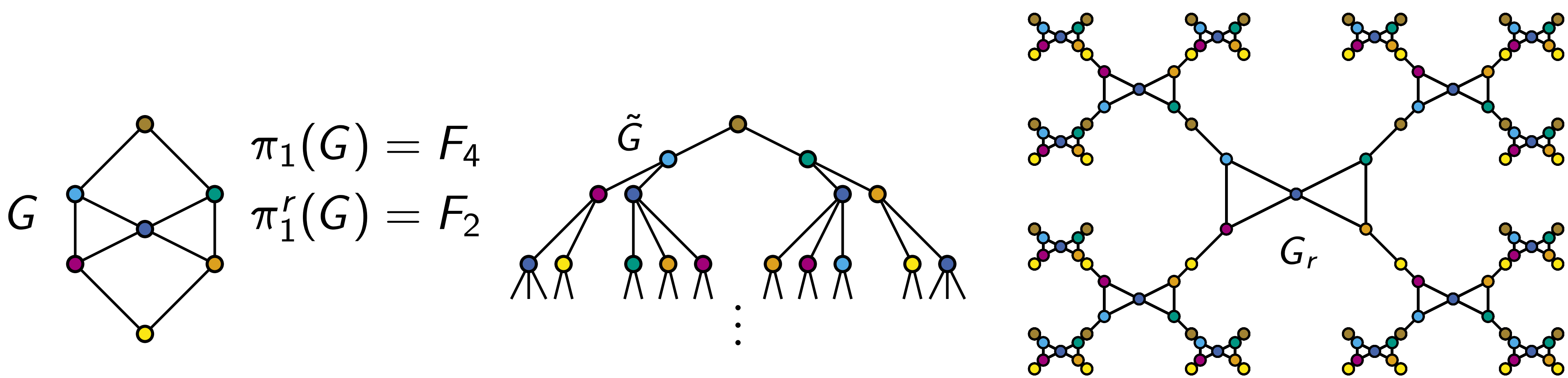
Canonical graph decompositions via coverings

REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*

Construction idea

- Use the fundamental group $\pi_1(G, x_0)$: The group of closed walks from x_0 up to local reductions.
- Let $\pi_1^r(G, x_0) \leq \pi_1(G, x_0)$ be generated by all closed walks of length $\leq r$.
- Consider the universal covering $\tilde{G}$. Its isometry group (of Deck transformations) is exactly $\pi_1(G, x_0)$.
- Define the r-local covering $G_r := \tilde{G}/\pi_1^r(G, x_0)$. Let D_r be its isometry group.



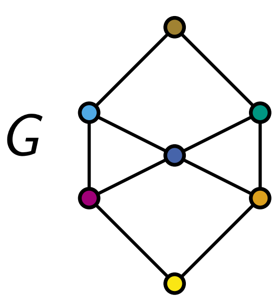
Canonical graph decompositions via coverings

REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

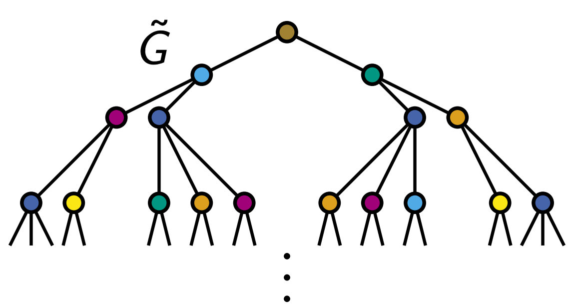
Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*

