Introduction to Weakly Nonlinear Analysis

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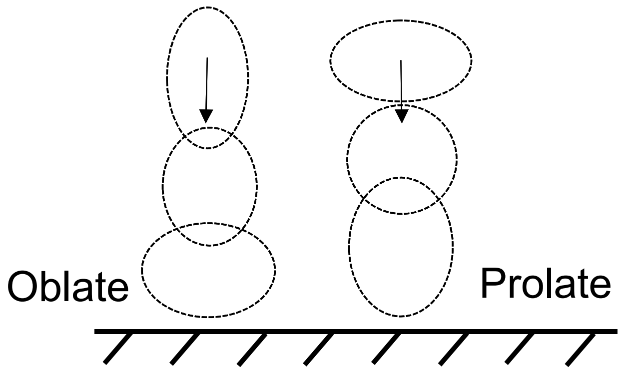
Case study – nonlinear drop shape oscillations

Upon formation and after collisions, drops are not spherical;
elastic systems → shape oscillations

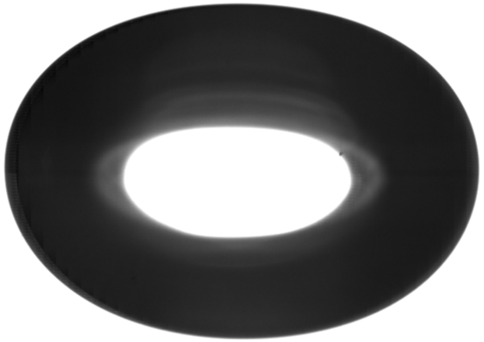
1. Spreading upon drop impact depends on drop shape



Applications
Ink-jet printing,
spray coating,
in-air microfluidics

2. Drop evaporation influenced by shape oscillations

$$\rho c_v \left(\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r} \frac{\partial T}{\partial \theta} \right) = \vec{\nabla} \cdot \left(k \vec{\nabla} T \right) + \Phi + \dot{q}_Q$$



Applications
Spray drying, fuel injection,
container-less materials processing
based on individual drops

Problem statement

Flow due to shape oscillations in a Newtonian, incompressible droplet in a vacuum
→ equations of motion

Mass balance – (also called the continuity equation)
for incompressible fluid requires solenoidal velocity field

$$\operatorname{div} \vec{u} = 0$$

Balance of linear momentum

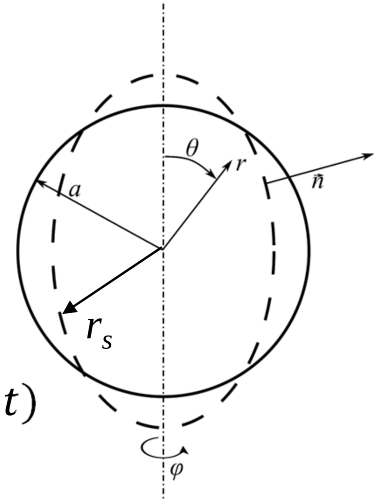
$$\rho \frac{d\vec{u}}{dt} = \nabla \cdot \Pi$$

where d/dt is the material derivative, and the Cauchy stress tensor

$$\Pi = -p\, I + \tau$$

(pressure p, unit tensor I and extra-stress tensor τ)

Problem statement – equations of motion



Flow in a Newtonian, incompressible droplet in a vacuum, equations of motion in spherical coordinates, axisymmetric, non-dimensionalised with a, $(\rho a^3/\sigma)^{1/2}$, $(\sigma/\rho a)^{1/2}$ and σ/a

$$r_s(\theta,t) = 1 + \eta(\theta,t)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta \sin\theta) = 0$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_r}{\partial\theta}\sin\theta\right)\right]$$

$$\frac{\partial u_\theta}{\partial t} + \frac{u_r}{r}\frac{\partial r u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} = -\frac{1}{r}\frac{\partial p}{\partial\theta} + Oh\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\right) + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right]$$

with the Ohnesorge number $Oh = \mu/(\sigma a\rho)^{1/2}$

$Oh \to 0$ inviscid case
Tsamopoulos & Brown (1983)
Zrnić & Brenn (2021)

Problem statement – boundary and initial conditions

Boundary conditions to be satisfied at the deformed surface $r = r_s(\theta, t) = 1 + \eta(\theta, t)$

Kinematic – rate of radial displacement $u_r = \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + \frac{u_\theta}{r}\frac{\partial\eta}{\partial\theta}$

Dynamic – zero shear stress $(\vec{n} \cdot \underline{\tau}) \times \vec{n} = \vec{0}$

Dynamic – zero normal stress $-p + Oh(\vec{n} \cdot \underline{\tau}) \cdot \vec{n} + (\vec{\nabla} \cdot \vec{n}) = 0$

Initial conditions

Deformed shape $r_s(\theta, 0) = 1 + \eta(\theta, 0)$

Zero rate of deformation change $\partial\eta/\partial t\,(\theta, 0) = 0$

Approach for solving the nonlinear problem

Solution by **Weakly Nonlinear Analysis**

Approach: expansion of the unknowns for a small parameter η_0

Example: the radial velocity component

$$u_r(r,\theta,t) = u_{r1}(r,\theta,t)\,\eta_0 + u_{r2}(r,\theta,t)\,\eta_0^2 + u_{r3}(r,\theta,t)\,\eta_0^3 + \cdots$$

Functions $u_{r1}(r,\theta,t)$, $u_{r2}(r,\theta,t)$, and $u_{r3}(r,\theta,t)$
become the new unknowns

To be applied to all field variables, including the drop surface shape

Approach for solving the nonlinear problem

Substituting the series expansions up to 3rd order,
e.g. the radial momentum equation

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} + Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial \theta}\sin\theta\right)\right]$$

becomes

$$\frac{\partial}{\partial t}(u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3) + (u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3)\frac{\partial}{\partial r}(u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3)$$

$$+(u_{\theta 1}\,\eta_0 + u_{\theta 2}\,\eta_0^2 + u_{\theta 3}\,\eta_0^3)\frac{1}{r}\frac{\partial}{\partial \theta}(u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3) - \frac{1}{r}(u_{\theta 1}\,\eta_0 + u_{\theta 2}\,\eta_0^2 + u_{\theta 3}\,\eta_0^3)^2$$

$$= -\frac{\partial}{\partial r}(p_1\,\eta_0 + p_2\,\eta_0^2 + p_3\,\eta_0^3) + Oh\left\{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}[r^2(u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3)]\right.$$

$$\left. + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial \theta}(u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3)\sin\theta\right]\right\}$$

Radial momentum equations up to 3rd order

Collecting the terms with equal powers of the expansion parameter η_0 yields

1st order

$$\frac{\partial u_{r1}}{\partial t}\eta_0 = -\frac{\partial p_1}{\partial r}\eta_0 + Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_{r1}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_{r1}}{\partial\theta}\sin\theta\right)\right]\eta_0$$

2nd order

$$\frac{\partial u_{r2}}{\partial t}\eta_0^2 + \left(u_{r1}\frac{\partial u_{r1}}{\partial r} + u_{\theta 1}\frac{1}{r}\frac{\partial u_{r1}}{\partial\theta} - \frac{u_{\theta 1}^2}{r}\right)\eta_0^2$$
$$= -\frac{\partial p_2}{\partial r}\eta_0^2 + Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_{r2}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_{r2}}{\partial\theta}\sin\theta\right)\right]\eta_0^2$$

