Covariant, contravariant, irrelevant?

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13.01.2023

Introduction

In continuum mechanics (especially in non-linear solid mechanics) it is well-known that vectors and also (pseudo-) scalars obey

- different transformation laws when changing the coordinate system

and different rules

- upon pushing them forward or pulling them back between configurations.

The vectorial objects are basically classified as covariant or contravariant, though there frequently occur objects which deviate from either transformation behavior.

Introduction (contd.)

For some objects these transformations are commonly derived from geometrical considerations, e.g. from deforming line-, area- or volume-elements.

For other objects, the transformation behavior seems to be more arbitrary and one sometimes considers differently transforming versions of „the same object“.

Questions to be addressed

Today we address the following questions:

- Do physical fields possess a natural type of variance?
- What does it mean to change the variance?
- How does the variance matter for formulating constitutive laws?

Tranformations in 3D

In three-dimensional space there occur

- four different transformation types for „vector fields“ (and tensor indices)
- three different transformation types for „scalar fields“.

Note: objects with different transformation behavior represent different geometrical entities.

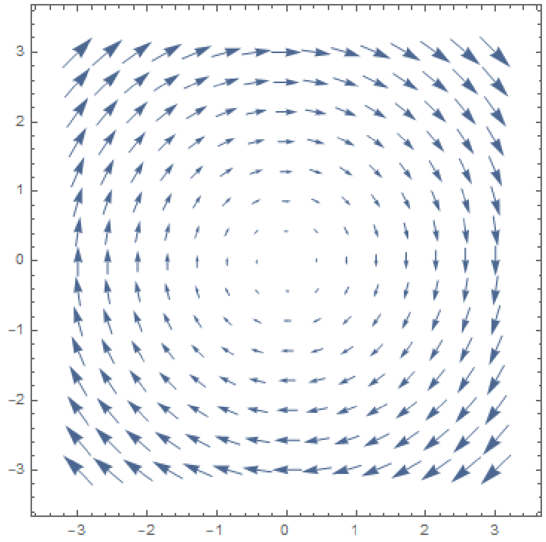
The four different „vector fields“ in 3D

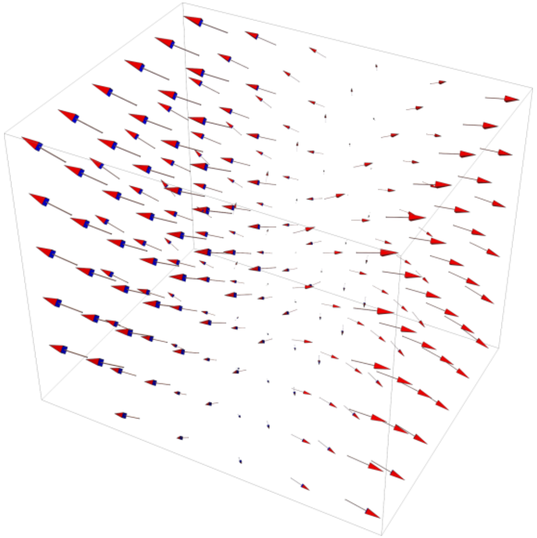
In 3D there are four different field objects which are characterized by three coefficients at each point:

1. (tangent) vector fields (infinitesimal line-elements)
2. differential one-forms or co-vector fields (densities of surfaces)
3. bi-vector fields (infinitesimal surface elements)
4. differential two-form (densities of curves)

Tangent vector fields

Vector fields can be thought of as indicating infinitesimal line elements (as if connecting neighboring points) attached to each point.