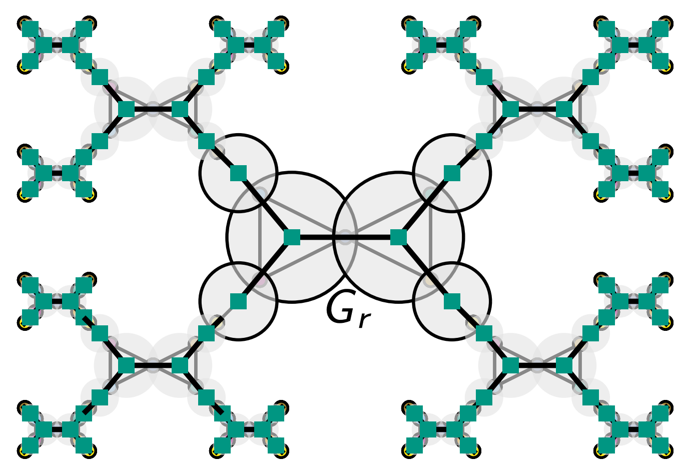
Construction idea

- Use the fundamental group $\pi_1(G, x_0)$: The group of closed walks from x_0 up to local reductions.
- Let $\pi_1^r(G, x_0) \leq \pi_1(G, x_0)$ be generated by all closed walks of length $\leq r$.
- Consider the universal covering $\tilde{G}$. Its isometry group (of Deck transformations) is exactly $\pi_1(G, x_0)$.
- Define the r-local covering $G_r := \tilde{G}/\pi_1^r(G, x_0)$. Let D_r be its isometry group.
- Find a canonical tree decomposition $(T_r, \mathcal{V}_{T_r})$ of G^r using the theory of ends and tangles.



$$\pi_1(G) = F_4$$
$$\pi_1^r(G) = F_2$$





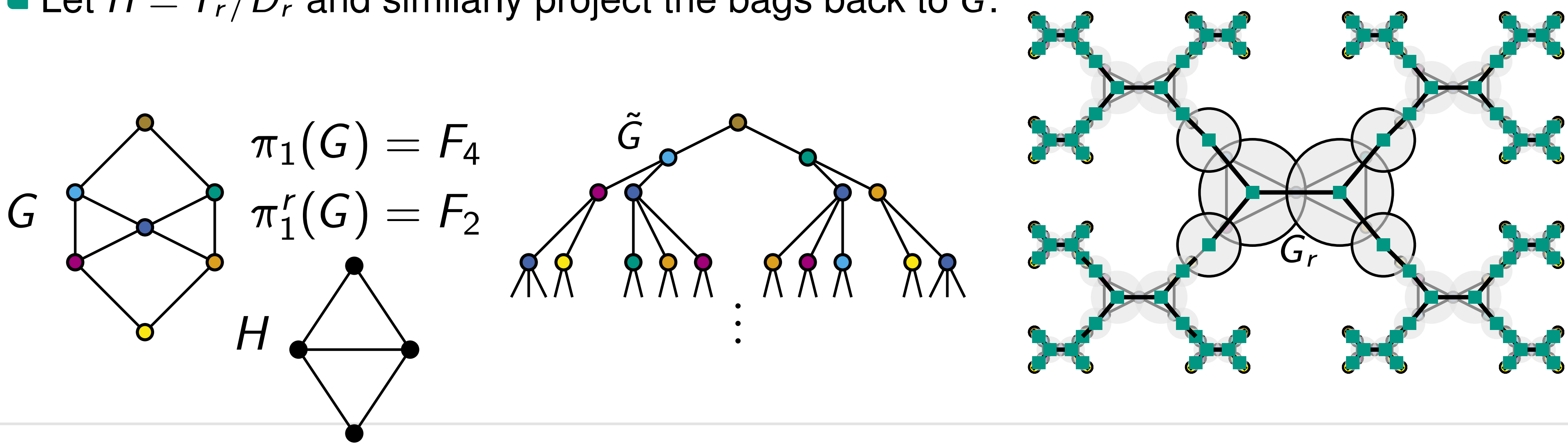
Canonical graph decompositions via coverings

REINHARD DIESTEL, RAPHAEL W. JACOBS, PAUL KNAPPE, AND JAN KURKOFKA, arχiv 2022

Theorem 1. *Let G be any finite graph, and $r > 0$ an integer. Then G has a unique canonical decomposition modelled on another finite graph $H = H(G, r)$ that displays its r-global structure.*

Construction idea

- Use the fundamental group $\pi_1(G, x_0)$: The group of closed walks from x_0 up to local reductions.
- Let $\pi_1^r(G, x_0) \leq \pi_1(G, x_0)$ be generated by all closed walks of length $\leq r$.
- Consider the universal covering $\tilde{G}$. Its isometry group (of Deck transformations) is exactly $\pi_1(G, x_0)$.
- Define the r-local covering $G_r := \tilde{G}/\pi_1^r(G, x_0)$. Let D_r be its isometry group.
- Find a canonical tree decomposition $(T_r, \mathcal{V}_{T_r})$ of G^r using the theory of ends and tangles.
- Let $H = T_r/D_r$ and similarly project the bags back to G.