3rd order

$$\frac{\partial u_{r3}}{\partial t}\eta_0^3 + \left(u_{r1}\frac{\partial u_{r2}}{\partial r} + u_{r2}\frac{\partial u_{r1}}{\partial r} + u_{\theta 1}\frac{1}{r}\frac{\partial u_{r2}}{\partial\theta} + u_{\theta 2}\frac{1}{r}\frac{\partial u_{r1}}{\partial\theta} - \frac{2u_{\theta 1}u_{\theta 2}}{r}\right)\eta_0^3$$
$$= -\frac{\partial p_3}{\partial r}\eta_0^3 + Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_{r3}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_{r3}}{\partial\theta}\sin\theta\right)\right]\eta_0^3$$

Boundary conditions – two steps needed

First step – substitute series expansions into the boundary conditions, e.g. kinematic

$$u_r = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \frac{u_\theta}{r}\frac{\partial \eta}{\partial \theta} \quad \text{at } r = r_s(\theta, t) = 1 + \eta(\theta, t) \quad \text{, where } \eta(\theta, t) = \eta_1 \eta_0 + \eta_2 \eta_0^2 + \eta_3 \eta_0^3 + \cdots$$

becomes (expansions up to 3rd order)

$$u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3 = \frac{\partial}{\partial t}(\eta_1\,\eta_0 + \eta_2\,\eta_0^2 + \eta_3\,\eta_0^3) + (u_{\theta 1}\,\eta_0 + u_{\theta 2}\,\eta_0^2 + u_{\theta 3}\,\eta_0^3)\frac{1}{r}\frac{\partial}{\partial \theta}(\eta_1\,\eta_0 + \eta_2\,\eta_0^2 + \eta_3\,\eta_0^3)$$

$$\text{at } r = 1 + \eta(\theta, t)$$

Second step – represent functions at the deformed surface by their values at the undeformed surface → Taylor expansion – e.g. for radial velocity component

$$u_r|_{r=1+\eta} = u_r|_{r=1} + \frac{1}{1!}\frac{\partial u_r}{\partial r}|_{r=1}\,\eta + \frac{1}{2!}\frac{\partial^2 u_r}{\partial r^2}|_{r=1}\,\eta^2 + \cdots$$

Boundary conditions – second step

Introducing these expansions into, e.g., the kinematic boundary condition

$$u_{r1}\,\eta_0 + u_{r2}\,\eta_0^2 + u_{r3}\,\eta_0^3 = \frac{\partial}{\partial t}(\eta_1\,\eta_0 + \eta_2\,\eta_0^2 + \eta_3\,\eta_0^3) + (u_{\theta 1}\,\eta_0 + u_{\theta 2}\,\eta_0^2 + u_{\theta 3}\,\eta_0^3)\frac{1}{r}\frac{\partial}{\partial \theta}(\eta_1\,\eta_0 + \eta_2\,\eta_0^2 + \eta_3\,\eta_0^3)$$

at $r = 1 + \eta(\theta, t)$ yields

$$\left[u_{r1} + \frac{\partial u_{r1}}{\partial r}(\eta_1\eta_0 + \eta_2\eta_0^2) + \frac{1}{2}\frac{\partial^2 u_{r1}}{\partial r^2}(\eta_1\eta_0)^2\right]\eta_0 + \left[u_{r2} + \frac{\partial u_{r2}}{\partial r}(\eta_1\eta_0 + \eta_2\eta_0^2)\right]\eta_0^2 + \left[u_{r3} + \frac{\partial u_{r3}}{\partial r}(\eta_1\eta_0 + \eta_2\eta_0^2)\right]\eta_0^3 =$$

$$\frac{\partial \eta_1}{\partial t}\eta_0 + \frac{\partial \eta_2}{\partial t}\eta_0^2 + \frac{\partial \eta_3}{\partial t}\eta_0^3 + \left[\left(\frac{u_{\theta 1}}{r} + \frac{\partial}{\partial r}\left(\frac{u_{\theta 1}}{r}\right)(\eta_1\eta_0 + \eta_2\eta_0^2)\right)\eta_0 + \left(\frac{u_{\theta 2}}{r} + \frac{\partial}{\partial r}\left(\frac{u_{\theta 2}}{r}\right)(\eta_1\eta_0 + \eta_2\eta_0^2)\right)\eta_0^2 +\right.$$

$$\left.\left(\frac{u_{\theta 3}}{r} + \frac{\partial}{\partial r}\left(\frac{u_{\theta 3}}{r}\right)(\eta_1\eta_0 + \eta_2\eta_0^2)\right)\eta_0^3\right]\left(\frac{\partial \eta_1}{\partial \theta}\eta_0 + \frac{\partial \eta_2}{\partial \theta}\eta_0^2 + \frac{\partial \eta_3}{\partial \theta}\eta_0^3\right) \qquad \text{at } r = 1$$

Kinematic boundary conditions up to 3rd order

Collecting the terms with equal powers of η_0 yields (e.g. kinematic)

1st order $$u_{r1}\eta_0 = \frac{\partial \eta_1}{\partial t}\eta_0 \qquad \text{at } r = 1$$

2nd order $$\left(u_{r2} + \frac{\partial u_{r1}}{\partial r}\eta_1\right)\eta_0^2 = \left(\frac{\partial \eta_2}{\partial t} + u_{\theta 1}\frac{1}{r}\frac{\partial \eta_1}{\partial \theta}\right)\eta_0^2 \qquad \text{at } r = 1$$

3rd order $$\left(u_{r3} + \frac{\partial u_{r1}}{\partial r}\eta_2 + \frac{\partial u_{r2}}{\partial r}\eta_1 + \frac{1}{2}\frac{\partial^2 u_{r1}}{\partial r^2}\eta_1^2\right)\eta_0^3 =$$
$$\left(\frac{\partial \eta_3}{\partial t} + u_{\theta 1}\frac{1}{r}\frac{\partial \eta_2}{\partial \theta} + u_{\theta 2}\frac{1}{r}\frac{\partial \eta_1}{\partial \theta} + \frac{\partial}{\partial r}\left(\frac{u_{\theta 1}}{r}\right)\eta_1\frac{\partial \eta_1}{\partial \theta}\right)\eta_0^3 \qquad \text{at } r = 1$$

Initial conditions

First initial condition is derived from the requirement that the drop volume must be conserved despite the deformation. For this purpose, we write

$$r_s(\theta,0) = \tilde{R} + \eta_0 P_m(\cos\theta)$$

Formulating the non-dimensional drop volume as

$$\frac{4\pi}{3} = -\frac{2\pi}{3} \int_{\cos\theta=1}^{-1} r_s^3(\cos\theta\,,0)\,d\cos\theta$$

Reveals the form of the initial surface shape as

$$r_s(\theta,0) = 1 + \eta_0 P_m(\cos\theta) - \eta_0^2 \frac{1}{2m+1} - \frac{\eta_0^3}{6} \int_{-1}^{1} P_m(\cos\theta)^3 d\cos\theta \mp \cdots$$

from where we read the surface shape contributions keeping the drop volume

Further condition: $\frac{\partial\eta}{\partial t}(\theta,0) = 0 \;\rightarrow\; \frac{\partial\eta_1}{\partial t}(\theta,0)\,\eta_0 + \frac{\partial\eta_2}{\partial t}(\theta,0)\,\eta_0^2 + \frac{\partial\eta_3}{\partial t}(\theta,0)\,\eta_0^3 = 0$