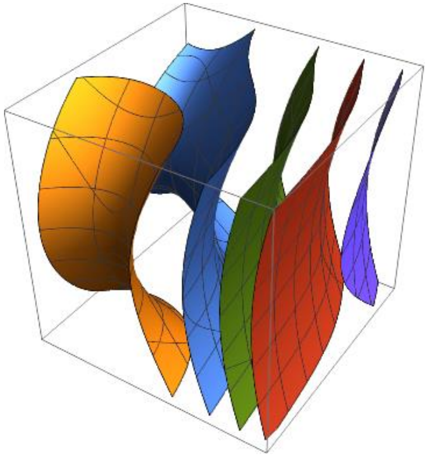


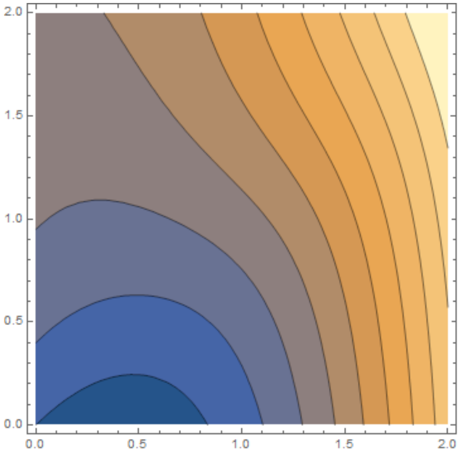
Differential one-form (co-vector, surface density)

Basic differential one-forms arise as differentials of functions.

They represent the density of isosurfaces.

In general, a one-form need not be the differential of a function.



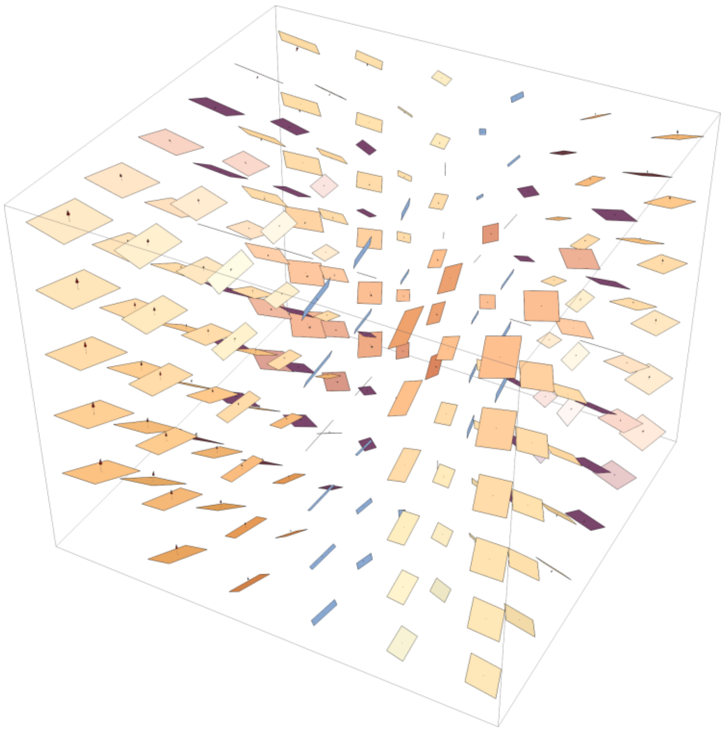


Bi-vector fields

Bi-vector fields represent infinitesimal oriented surface elements.

The surface elements have no shape but only orientation and „surface content“.

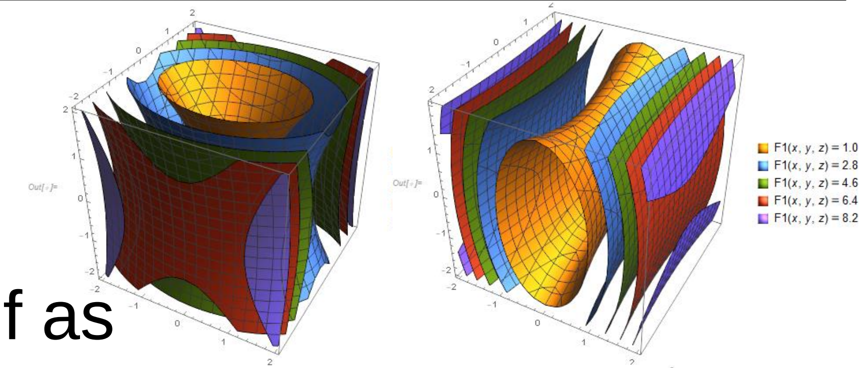
Elementary bi-vectors are surfaces spanned by two line segments (vector-fields).

