Quasi-One-Dimensional Flow Analysis

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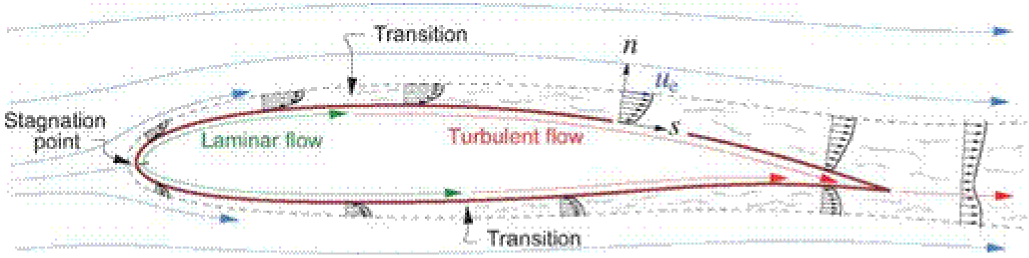
Cases of application – flow in slender geometries

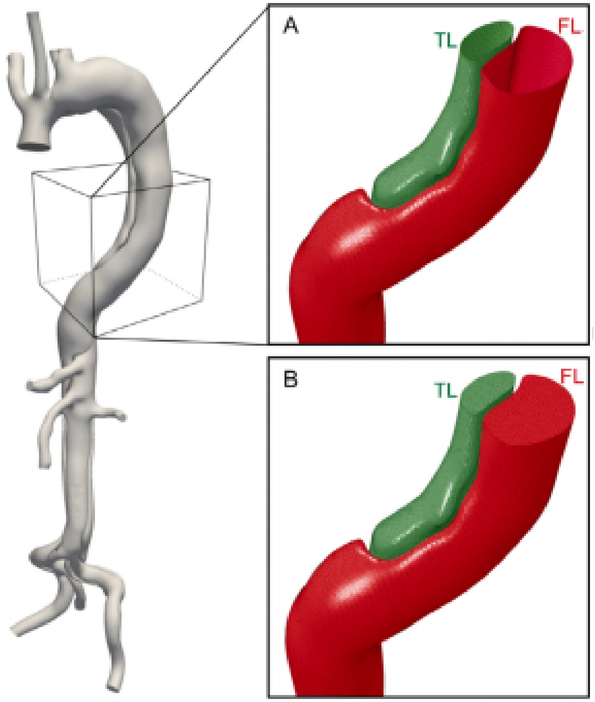
Flow in slender geometries (boundary layers, pipe flows, liquid jets in a gas, thin films) may be conveniently analysed as quasi-one-dimensional

Boundary layers

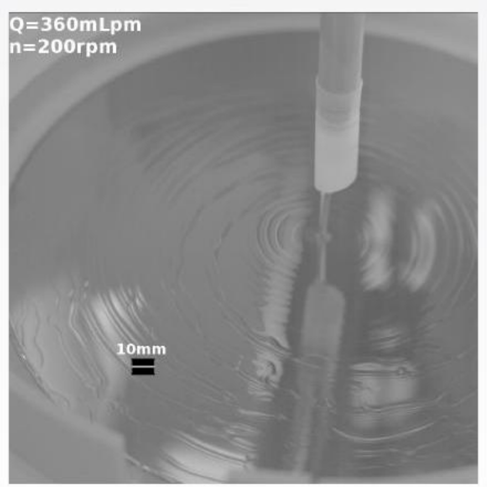


From heli-air.net

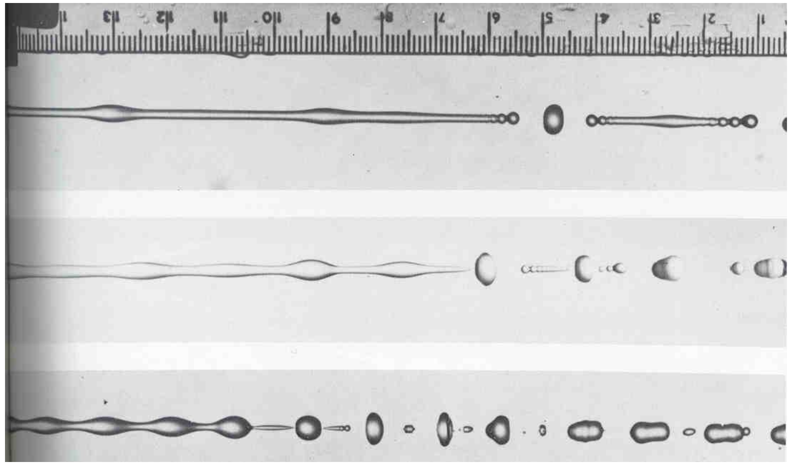
Slender ducts (blood vessels, pipes, tunnels etc.)



Thin films on substrates



Liquid jets

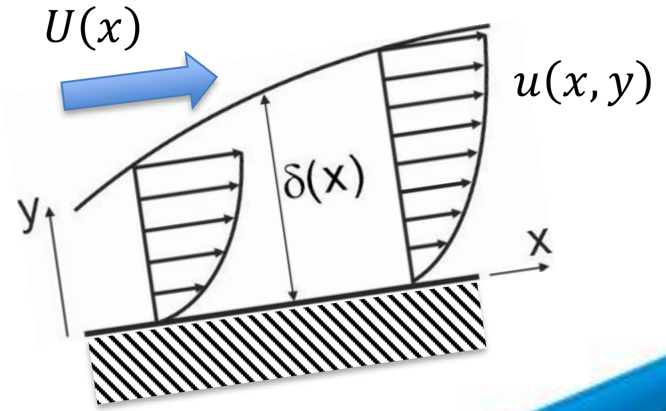


Rutland & Jameson JFM 1971

Three case studies

We look at three cases, using two different methods

1. Boundary-layer flow
   (near-wall external flow)



2. Flow through slender ducts
   (blood vessels, pipes, tunnels)

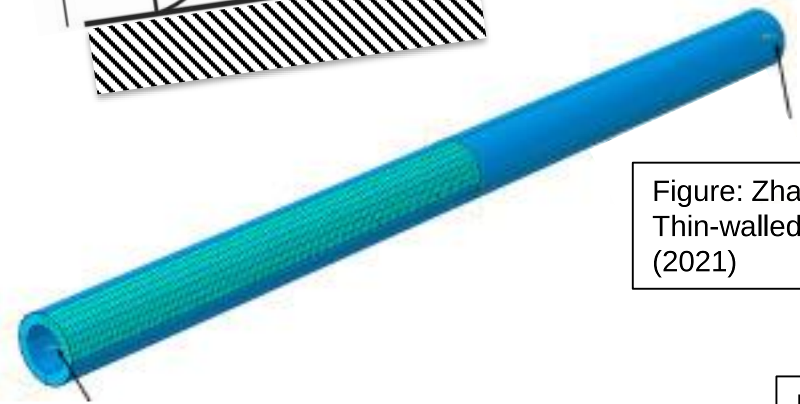
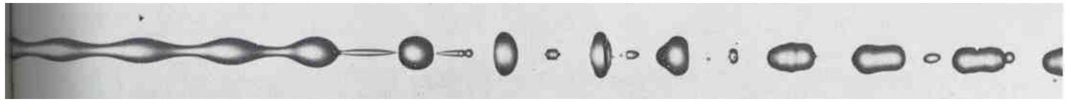