Equations of motion – first order

Continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_{r1}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_{\theta 1}\sin\theta\right) = 0$$

Momentum equation – radial

$$\frac{\partial u_{r1}}{\partial t} - Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 u_{r1}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_{r1}}{\partial\theta}\sin\theta\right)\right] + \frac{\partial p_1}{\partial r} = 0$$

Momentum equation – polar

$$\frac{\partial u_{\theta 1}}{\partial t} - Oh\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_{\theta 1}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(u_{\theta 1}\sin\theta\right)\right) + \frac{2}{r^2}\frac{\partial u_{r1}}{\partial\theta}\right] + \frac{1}{r}\frac{\partial p_1}{\partial\theta} = 0$$

Boundary and initial conditions – first order

Kinematic boundary condition $u_{r1} = \frac{\partial \eta_1}{\partial t}$ at r = 1

Dynamic boundary condition – zero shear stress

$$r\frac{\partial}{\partial r}\left(\frac{u_{\theta 1}}{r}\right) + \frac{1}{r}\frac{\partial u_{r1}}{\partial \theta} = 0 \qquad \text{at } r = 1$$

Dynamic boundary condition – zero normal stress

$$-p_1 + 2Oh\frac{\partial u_{r1}}{\partial r} - \left(2\eta_1 + \frac{\partial \eta_1}{\partial \theta}\cot\theta + \frac{\partial^2 \eta_1}{\partial \theta^2}\right) = 0 \qquad \text{at } r = 1$$

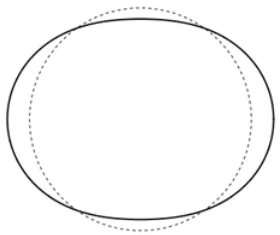
Initial conditions

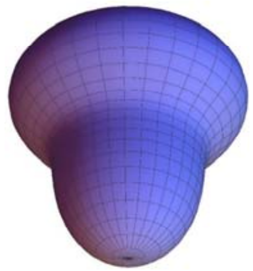
$$\eta_1(\theta, 0) = P_m(\cos\theta)$$

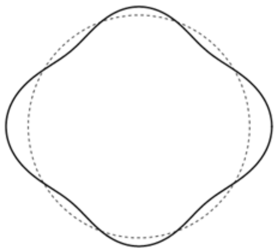
Shape according to Legendre polynomial

$$\frac{\partial \eta_1}{\partial t}(\theta, 0) = 0$$

Zero rate of shape change


m = 2


m = 3


m = 4

First-order solutions – by Stokes stream function

The two-dimensional velocity field represented by Stokes stream function ψ
We define the first-order velocity components as the derivatives

$$u_{r1} = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \qquad u_{\theta 1} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

This velocity field is solenoidal, i.e., it satisfies the continuity equation for the incompressible fluid.
Substitute this into the 1st-order vectorial momentum equation and take its curl

→ $\left(\frac{1}{Oh}\frac{\partial}{\partial t} - E^2\right)(E^2\psi) = 0$ where $E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$

→ Solution $\psi = \psi_1 + \psi_2 = C_{1m} r^{m+1} \sin^2\theta P'_m(\cos\theta)\exp(-\alpha_m t)$

$$+C_{2m} qr\, j_m(qr) \sin^2\theta P'_m(\cos\theta)\exp(-\alpha_m t) \qquad q = \sqrt{\frac{\alpha_m}{Oh}}$$

First-order solutions – velocity and pressure fields

We seek first-order surface deformation in the form $\eta_1(\theta,t) = \hat{\eta}_1 P_m(\cos\theta) e^{-\alpha_m t}$

Velocity and pressure are found as

$$u_{r1} = -\left[C_{1m} r^{m-1} + C_{2m} q^2 \frac{j_m(qr)}{qr}\right] m(m+1) P_m(\cos\theta)\, e^{-\alpha_m t} \quad \text{, where } q = \sqrt{\frac{\alpha_m}{Oh}}$$

$$u_{\theta 1} = \left[C_{1m}(m+1) r^{m-1} + C_{2m} q^2 \left((m+1)\frac{j_m(qr)}{qr} - j_{m+1}(qr)\right)\right] \sin\theta\, P_m'(\cos\theta)\, e^{-\alpha_m t}$$

$$p_1 = -C_{1m}(m+1)\alpha_m r^m P_m e^{-\alpha_m t}$$

C_{1m} and C_{2m} determined by first-order kinematic and zero shear stress BCs

$$C_{1m} = \frac{\hat{\eta}_1 \alpha_m}{m(m+1)}\left[1 + \frac{2(m^2-1)}{2q j_{m+1}(q)/j_m(q) - q^2}\right]$$

$$C_{2m} = -\frac{2(m-1)\hat{\eta}_1 \alpha_m}{mq[2q j_{m+1}(q) - q^2 j_m(q)]}$$

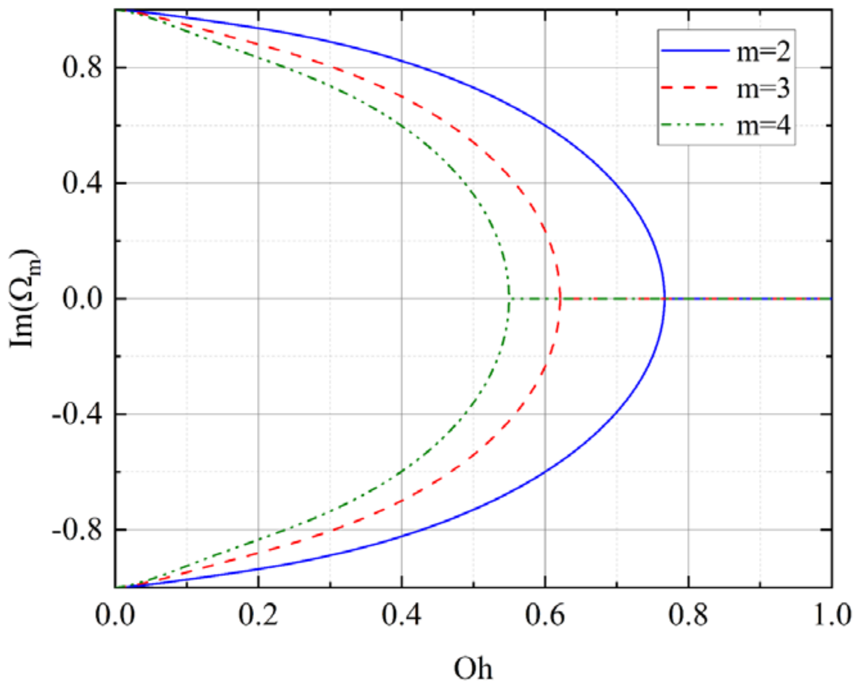
First-order solutions – characteristic equation