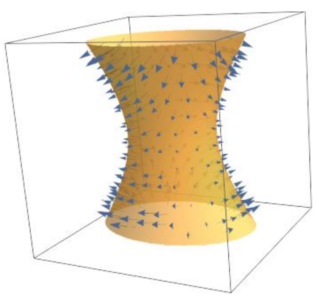


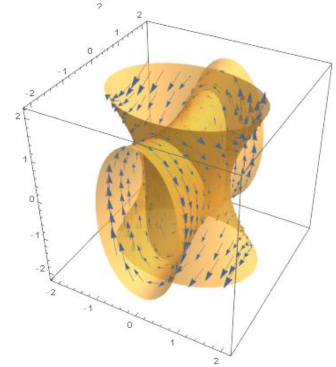
Two-forms
(line densities)

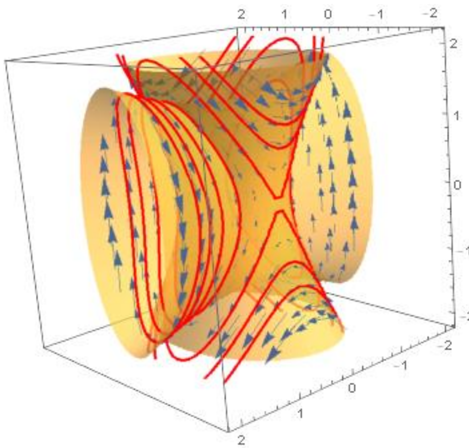
Two-forms may be thought of as density of curves or curve-segments.

Elementary line-densities arise as density of intersection lines from two surface densities (differential one-forms).









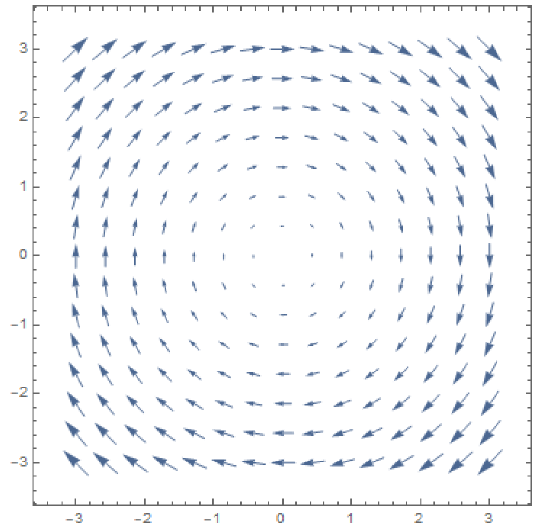
Example

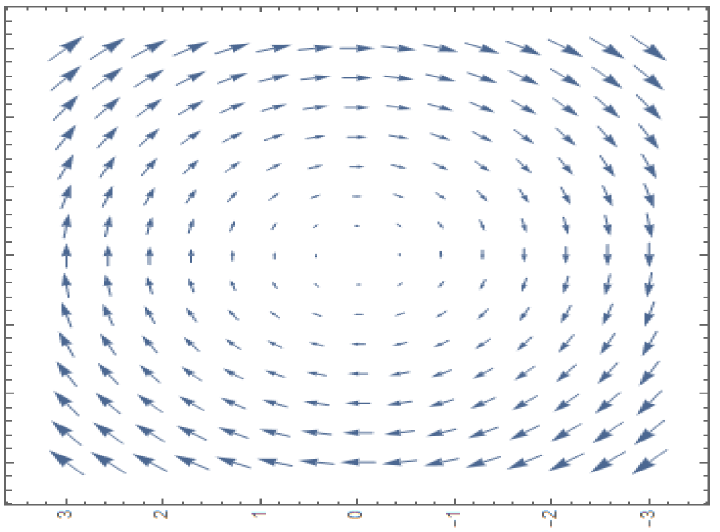
- vector fields
  - displacement fields
  - velocity fields
- differential one-forms
  - (total) differentials of functions
  - forces
  - electrical field
- bi-vector fields
  - surface elements
  - normal vectors to surfaces
- two-forms
  - currents (mass, charge, etc.)
  - magnetic field

Transformation of vector fields

Line elements deform with space.

In the example, horizontal vectors get stretched while vertical ones remain unchanged.