Figure: Zhao et al.
Thin-walled structures
(2021)

3. Free liquid jets
   (drop and spray formation,
   ink-jet printing)



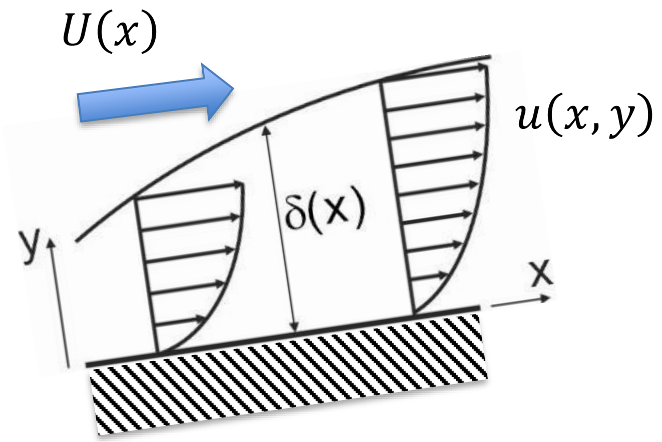
Rutland &
Jameson
JFM 1971

Plane boundary layer – problem statement in 2D

Flow is governed by the 2D equations of motion in boundary-layer form

We assume steady state, large Froude number, incompressible Newtonian fluid

Formulation in Cartesian coordinates



Mass balance

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Balance of linear momentum in boundary-layer form

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\equiv\rho\left(\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2}$$

$$0\approx-\frac{\partial p}{\partial y}$$

Boundary conditions

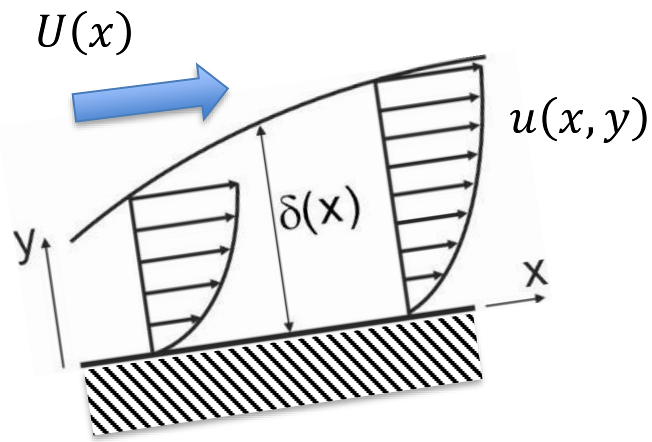
$y=0$: $u=v=0$

$y\rightarrow\infty$: $u=U(x), p+\frac{\rho U^2(x)}{2}=\text{constant}$

Boundary layer – quasi-one-dimensional mass balance

Quasi-one-dimensional form of mass balance is obtained by integration over the boundary-layer thickness.

For thin films this is called “depth averaging”.



Mass balance

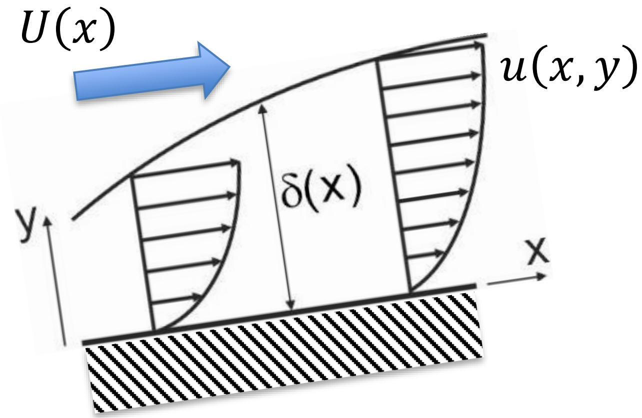
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Bigg| \int_{y=0}^{\delta(x)} dy$$

$$\int_{y=0}^{\delta(x)} \frac{\partial u}{\partial x} dy + \int_{y=0}^{\delta(x)} \frac{\partial v}{\partial y} dy = 0$$

Boundary layer – quasi-one-dimensional mass balance

For further developing the quasi-one-dimensional form of the mass balance

$$\int_{y=0}^{\delta(x)} \frac{\partial u}{\partial x} dy + \int_{y=0}^{\delta(x)} \frac{\partial v}{\partial y} dy = 0$$



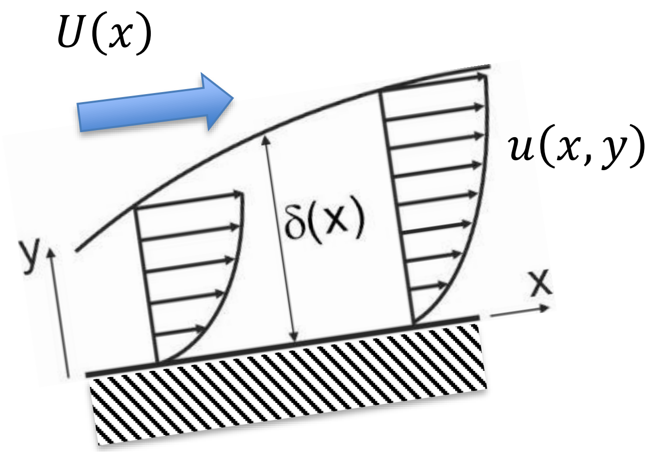
it is the intention to exchange integration and differentiation $\partial/\partial x$ to achieve a statement about spatial change of a volume flux in the flow direction.
For this we apply the Leibniz rule for derivatives of definite integrals:

$$\frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} F(x,y)\, dy = \int_{y=0}^{\delta(x)} \frac{\partial F(x,y)}{\partial x} dy + \frac{d\delta(x)}{dx} F\big(x,\delta(x)\big) - \frac{d0}{dx} F(x,0)$$