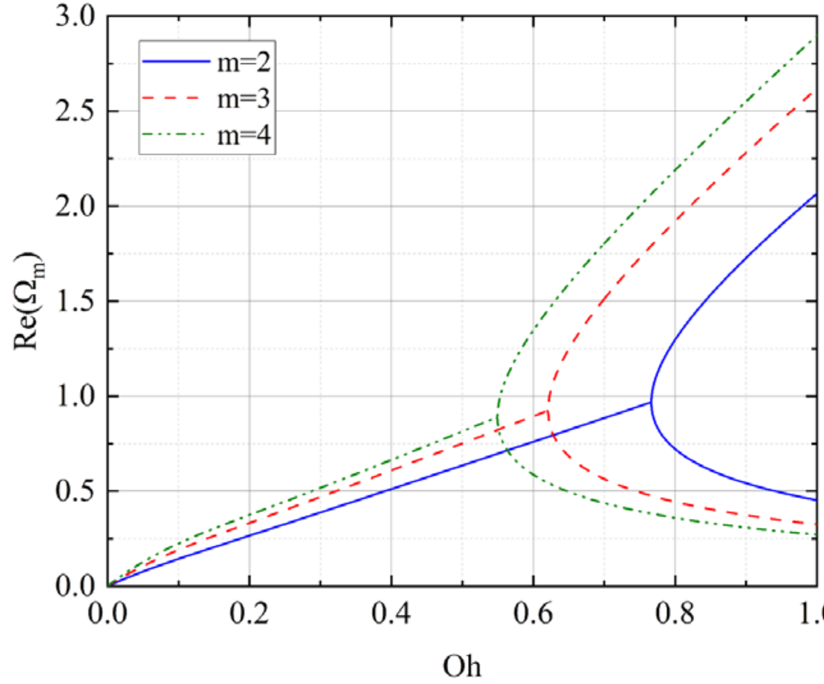
Zero normal stress BC yields characteristic equation of the drop $\left(q = \sqrt{\frac{\alpha_m}{Oh}}\right)$ $\left(Oh = \frac{\mu}{\sqrt{\sigma a \rho}}\right)$

$$\frac{\alpha_{m,0}^2}{\alpha_m^2} = \frac{2(m^2-1)}{q^2 - 2q\, j_{m+1}/j_m} - 1 + \frac{2m(m-1)}{q^2}\left[1 + \frac{2(m+1)j_{m+1}/j_m}{2j_{m+1}/j_m - q}\right]$$

, where $\alpha_{m,0} = \left[m(m-1)(m+2)\frac{\sigma}{\rho a^3}\right]^{1/2}$

$$\Omega_m = \frac{\alpha_m}{\alpha_{m,0}}$$





→ Complex conjugate solutions, therefore two different time behaviours

First-order solutions – damped drop-shape oscillation

Drop surface shape

$$r_s(\theta,t) = 1 + \eta_0\eta_1(\theta,t)$$

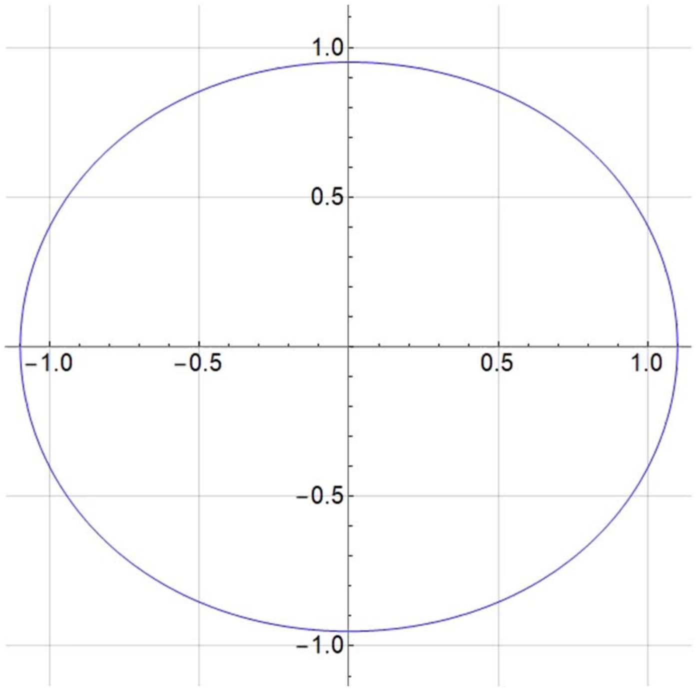
as a function of time for
η_0 = 0.1, m = 2, Oh = 0.02

where

$$\eta_1(\theta,t) = \left(\hat{\eta}_1^{(p)} e^{-\alpha_m^{(p)} t} + \hat{\eta}_1^{(n)} e^{-\alpha_m^{(n)} t}\right) P_m(\cos\theta)$$

and

$$\hat{\eta}_1^{(p)} = -\frac{\alpha_m^{(n)}}{\alpha_m^{(p)} - \alpha_m^{(n)}}, \qquad \hat{\eta}_1^{(n)} = \frac{\alpha_m^{(p)}}{\alpha_m^{(p)} - \alpha_m^{(n)}}$$



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Equations of motion – second order

Continuity equation $$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_{r2}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_{\theta 2}\sin\theta) = 0$$

Momentum equation – radial

$$\frac{\partial u_{r2}}{\partial t} - Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_{r2}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_{r2}}{\partial\theta}\sin\theta\right)\right] + \frac{\partial p_2}{\partial r} =$$

$$-u_{r1}\frac{\partial u_{r1}}{\partial r} - u_{\theta 1}\frac{1}{r}\frac{\partial u_{r1}}{\partial\theta} + \frac{u_{\theta 1}^2}{r}$$

Momentum equation – polar

$$\frac{\partial u_{\theta 2}}{\partial t} - Oh\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_{\theta 2}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_{\theta 2}\sin\theta)\right) + \frac{2}{r^2}\frac{\partial u_{r2}}{\partial\theta}\right] + \frac{1}{r}\frac{\partial p_2}{\partial\theta} =$$

$$-u_{r1}\frac{\partial u_{\theta 1}}{\partial r} - u_{\theta 1}\frac{1}{r}\frac{\partial u_{\theta 1}}{\partial\theta} - \frac{u_{r1}u_{\theta 1}}{r}$$

Boundary and initial conditions – second order

Kinematic boundary condition $$u_{r2} - \frac{\partial \eta_2}{\partial t} = u_{\theta 1} \frac{1}{r} \frac{\partial \eta_1}{\partial \theta} - \frac{\partial u_{r1}}{\partial r} \eta_1$$ at r = 1

Dynamic boundary condition – zero shear stress

$$r \frac{\partial}{\partial r} \left(\frac{u_{\theta 2}}{r} \right) + \frac{1}{r} \frac{\partial u_{r2}}{\partial \theta} = -\eta_1 \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \left(\frac{u_{\theta 1}}{r} \right) + \frac{1}{r} \frac{\partial u_{r1}}{\partial \theta} \right) - 2 \left(r \frac{\partial}{\partial r} \left(\frac{u_{r1}}{r} \right) - \frac{1}{r} \frac{\partial u_{\theta 1}}{\partial \theta} \right) \frac{1}{r} \frac{\partial \eta_1}{\partial \theta}$$ at r = 1