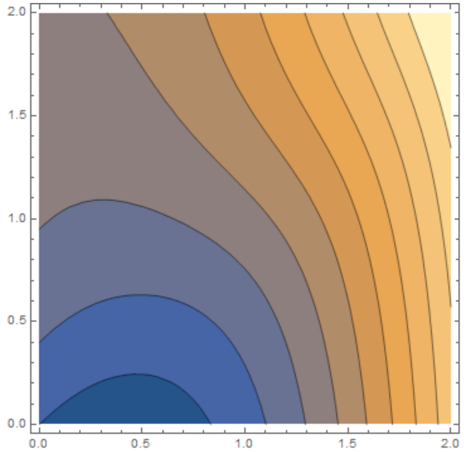


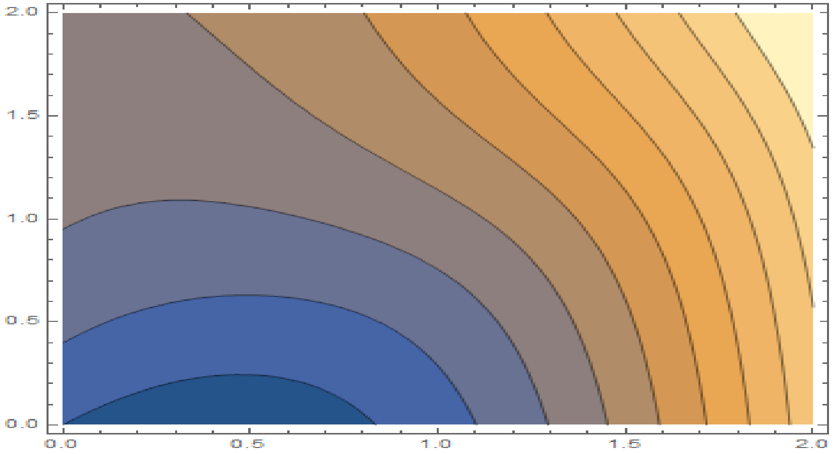


Transformation of differential one-forms

Differential one-forms deform invers to tangential vectors.

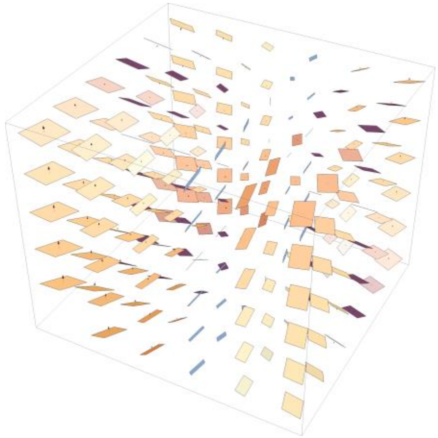
If isosurfaces are pulled apart, the differential (gradient) decreases.

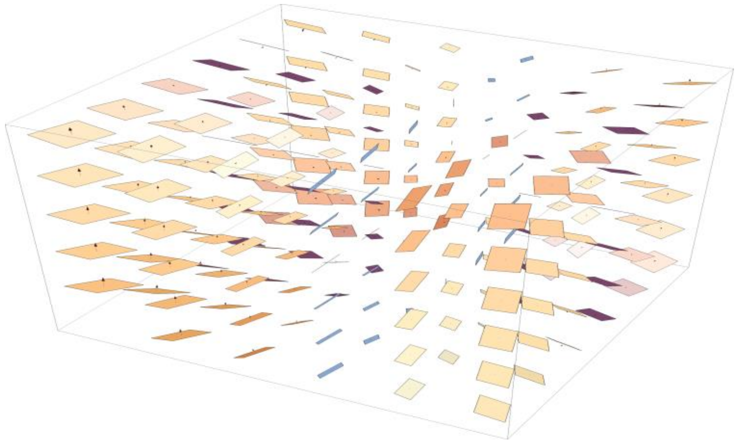


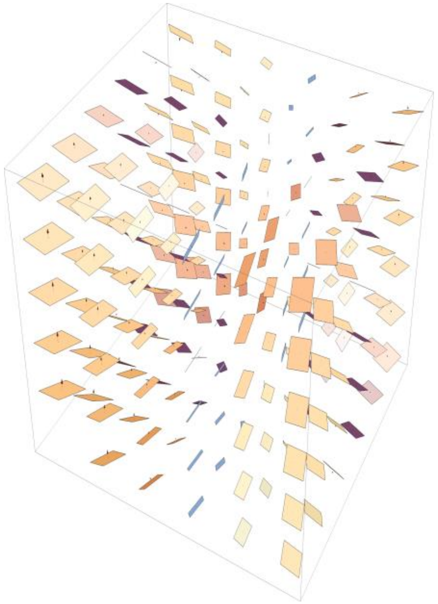


Transformation of bi-vector fields

Bi-vectors reflect only deformations within their local plane.