Boundary layer – quasi-one-dimensional mass balance

Applying the Leibniz rule turns the quasi-one-dimensional mass balance into

$$\frac{\partial}{\partial x}\int_{y=0}^{\delta(x)} u dy - u\Big|_{y=\delta(x)} \frac{d\delta(x)}{dx} + v\Big|_{y=\delta(x)} = 0$$

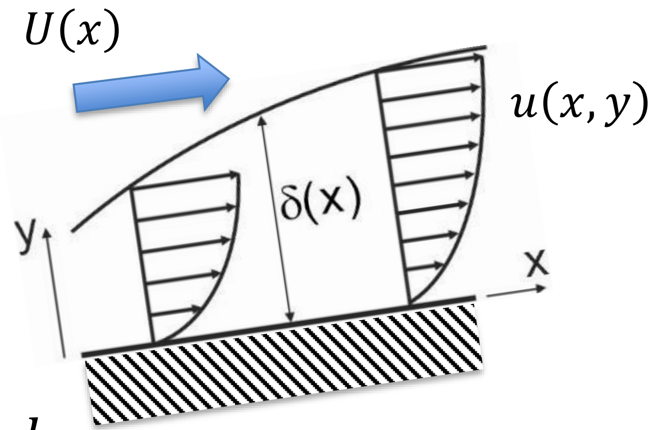


This means that

$$-u\Big|_{y=\delta(x)} \frac{d\delta(x)}{dx} + v\Big|_{y=\delta(x)} = -\frac{\partial}{\partial x}\int_{y=0}^{\delta(x)} u dy$$

Bound lay – quasi-one-dimensional x momentum balance

Quasi-one-dimensional form of x momentum balance is obtained by integration over the boundary-layer thickness



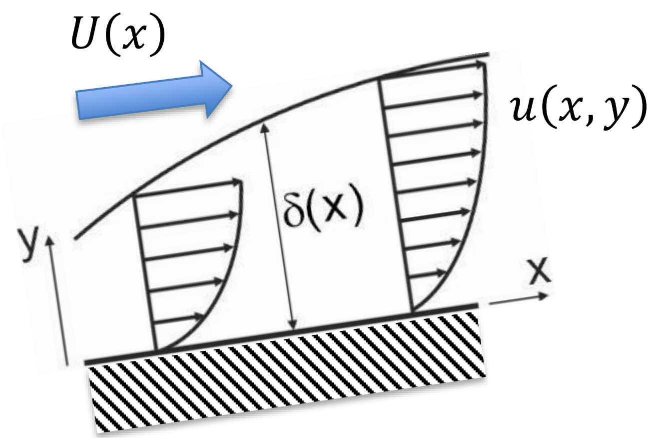
Balance of linear x momentum in boundary-layer form

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\equiv\rho\left(\frac{\partial uu}{\partial x}+\frac{\partial uv}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2}\quad\Bigg|\int_{y=0}^{\delta(x)} dy$$

$$\rho\int_{y=0}^{\delta(x)}\frac{\partial uu}{\partial x}dy+\rho\int_{y=0}^{\delta(x)}\frac{\partial uv}{\partial y}dy=-\int_{y=0}^{\delta(x)}\frac{dp}{dx}dy+\mu\int_{y=0}^{\delta(x)}\frac{\partial^2 u}{\partial y^2}dy$$

Bound lay – quasi-one-dimensional x momentum balance

Since the flow outside the boundary layer is potential, i.e. irrotational, i.e. inviscid, Bernoulli's equation states that



$$p + \rho \frac{U^2(x)}{2} = \text{constant; therefore } -\frac{dp}{dx} = \rho U \frac{dU}{dx}$$

Again with the Leibniz rule we obtain

$$\rho \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} uu dy - \rho\, uu\Big|_{y=\delta(x)} \frac{d\delta(x)}{dx} + \rho\, uv\Big|_{y=\delta(x)} = \rho \int_{y=0}^{\delta(x)} U \frac{dU}{dx} dy - \underbrace{\mu \frac{\partial u}{\partial y}\Big|_{y=0}}_{\equiv \tau_w}$$

Using the mass balance, we obtain furthermore

$$\rho \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} uu dy - \rho\, \underbrace{u\Big|_{y=\delta(x)}}_{\equiv U(x)} \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} u dy = \rho \int_{y=0}^{\delta(x)} U \frac{dU}{dx} dy - \tau_w$$

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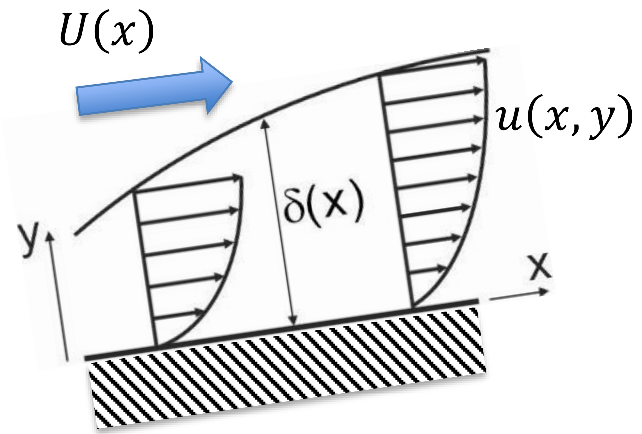
Bound lay – quasi-one-dimensional x momentum balance

This equation …

$$\rho \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} u u dy - \rho \underbrace{u\Big|_{y=\delta(x)}}_{\equiv U(x)} \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} u dy = \rho \int_{y=0}^{\delta(x)} U \frac{dU}{dx} dy - \tau_w$$