Dynamic boundary condition – zero normal stress

$$-p_2 + 2Oh \frac{\partial u_{r2}}{\partial r} - \left(2\eta_2 + \frac{\partial \eta_2}{\partial \theta} \cot\theta + \frac{\partial^2 \eta_2}{\partial \theta^2} \right) = \eta_1 \frac{\partial p_1}{\partial r} - 2Oh \left[\eta_1 \frac{\partial^2 u_{r1}}{\partial r^2} - \frac{1}{r} \frac{\partial \eta_1}{\partial \theta} \left(r \frac{\partial}{\partial r} \left(\frac{u_{\theta 1}}{r} \right) + \frac{1}{r} \frac{\partial u_{r1}}{\partial \theta} \right) \right]$$

$$- \left(2\eta_1^2 + 2\eta_1 \frac{\partial \eta_1}{\partial \theta} \cot\theta + 2\eta_1 \frac{\partial^2 \eta_1}{\partial \theta^2} \right)$$ at r = 1

Initial conditions $$\eta_2(\theta, 0) = -\frac{1}{2m+1}$$ $$\frac{\partial \eta_2}{\partial t}(\theta, 0) = 0$$

Approach to a second-order solution

Start from the pressure p_2. The solution is composed as $p_2 = p_{21} + p_{22}$, solving the inhomogeneous and the homogeneous equations, respectively

Start from the second-order momentum equation

Radial $$\frac{\partial u_{r2}}{\partial t} - Oh\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_{r2}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial u_{r2}}{\partial\theta}\sin\theta\right)\right] + \frac{\partial p_{21}}{\partial r} = -u_{r1}\frac{\partial u_{r1}}{\partial r} - u_{\theta 1}\frac{1}{r}\frac{\partial u_{r1}}{\partial\theta} + \frac{u_{\theta 1}^2}{r}$$

Polar $$\frac{\partial u_{\theta 2}}{\partial t} - Oh\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_{\theta 2}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_{\theta 2}\sin\theta)\right) + \frac{2}{r^2}\frac{\partial u_{r2}}{\partial\theta}\right] + \frac{1}{r}\frac{\partial p_{21}}{\partial\theta} = -u_{r1}\frac{\partial u_{\theta 1}}{\partial r} - u_{\theta 1}\frac{1}{r}\frac{\partial u_{\theta 1}}{\partial\theta} - \frac{u_{r1}u_{\theta 1}}{r}$$

Vectorial second-order momentum equation, in symbolic formulation

$$\frac{\partial\vec{u}_2}{\partial t} - Oh\Delta\vec{u}_2 + \vec{\nabla}p_{21} = -(\vec{u}_1\cdot\vec{\nabla})\vec{u}_1 = -\vec{\nabla}\frac{\vec{u}_1^2}{2} + \vec{u}_1\times(\vec{\nabla}\times\vec{u}_1)$$

$$\rightarrow \frac{\partial\vec{u}_2}{\partial t} - Oh\,\Delta\vec{u}_2 + \vec{\nabla}\left(p_{21} + \frac{\vec{u}_1^2}{2}\right) = \vec{u}_1\times(\vec{\nabla}\times\vec{u}_1) = \frac{1}{Oh}\frac{1}{r^2\sin^2\theta}\frac{\partial\psi_2}{\partial t}\vec{\nabla}\psi$$

Approach to a second-order solution

The divergence of this equation generates an equation determining pressure p_2
This would be particularly elegant if the Helmholtz decomposition of the right-hand vector was known

$$\frac{\partial \vec{u}_2}{\partial t} - Oh\,\Delta \vec{u}_2 + \vec{\nabla}\left(p_{21} + \frac{\vec{u}_1^2}{2}\right) = \frac{1}{Oh}\frac{1}{r^2 \sin^2\theta}\frac{\partial \psi_2}{\partial t}\vec{\nabla}\psi = -\vec{\nabla}\phi + \vec{\nabla}\times\vec{F}$$

Taking the divergence of this equation would then yield the Laplace equation

$$\Delta\left(p_{21} + \frac{\vec{u}_1^2}{2} + \phi\right) = 0$$

for another modified second-order pressure. But we do not have the Helmholtz decomposition. Therefore we apply brute force – series expansions of the rhs.

Approach to a second-order solution

Naming $\wp_{21} \coloneqq p_{21} + \vec{u}_1^2/2$, we write the Poisson equation for $\wp_{21}$ as

$$r^2 \Delta \wp_{21} = -q^6 C_{2m}^2 \left[\left(\sum_{k=0}^{N} c_{km} (qr)^{2k+2m-2} - \sum_{k=0}^{N} b_{km} (qr)^{2k+2m} + \frac{C_{1m}}{C_{2m}} \frac{m+1}{q^{m+1}} \sum_{k=0}^{N} (2k+m+1) a_{km} (qr)^{2k+2m-2} \right) \times \right.$$

$$\left. \sum_{l=0}^{2m} C_{3l} P_l(x) + \left(\sum_{k=0}^{N} b_{km} (qr)^{2k+2m-2} + \frac{C_{1m}}{C_{2m}} \frac{1}{q^{m+1}} \sum_{k=0}^{N} a_{km} (qr)^{2k+2m-2} \right) \times m^2 (m+1)^2 \sum_{l=0}^{2m} C_{4l} P_l(x) \right] e^{-2\alpha_m t}$$

$$(x \coloneqq \cos\theta)$$

where the various series expansions represent the spherical Bessel and Legendre functions involved in the right-hand side. This kind of Poisson equation can be solved analytically, satisfying boundary conditions.

With the pressure known, the radial and polar velocity components are found. Finally, the drop shape is determined.

Zrnić, Berglez & Brenn, 2022

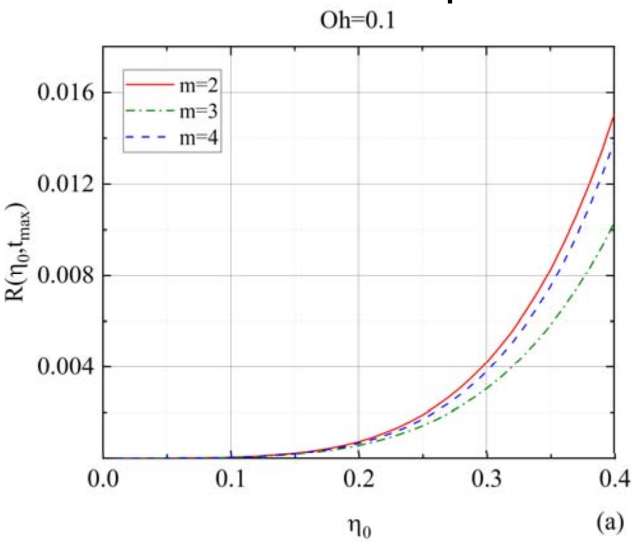
Second-order solutions “22” and third-order solutions

Solutions “22” satisfy the homogeneous equations of motion and exhibit time behaviour different from the solutions “21”;
→ represents quasi-periodic motion