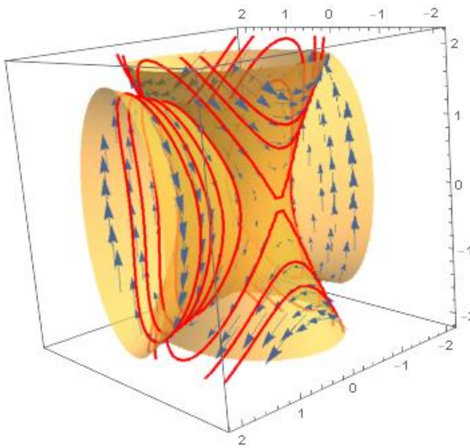


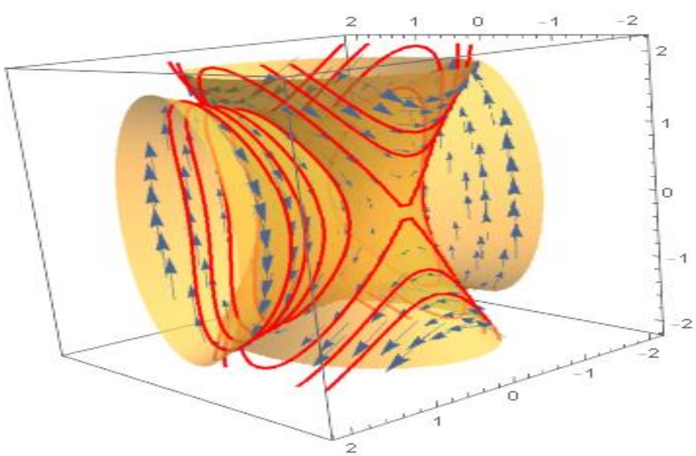


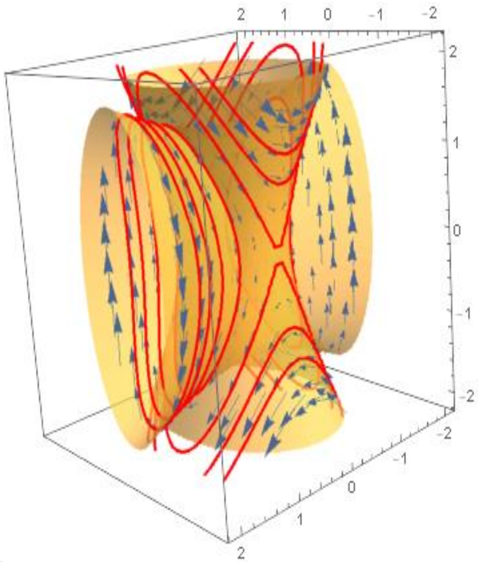


Transformation of two-forms (line-densities)

Line densities reflect the (inverse of) the deformation in a plane perpendicular to the local line-direction.



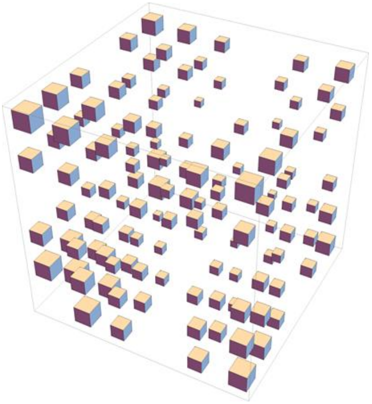


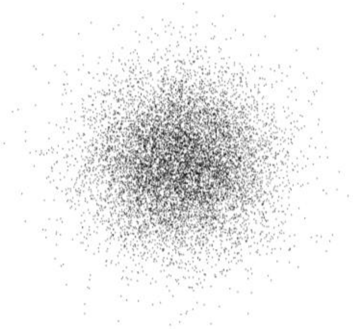


Different „scalars“

Besides the different types of vector quantities there are three different quantities locally characterized by a number:

1. (true) scalars
2. tri-vector fields or pseudo-scalars
3. three-forms





Examples

- scalars
  - temperature
  - potential (potential energy, electric potentials)
- tri-vectors
  - volume-elements
- three-forms
  - mass density
  - charge density
  - energy density
  - …

Coordinates and bases

Charts and coordinates

Let B be the domain of interest. Local coordinates are obtained from a so-called chart map $\varphi\colon B \to \mathbb{R}^3$ which assigns to each point $\boldsymbol{p} \in B$ bijectively three numbers,

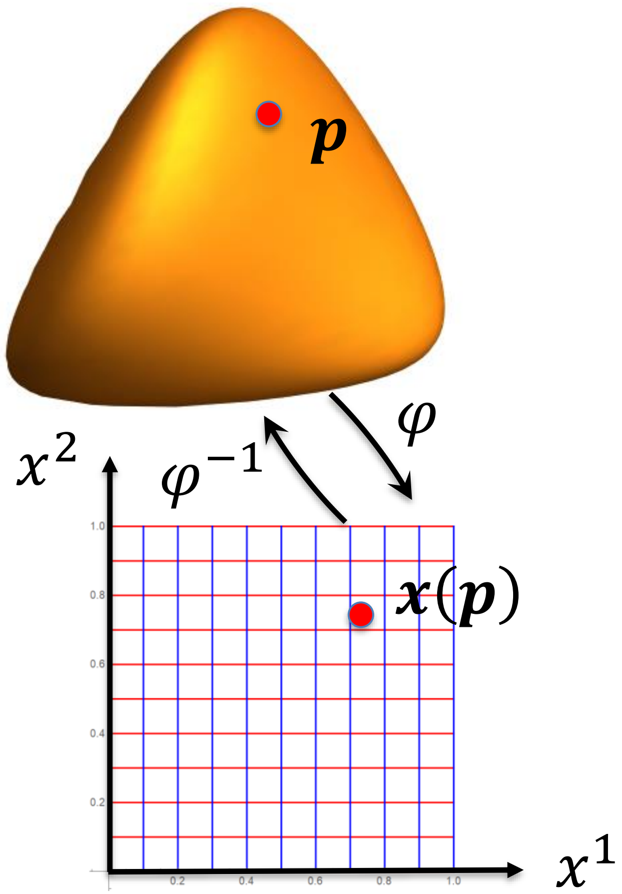
$$\varphi(\boldsymbol{p}) = \left(x^1(\boldsymbol{p}), x^2(\boldsymbol{p}), x^3(\boldsymbol{p})\right) = \boldsymbol{x}(\boldsymbol{p}).$$

The inverse chart map $\varphi^{-1}\colon \mathbb{R}^3 \to B$ assigns to points $\boldsymbol{x} \in \mathbb{R}^3$ its preimage in B,