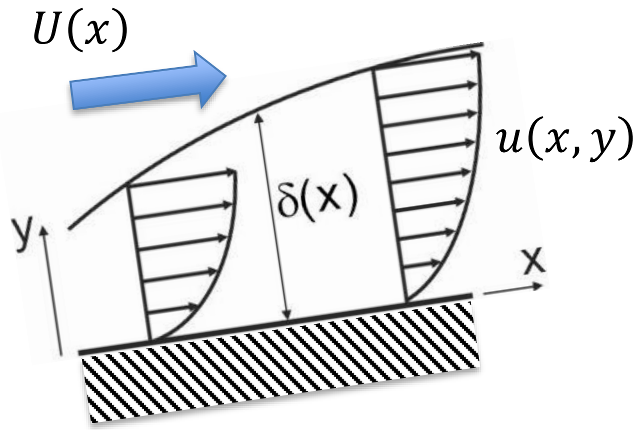


… we further develop into

$$\rho \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} u u dy - \rho \frac{\partial}{\partial x} \int_{y=0}^{\delta(x)} u U dy + \rho \frac{dU}{dx} \int_{y=0}^{\delta(x)} u dy = \rho \int_{y=0}^{\delta(x)} U \frac{dU}{dx} dy - \tau_w$$

$$\frac{\partial}{\partial x}\left[U^2 \underbrace{\int_{y=0}^{\delta(x)} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy}_{:=\delta_2} \right] + U \frac{dU}{dx} \underbrace{\int_{y=0}^{\delta(x)} \left(1 - \frac{u}{U}\right) dy}_{:=\delta_1} = \frac{\tau_w}{\rho}$$

Bound lay – quasi-one-dimensional x momentum balance



In the formulation

$$\frac{\partial}{\partial x}(U^2\delta_2) + U\frac{dU}{dx}\delta_1 = \frac{\tau_w}{\rho}$$

… this is called the integral momentum balance of the boundary layer. Displacement and momentum loss thicknesses, δ_1 and δ_2, usually defined with integration to ∞.

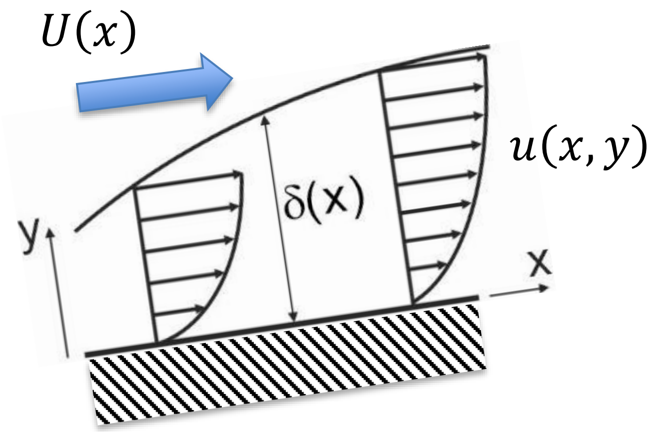
Frequently used for flow analysis with the *Pohlhausen method*:
Represent the velocity profile $u/U(x)$ by a polynomial, defining $\tilde{\eta} = y/\delta(x)$
Polynomial must satisfy boundary conditions.
Example: boundary-layer flow along a flat plate, → $U(x) = U_\infty = \text{constant}$

$$\frac{u}{U}(\tilde{\eta}) = 2\tilde{\eta} - \tilde{\eta}^2 \qquad \delta_2(x) = \frac{2}{15}\delta(x) \qquad \frac{\tau_w}{\rho} = 2\frac{\nu U_\infty}{\delta(x)} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial x}\left(U_\infty^2\delta(x)\frac{2}{15}\right) = 2\frac{\nu U_\infty}{\delta(x)}}$$

Bound lay – quasi-one-dimensional x momentum balance

This yields the boundary-layer thickness

$$\delta(x) = \sqrt{30}\sqrt{\nu x/U_\infty} \approx 5.477\sqrt{\nu x/U_\infty}$$



… which compares favorably to the exact Blasius solution based on self-similarity of the flow, which reads

$$\delta(x) = 5\sqrt{\nu x/U_\infty}$$

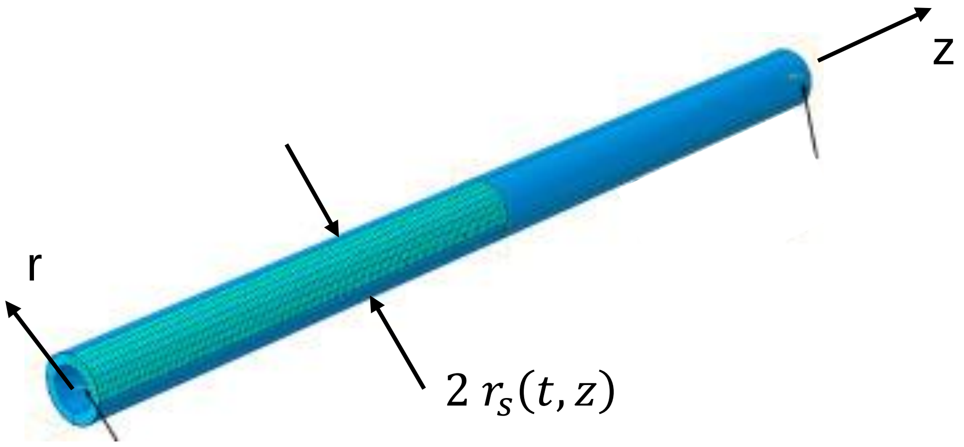
… , but with the agreement that $y = \delta(x)$ where $u/U_\infty = 0.99$.

Taking u/U_∞ closer to unity ($u/U_\infty = 0.996$) yields from the Blasius solution

$$\delta(x) = 5.4\sqrt{\nu x/U_\infty}$$ → little deviation, good agreement

Flow through a slender duct – problem statement in 3D

We assume axissymmetry, no swirl, large Froude number
Pipe wall may be flexible
Formulation in cylindrical coordinates (r, ϕ, z)



Mass balance

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

Balance of linear z momentum

$$\frac{\partial}{\partial t}(\rho u_z) + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r u_z) + \frac{\partial}{\partial z}(\rho u_z u_z) = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r \tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}$$

(balances of r and ϕ momentum become trivial)

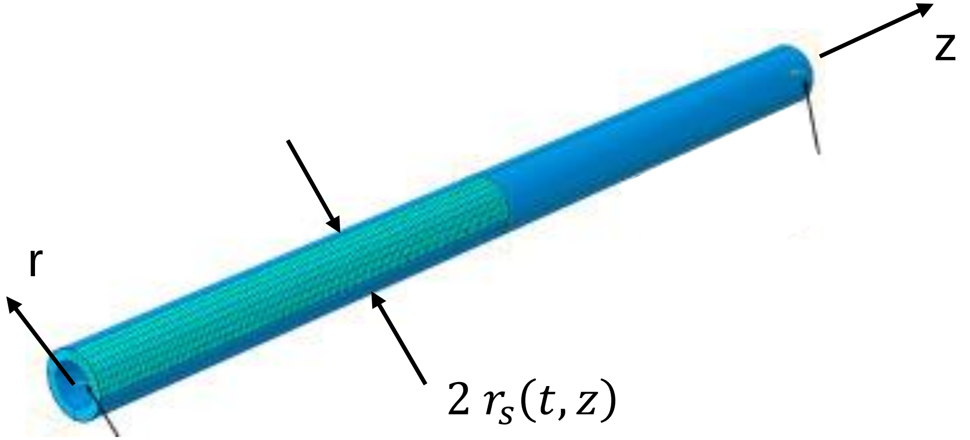
Figure: Zhao et al. Thin-walled structures (2021)

Kinematic boundary condition

This problem is solved subject to boundary conditions.