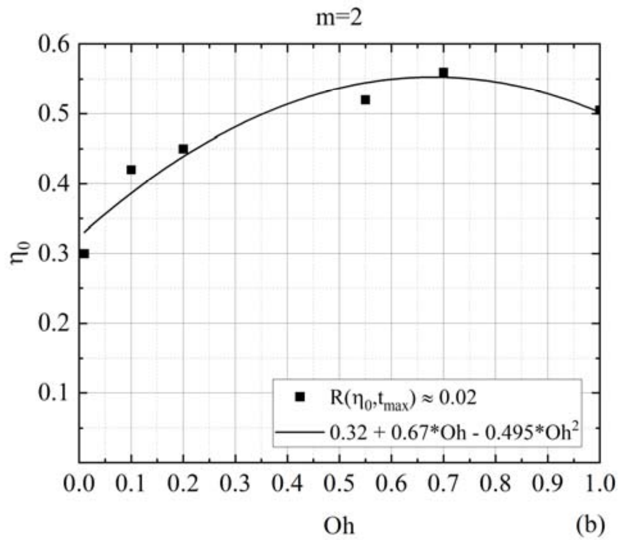
Third-order solutions are obtained using the same methods

Zrnić, Berglez & Brenn, 2022

Truncated series expansions → check drop volume conservation: $R=[V(t)-V_s]/V_s$



Relative volume deviation $R = f(\eta_0,t)$ for different m



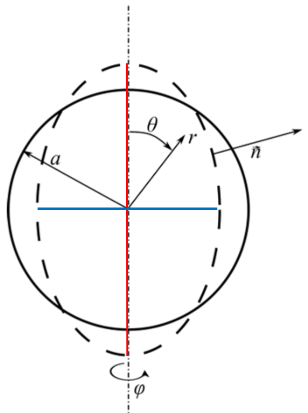
Maximum allowable η_0 for R=2 % at m=2 for different Oh

Verification – oscillations for $m = 2$

Radial position of points on the surface of the axisymmetric drop

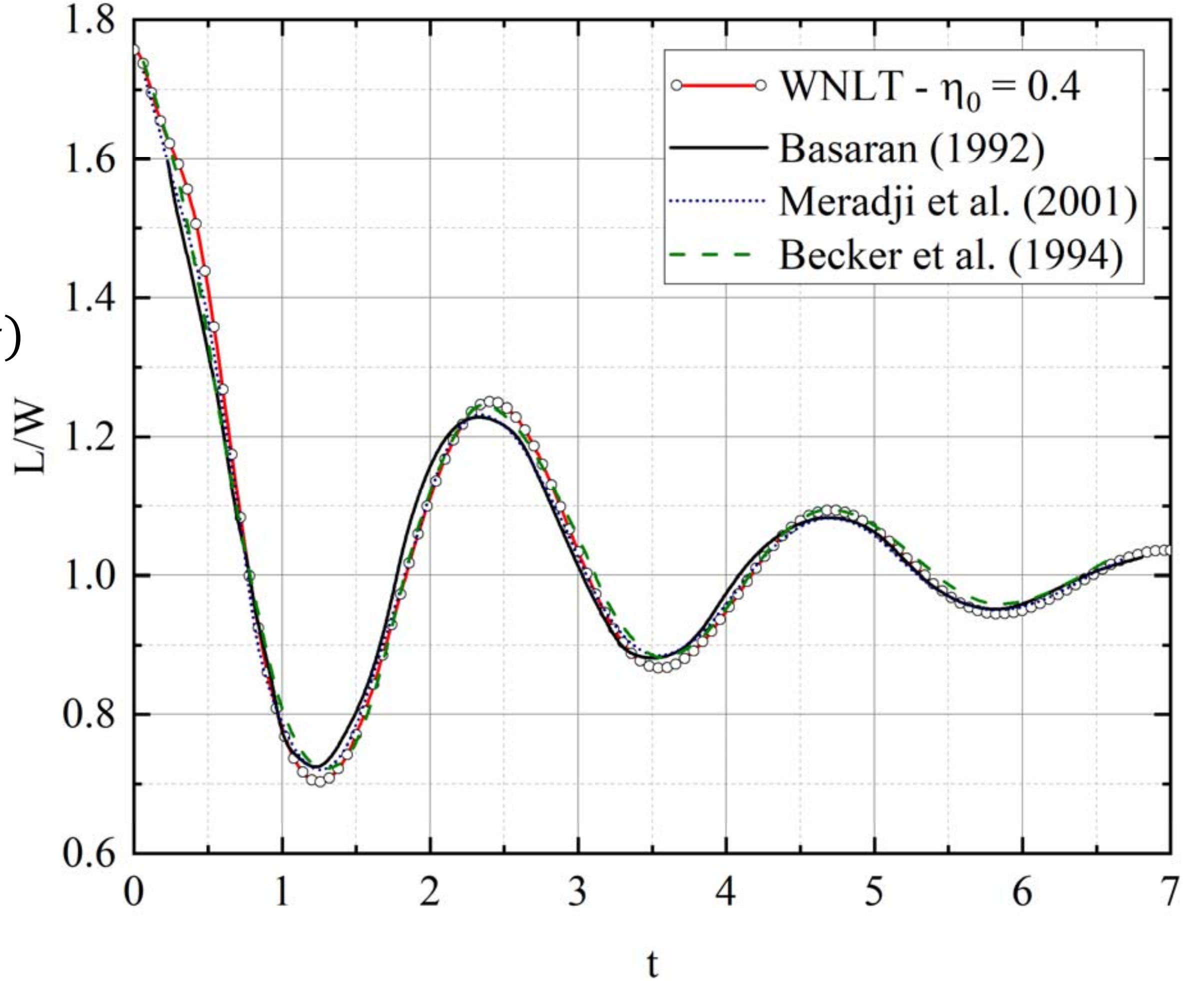
$$r_s(\theta,t) = 1 + \eta_0\eta_1(\theta,t) + \eta_0^2\eta_2(\theta,t) + \eta_0^3\eta_3(\theta,t)$$

The aspect ratio of the drop in time calculated as



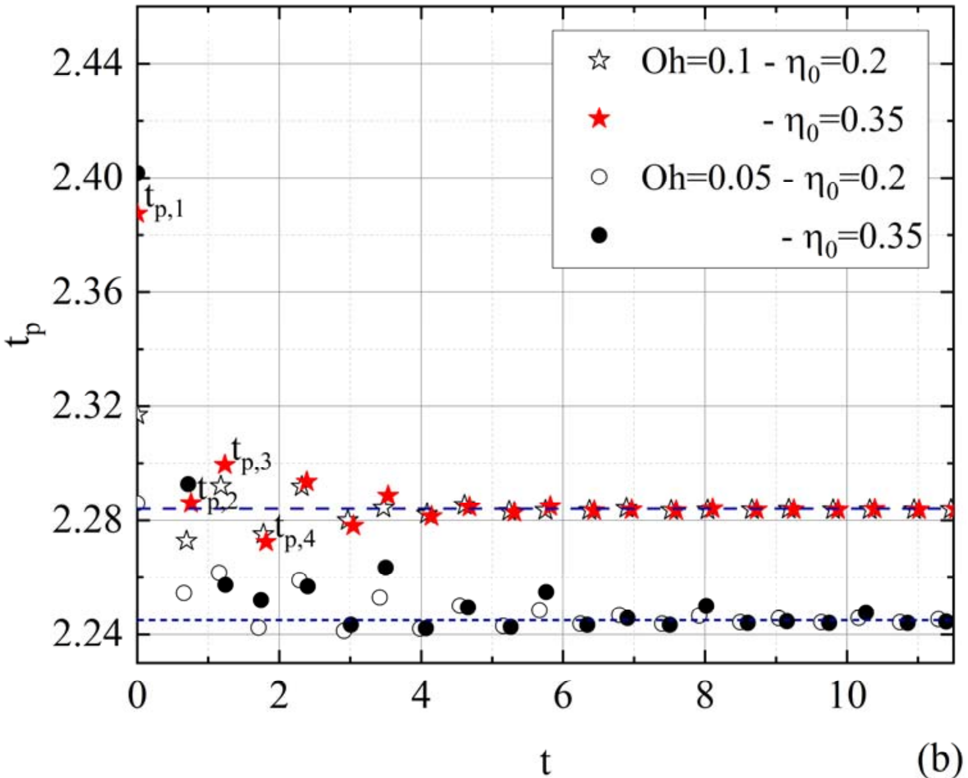
$$\frac{L}{W}(t) = \frac{r_s(0,t)}{r_s\left(\frac{\pi}{2},t\right)}$$