Closing thoughts

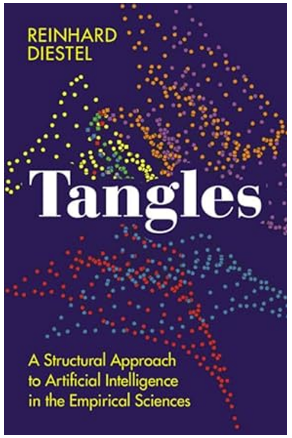
- The canonical tree decomposition on the covering space is somewhat mysterious to me.

Closing thoughts

- The canonical tree decomposition on the covering space is somewhat mysterious to me.
- Nice properties of H
  - H respects all symmetries of G.
  - For $r = |G|$, H is a tree.

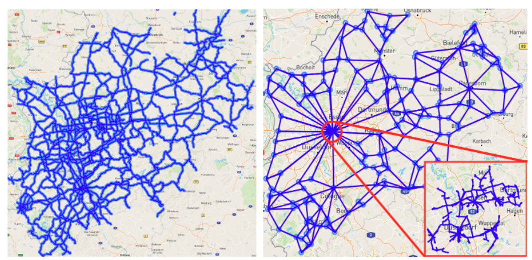
Closing thoughts

- The canonical tree decomposition on the covering space is somewhat mysterious to me.
- Nice properties of H
  - H respects all symmetries of G.
  - For $r = |G|$, H is a tree.
- Tangles!

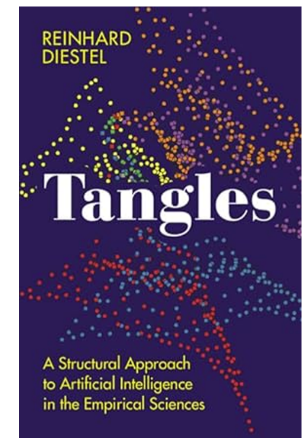


Closing thoughts

- The canonical tree decomposition on the covering space is somewhat mysterious to me.
- Nice properties of H
  - H respects all symmetries of G.
  - For $r = |G|$, H is a tree.
- Tangles!
- Applications?

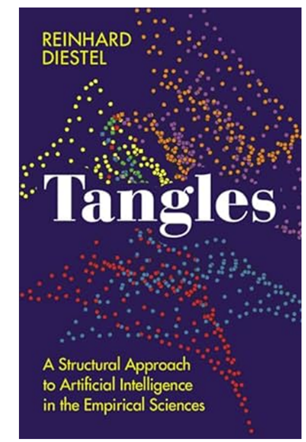


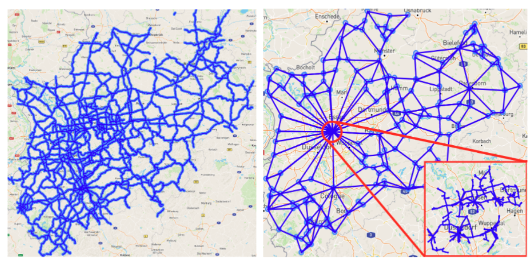
(a) (b)



Closing thoughts

- The canonical tree decomposition on the covering space is somewhat mysterious to me.
- Nice properties of H
  - H respects all symmetries of G.
  - For $r = |G|$, H is a tree.
- Tangles!
- Applications?





(a) (b)

1.6. **Applications.** We think of the decomposition theory advanced in this paper, and in particular of Theorem 1, not so much as a tool with which to attack existing problems in graph theory, but as a natural way to view graphs and to analyse their structure from first principles. We do believe that our decompositions have the potential to interact with other graph invariants, by splitting them into a local and a global aspect as indicated after the statement of Theorem 1. But how exactly this can happen may not be straightforward, and may require non-trivial structural analysis. This will be worthwhile if, but also only if, our new invariant of $H(G, r)$ and the associated decomposition of G are considered as a natural lens through which graph structure may be viewed. We believe they are.