Kinematic boundary condition emerges from material duct surface

$$F(t,r,z) = r - r_s(t,z)$$



The material surface implies that

$$\frac{\partial F}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} F\right) = 0$$

so that at the surface we have

$$-\frac{\partial r_s}{\partial t} + \left[u_r \frac{\partial F}{\partial r} + u_z \frac{\partial F}{\partial z}\right]\Bigg|_{r=r_s} = -\frac{\partial r_s}{\partial t} + \left[u_r - u_z \frac{\partial r_s}{\partial z}\right]\Bigg|_{r=r_s} = 0$$

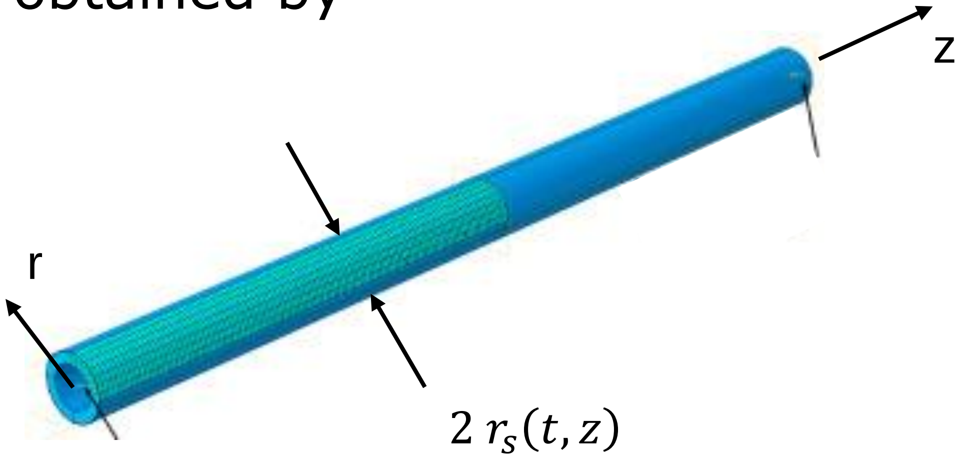
For rigid pipe walls, this is replaced by the no-slip condition

Figure: Zhao et al. Thin-walled structures (2021)

Duct flow – quasi-one-dimensional mass balance

Quasi-one-dimensional form of mass balance is obtained by integration over the duct cross section

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial}{\partial z}(\rho u_z) = 0 \quad \Big| \int dA$$



$$2\pi \int_{r=0}^{r_s(t,z)} \frac{\partial \rho}{\partial t} r dr + 2\pi \int_{r=0}^{r_s(t,z)} \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) r dr + 2\pi \int_{r=0}^{r_s(t,z)} \frac{\partial}{\partial z}(\rho u_z) r dr = 0$$

where we intentionally leave the factors 2π and the density ρ for the moment, although they are constants that may drop out

Figure: Zhao et al. Thin-walled structures (2021)

Duct flow – quasi-one-dimensional mass balance

It is the intention to exchange integration and differentiation, where appropriate. For this apply the Leibniz rule for deriving definite integrals.

$$\frac{\partial}{\partial z}\int_{r=0}^{r_s(t,z)} F(z,r)\,dr = \int_{r=0}^{r_s(t,z)} \frac{\partial F(z,r)}{\partial z}dr + \frac{\partial r_s(t,z)}{\partial z}F\big(z,r_s(z)\big) - \frac{\partial 0}{\partial z}F(z,0)$$

Applying this rule, we obtain the following form of the mass balance

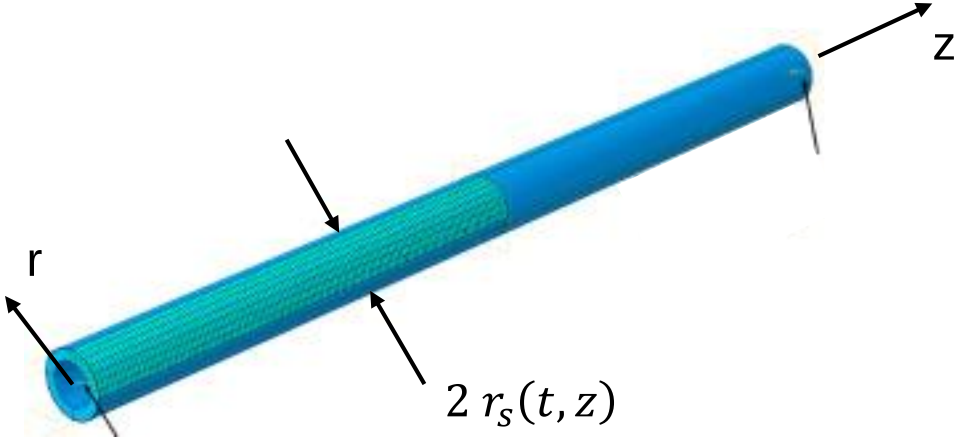
$$2\pi\frac{\partial}{\partial t}\int_{r=0}^{r_s(t,z)} \rho r dr - 2\pi\rho r_s\frac{\partial r_s(t,z)}{\partial t} + 2\pi\rho r_s\, u_r\Big|_{r=r_s(t,z)} + 2\pi\frac{\partial}{\partial z}\int_{r=0}^{r_s(t,z)} \rho u_z r dr - 2\pi\rho r_s\frac{\partial r_s(t,z)}{\partial z}u_z\Big|_{r=r_s(t,z)} = 0$$

The three terms without “integral” are the left-hand side of the kinematic boundary condition, times $2\pi\rho r_s$, so that they cancel from the equation.