$Oh = 0.1$

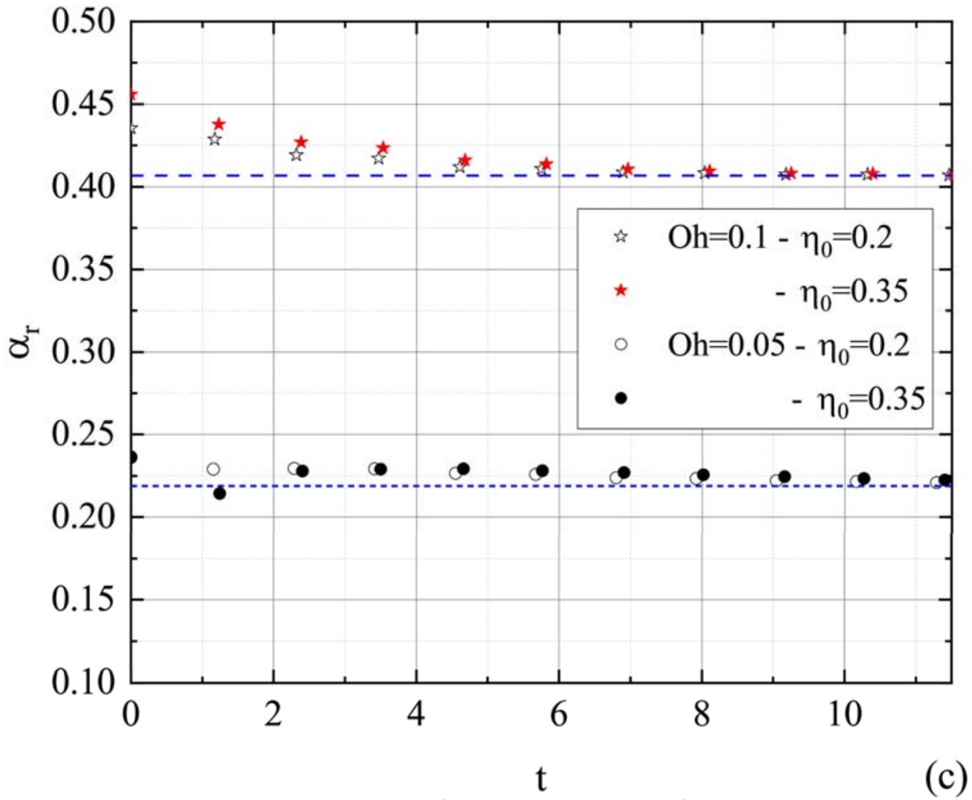


Period length and damping factor for m=2

Complex angular frequency is $\alpha_m = i\ 2\pi/t_p + \alpha_r$



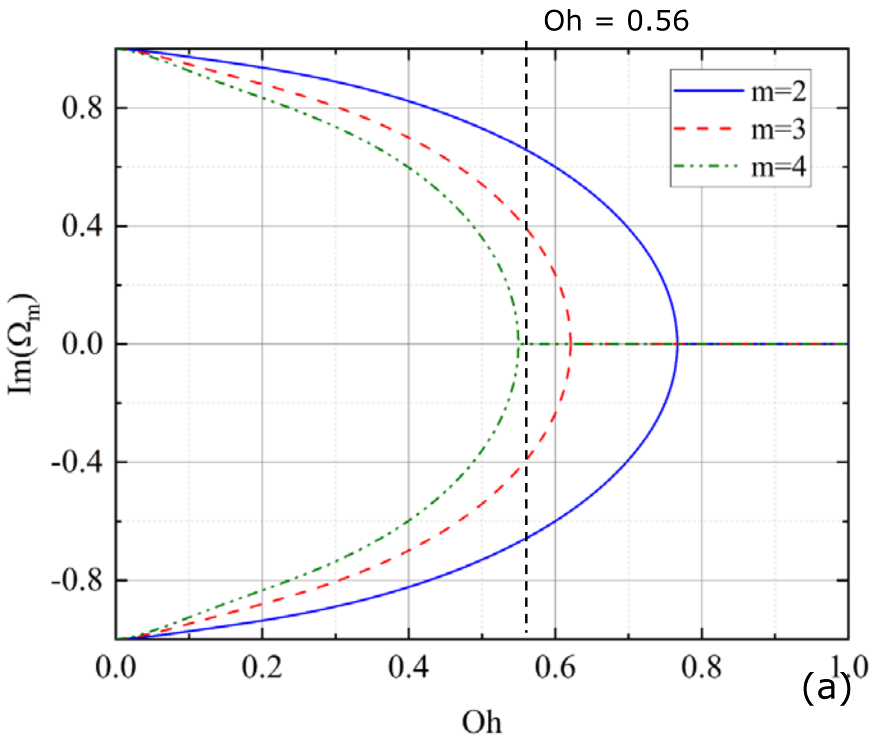
Oscillation period $t_p = f(t)$

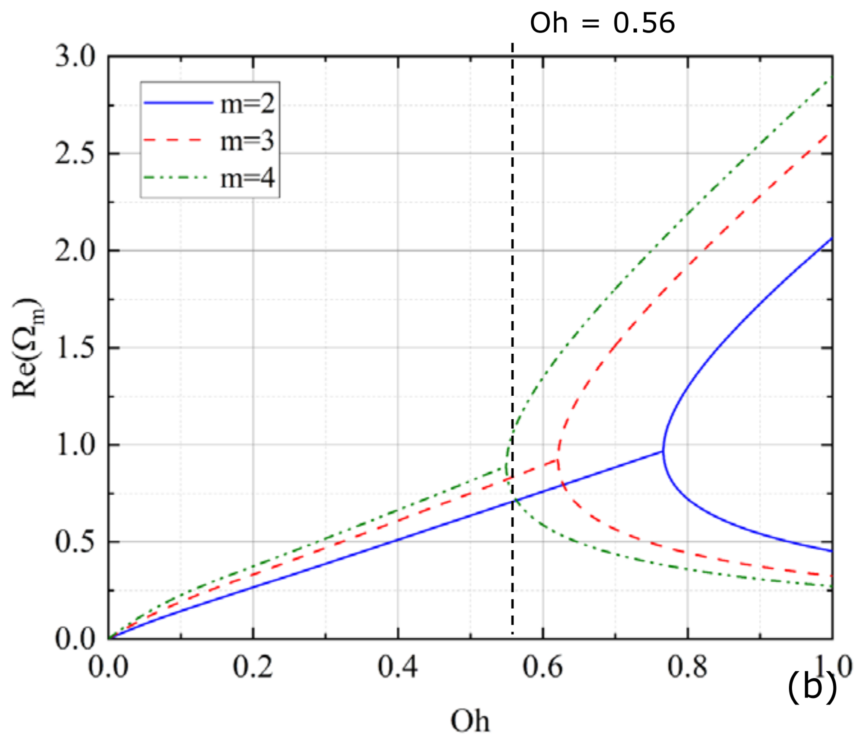


Damping factor $\alpha_r = f(t)$

Oscillations with supercritical Oh = 0.56 at m=4

$$\Omega_\mathrm{m} = \frac{\alpha_m}{\alpha_{m,0}}$$

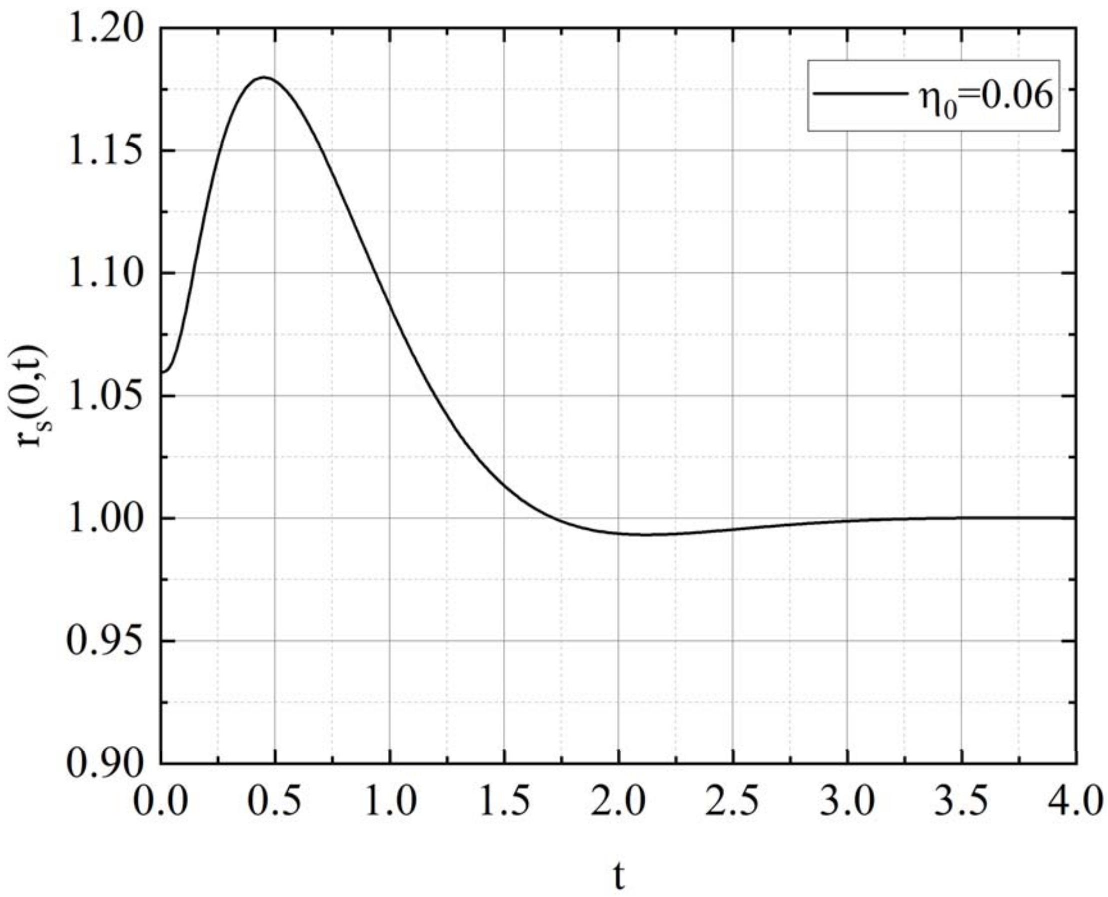


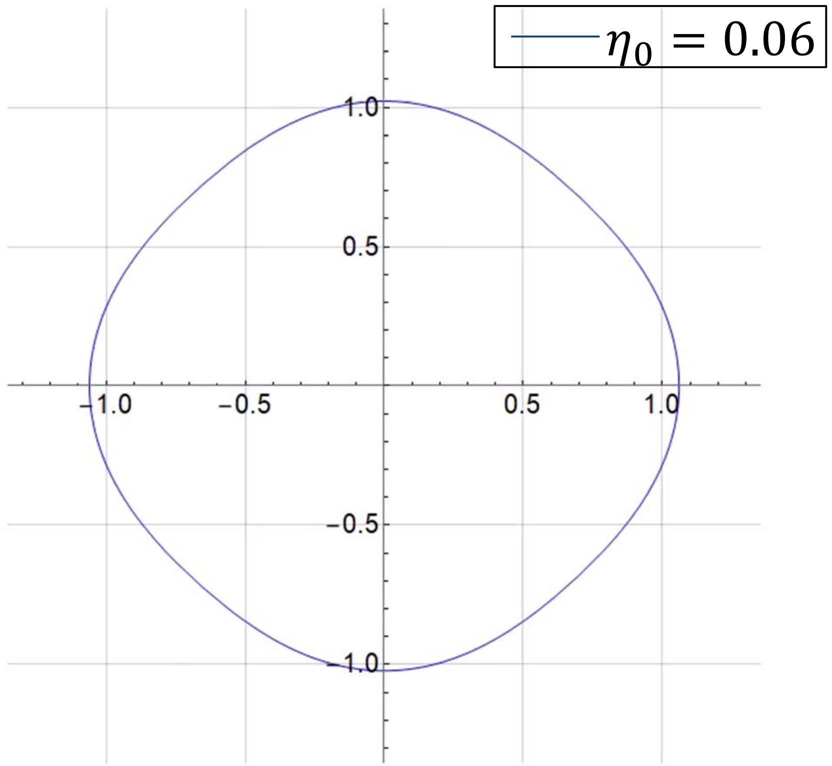


(a)– oscillation frequency and (b) – damping factor of a drop, which is linearly supercritical at m=4

→ Linear analysis predicts aperiodic behaviour

Oscillations with supercritical $Oh = 0.56$ at m=4





The droplet meridional section on the video is rotated by $90°$ and shown for $0 < t < 3$

Summary

- Weakly nonlinear analysis is a method of successive approximation to a problem solution
- The analysis builds on series expansions of the unknowns of a physical problem – here of the flow field variables
- At each approximation order ≥ 2, the unknowns are governed by linear differential equations for the respective order, with nonlinear terms in the solutions of the lower orders
- The solutions may represent nonlinear system behaviour, but with the limitation to moderate deformations, since
  - volume is not conserved inherently and
  - boundary conditions satisfied on the deformed system boundaries are represented as Taylor series
- In the present case studied, nonlinear system behaviour such as mode coupling and quasi-periodicity of the oscillatory motion is captured by the analysis

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