Duct flow – quasi-one-dimensional mass balance

Applying this boundary condition, the following terms are left from the integrated mass balance.

$$2\pi \frac{\partial}{\partial t} \int_{r=0}^{r_s(t,z)} \rho r dr + 2\pi \frac{\partial}{\partial z} \int_{r=0}^{r_s(t,z)} \rho u_z r dr = 0$$



They may be further developed into

$$\boxed{\frac{\partial \rho A}{\partial t} + \frac{\partial \rho \bar{u}_z A}{\partial z} = 0}$$

Quasi-one-dimensional mass balance for the flow of a compressible fluid through a flexible duct, mass flux $\rho \bar{u}_z$

The equation states that a change of liquid mass flow rate $\rho \bar{u}_z A$ with the z coordinate along the duct, in case of an incompressible fluid, is possible only if the duct cross section can vary in time (e.g., elastic walls)

Duct flow – quasi-one-dimensional z momentum balance

Quasi-one-dimensional form of z momentum balance is obtained by integration over the jet cross section

$$\frac{\partial}{\partial t}(\rho u_z) + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r u_z) + \frac{\partial}{\partial z}(\rho u_z u_z) = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z} \qquad \Big| \int dA$$

$$2\pi \int_{r=0}^{r_s(t,z)} \frac{\partial}{\partial t}(\rho u_z) r dr + 2\pi \int_{r=0}^{r_s(t,z)} \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r u_z) r dr + 2\pi \int_{r=0}^{r_s(t,z)} \frac{\partial}{\partial z}(\rho u_z u_z) r dr =$$

$$= -2\pi \int_{r=0}^{r_s(t,z)} \frac{\partial p}{\partial z} r dr + 2\pi \int_{r=0}^{r_s(t,z)} \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) r dr + 2\pi \int_{r=0}^{r_s(t,z)} \frac{\partial \tau_{zz}}{\partial z} r dr$$

We keep the factor 2π, although it is a constant that eventually drops out

Duct flow – quasi-one-dimensional z momentum balance

Again, applying the Leibniz rule, we exchange differentiation and integration to obtain

$$2\pi \frac{\partial}{\partial t} \int\limits_{r=0}^{r_s(t,z)} \rho u_z r dr - 2\pi\rho\, u_z\Big|_{r=r_s} r_s \frac{\partial r_s}{\partial t} + 2\pi\rho r_s\, (u_r u_z)\Big|_{r=r_s} - 0 + 2\pi \frac{\partial}{\partial z} \int\limits_{r=0}^{r_s(t,z)} (\rho u_z u_z) r dr - 2\pi\rho r_s\, (u_z u_z)\Big|_{r=r_s} \frac{\partial r_s}{\partial z} =$$

$$= -2\pi \frac{\partial}{\partial z} \int\limits_{r=0}^{r_s(t,z)} p r dr + 2\pi r_s\, p\Big|_{r=r_s} \frac{\partial r_s}{\partial z} + 2\pi r_s\, \tau_{rz}\Big|_{r=r_s} + 2\pi \frac{\partial}{\partial z} \int\limits_{r=0}^{r_s(t,z)} \tau_{zz} r dr - 2\pi r_s\, \tau_{zz}\Big|_{r=r_s} \frac{\partial r_s}{\partial z}$$

The three terms on the left-hand side without “integral” cancel because of the kinematic boundary condition (but also due to the no-slip condition at the wall).

Duct flow – quasi-one-dimensional z momentum balance

We develop this equation further into

$$\frac{\partial \rho \bar{u}_z A}{\partial t} + \frac{\partial \rho \bar{u}_z^2 A}{\partial z} = -\frac{\partial \bar{p} A}{\partial z} + \frac{\partial \bar{\tau}_{zz} A}{\partial z} - 2\pi r_s \left[\left(-p\Big|_{r=r_s} + \tau_{zz}\Big|_{r=r_s} \right) \frac{\partial r_s}{\partial z} - \tau_{rz}\Big|_{r=r_s} \right]$$

Since $\tau_{zz} = 0$ at the wall, the right-hand side simplifies to the form

$$\boxed{\frac{\partial \rho \bar{u}_z A}{\partial t} + \frac{\partial \rho \bar{u}_z^2 A}{\partial z} = -\frac{\partial \bar{p} A}{\partial z} + \frac{\partial \bar{\tau}_{zz} A}{\partial z} + 2\pi r_s \, p\Big|_{r=r_s} \frac{\partial r_s}{\partial z} + 2\pi r_s \, \tau_{rz}\Big|_{r=r_s}}$$

Model for viscous stresses: Newtonian

→ Relationship between stress and rate of deformation

$$\bar{\tau}_{zz} = 2\mu \frac{\partial \bar{u}_z}{\partial z} \qquad \tau_{rz}\Big|_{r=r_s} \equiv \tau_w = \frac{\lambda(Re, k_s/r_s)}{8} \rho \bar{u}_z^2$$

Assuming the cross section $\pi r_s^2(z)$ to be known, a system of two quasi-one-dimensional equations for the unknowns $\bar{u}_z(z,t)$ and $\bar{p}(z,t)$ results

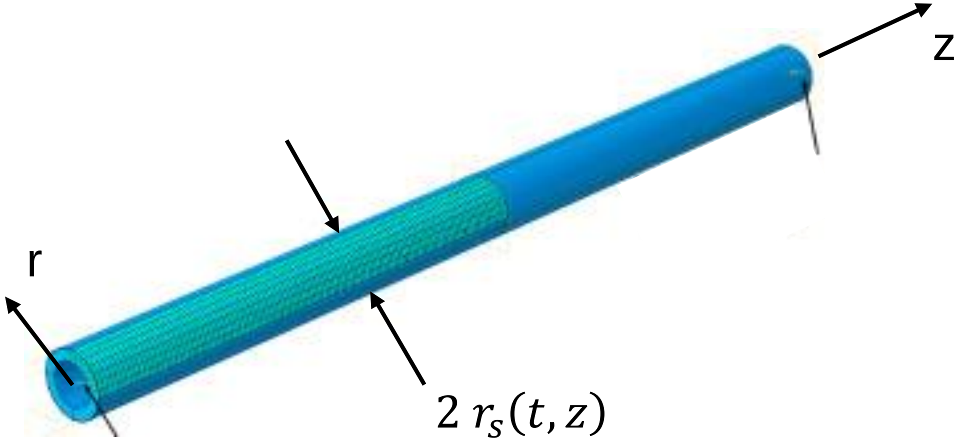
Special case of quasi-one-dimensional duct flow

The quasi-one-dimensional equations

Continuity equation

$$\boxed{\frac{\partial \rho A}{\partial t} + \frac{\partial \rho \bar{u}_z A}{\partial z} = 0}$$

and



Balance of the z momentum

$$\boxed{\frac{\partial \rho \bar{u}_z A}{\partial t} + \frac{\partial \rho \bar{u}_z^2 A}{\partial z} = -\frac{\partial \bar{p} A}{\partial z} + \frac{\partial \bar{\tau}_{zz} A}{\partial z} + 2\pi r_s\, p\Big|_{r=r_s} \frac{\partial r_s}{\partial z} + 2\pi r_s\, \tau_{rz}\Big|_{r=r_s}}$$

Two equations in the two unknowns $\bar{p}(z,t)$ and $\bar{u}_z(z,t)$, i.e. pressure and cross-sectional mean of the z velocity.

For variable r_s due to wall elasticity, additional material law needed

For steady, incompressible flow through a cylindrical duct (pipe), eqs. reduce to

$\boxed{\frac{\partial \bar{u}_z}{\partial z} = 0}$ and $\boxed{0 = -\frac{\partial \bar{p}}{\partial z} + \frac{2}{r_s} \tau_{rz}\Big|_{r=r_s}}$, the latter better known as $\boxed{\tau_{rz}\Big|_{r=r_s} \equiv \tau_w = \frac{\partial \bar{p}}{\partial z} \frac{r_s}{2}}$

Liquid jet in vacuum – problem statement in 3D

Flow is governed by the equations of motion
We assume axissymmetry, no swirl, large Froude number
Formulation in cylindrical coordinates (r, ϕ, z), surface shape $r_s(t, z)$

Mass balance

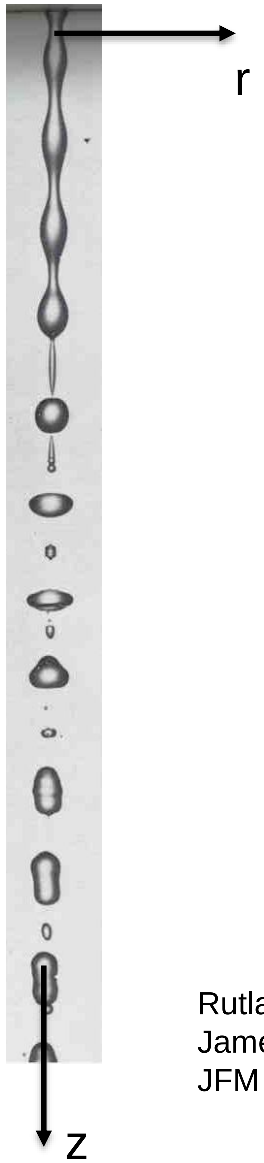
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

Balance of linear r momentum

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{\tau_{\phi\phi}}{r} + \frac{\partial \tau_{rz}}{\partial z}$$

Balance of linear z momentum

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}$$

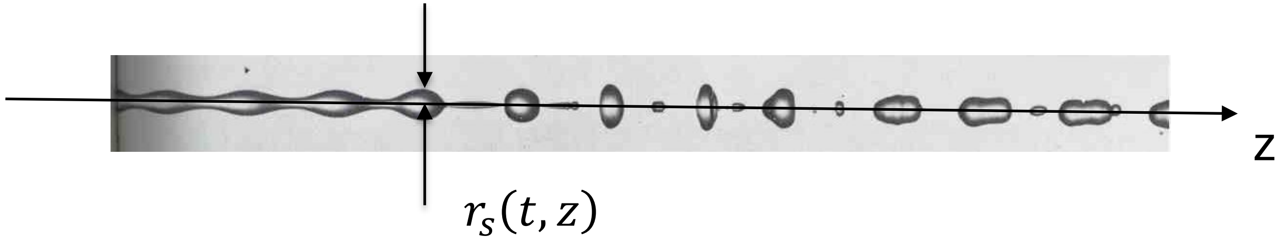


Rutland & Jameson JFM 1971

Boundary conditions for the jet – kinematic

Kinematic boundary condition emerges from material jet surface

$$F(t,r,z) = r - r_s(t,z)$$



The material surface implies that

$$\frac{\partial F}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} F\right) = 0$$

so that at the surface we have

$$-\frac{\partial r_s}{\partial t} + \left[u_r \frac{\partial F}{\partial r} + u_z \frac{\partial F}{\partial z}\right]\Bigg|_{r=r_s} = -\frac{\partial r_s}{\partial t} + \left[u_r - u_z \frac{\partial r_s}{\partial z}\right]\Bigg|_{r=r_s} = 0$$

Boundary conditions for the jet – dynamic

Dynamic boundary conditions for the jet surface

- No shear stress on the jet surface, since the jet moves in a vacuum

$(\tau \cdot \vec{n}) \times \vec{n} = \vec{0}$ This reduces to $\tau_{rz}\left[1-\left(\frac{\partial r_s}{\partial z}\right)^2\right] = (\tau_{zz}-\tau_{rr})\frac{\partial r_s}{\partial z}$ at $r = r_s$

- Normal stress on the jet surface "inside" differs from "outside" by capillary stress; stress outside neglected (vacuum)

$-p + (\tau \cdot \vec{n}) \cdot \vec{n} + \sigma(\vec{\nabla} \cdot \vec{n}) = 0$ This becomes $-p + \frac{\tau_{rr} - 2\tau_{rz}\,\partial r_s/\partial z + \tau_{zz}(\partial r_s/\partial z)^2}{1+(\partial r_s/\partial z)^2} + \sigma(\vec{\nabla} \cdot \vec{n}) = 0$

(viscous stress tensor τ, outward pointing normal-unit vector $\vec{n}$, at $r = r_s$
pressure p, surface tension σ, surface curvature $(\vec{\nabla} \cdot \vec{n})$)

where $(\vec{\nabla} \cdot \vec{n}) = \frac{1}{r_s}\frac{1}{\sqrt{1+(\partial r_s/\partial z)^2}} - \frac{\partial^2 r_s/\partial z^2}{\sqrt{1+(\partial r_s/\partial z)^2}}$, since $\vec{n} = \frac{1}{|\vec{\nabla} F|}\vec{\nabla} F$ with F from slide 23

Expansion in power series for radial coordinate

Idea: Jet is slender, dimensions in radial direction much smaller than in axial direction. Therefore expand velocities and pressure in a power series with respect to the radial coordinate.

For symmetry reasons we set

$$u_z(t,r,z) = u_0(t,z) + u_2(t,z)\ r^2 + \cdots$$

From continuity equation it follows that

$$u_r(t,r,z) = -\frac{1}{2}\frac{\partial u_0(t,z)}{\partial z}\ r - \frac{1}{4}\frac{\partial u_2(t,z)}{\partial z} r^3 - \cdots$$

Furthermore

$$p(t,r,z) = p_0(t,z) + p_2(t,z)\ r^2 + \cdots$$

Eggers & Dupont, 1994

Quasi-one-dimensional equations

Introducing expansions into r momentum equation, the equation is identically satisfied at lowest order in r.

Introduce expansions into z momentum equation and boundary conditions, z momentum balance becomes at lowest order in r

$$\rho\left(\frac{\partial u_0}{\partial t}+u_0\frac{\partial u_0}{\partial z}\right)=-\frac{\partial p_0}{\partial z}+\mu\left(4u_2+\frac{\partial^2 u_0}{\partial z^2}\right)\qquad (*)$$

Boundary condition for normal stress becomes at lowest order in r (since $\partial r_s/\partial z$ is $O(r)$)

$$-p_0-\mu\frac{\partial u_0}{\partial z}+\sigma\left(\vec{\nabla}\cdot\vec{n}\right)=0$$

and boundary condition for shear stress becomes at lowest order in r

$$-\frac{\partial u_0}{\partial z}\frac{\partial r_s}{\partial z}+2u_2 r_s-\frac{1}{2}\frac{\partial^2 u_0}{\partial z^2}r_s-2\frac{\partial u_0}{\partial z}\frac{\partial r_s}{\partial z}=0$$

Eggers & Dupont, 1994

Quasi-one-dimensional equation

Eliminate p_0 and u_2 from z momentum balance ($*$) by boundary conditions to obtain

$$\boxed{\rho r_s^2 \left(\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial z} \right) = 3\mu \frac{\partial}{\partial z} \left(\frac{\partial u_0}{\partial z} r_s^2 \right) - \sigma r_s^2 \frac{\partial}{\partial z} (\vec{\nabla} \cdot \vec{n})}$$

The jet surface curvature was given on slide 24 as

$$(\vec{\nabla} \cdot \vec{n}) = \frac{1}{r_s} \frac{1}{\sqrt{1 + (\partial r_s/\partial z)^2}} - \frac{\partial^2 r_s/\partial z^2}{\sqrt{1 + (\partial r_s/\partial z)^2}}$$

Furthermore, the kinematic boundary condition reads to lowest order in r

$$\boxed{\frac{\partial r_s}{\partial t} + u_0 \frac{\partial r_s}{\partial z} + \frac{1}{2} \frac{\partial u_0}{\partial z} r_s = 0}$$

Eggers & Dupont, 1994

These are two equations in the two unknowns $u_0(z,t)$ and $r_s(z,t)$

Perturbation approach for stability analysis

Quasi-one-dimensional equations re-written using a perturbation formulation of r_s and u_0, assuming temporal instability as per, e.g.,

$r_s(z,t) = r_{s0}\left(1 + \epsilon A_0 e^{-\alpha t + i\tilde{k}z}\right)$ (real wave number $\tilde{k} = 2\pi/\lambda$, complex rate factor α)

yield the dispersion relation

$$\alpha^2 - \frac{3\mu}{\rho r_{s0}^2} k^2 \alpha - \frac{\sigma}{2\rho r_{s0}^3} k^2 (1 - k^2) = 0$$

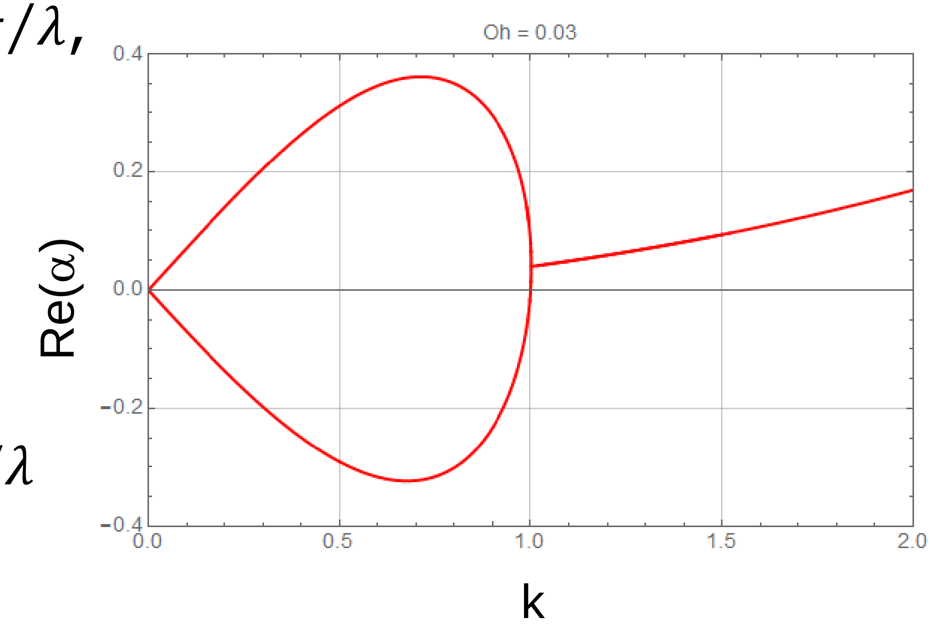
Yarin 1993

for the jet, representing a relationship $\alpha = f(k)$ between growth rate and wave number $\mathrm{k} = 2\pi r_{s0}/\lambda$ of the disturbance (wavelength λ).

Wavenumber at maximum disturbance growth rate inviscid

– from exact analysis (Rayleigh 1878): at $k_{opt} = 0.691$

– from quasi-one-dimensional analysis: at $k_{opt} = 0.707$ ($\Delta = 2.3\%$)



Predicted real parts of the rate factor α for $Oh \equiv \mu/(\sigma r_{s0}\rho)^{1/2} = 0.03$

Goedde & Yuen, 1970

González & García, 2009

Summary

- Slender flow fields (in boundary layers, slender ducts, thin films, free jets) may be analysed solving quasi-one-dimensional equations of motion
- Quasi-one-dimensional equations may be derived by
  - integration over flow cross section, or by
  - series expansion w.r.t. coordinate transverse to the flow direction
- In boundary-layer flow, boundary layer thickness may be well represented by the quasi-one-dimensional approach
- In duct flow, the quasi-one-dimensional equations represent balances of cross-sectional mean variables and rates of throughput through the cross-sections
- In free-surface flow (free jets and films on substrates), the equations represent remarkably well the transport processes (not discussed here) and flow stability properties
- The concept of quasi-one-dimensional analysis can be applied to the energy equation also – which was not presented here for time constraints

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