$$\varphi^{-1}(x^1, x^2, x^3) = \boldsymbol{p}(\boldsymbol{x}).$$

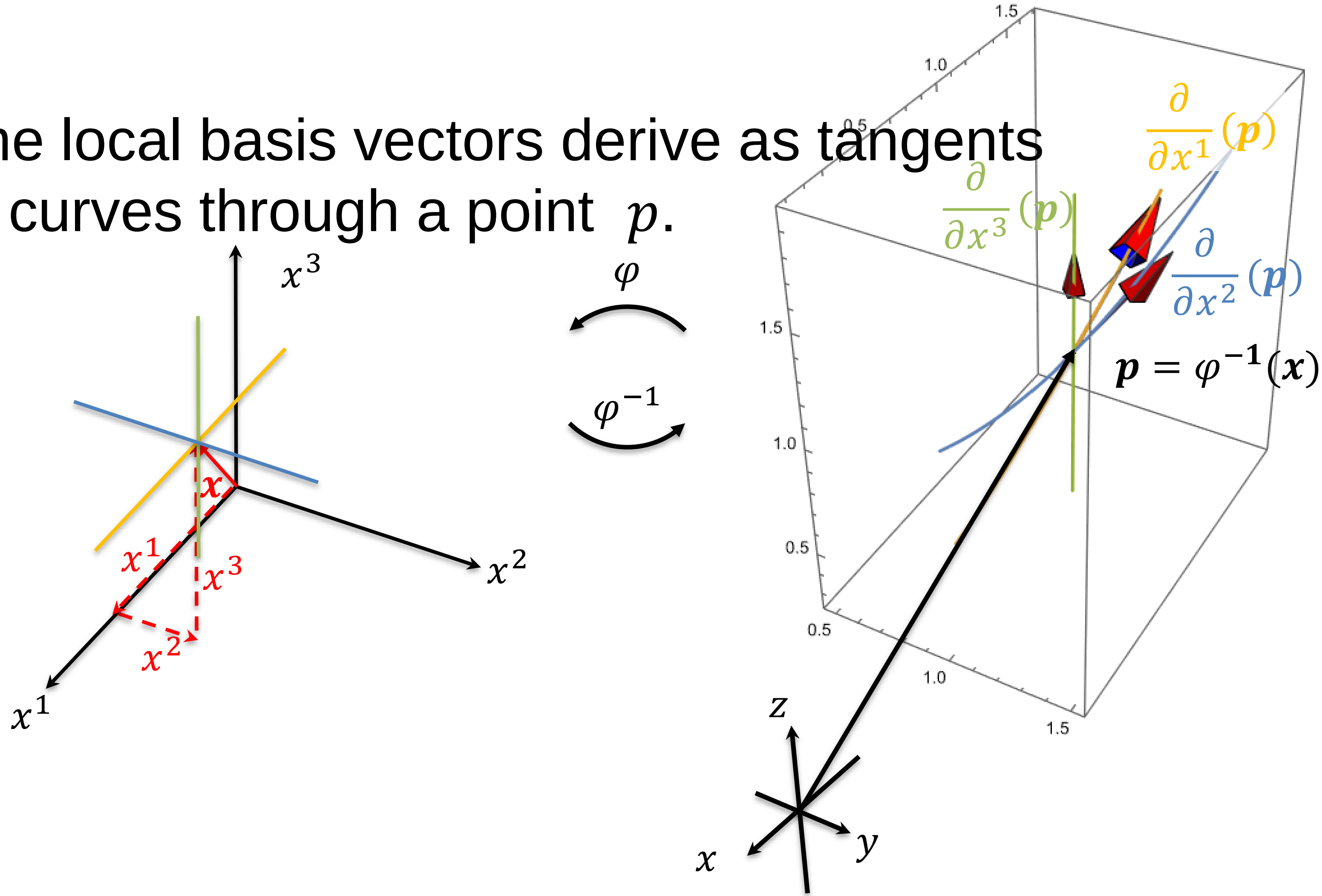
Remark:

- Upper indices are no exponents!



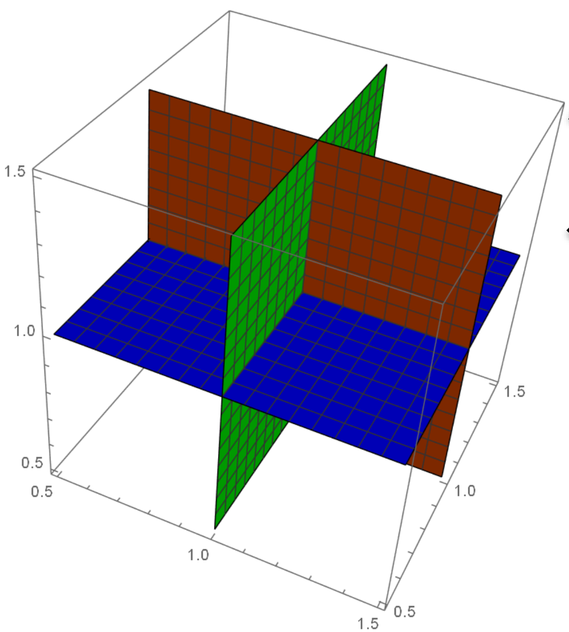
Canonical tangent vector basis

The local basis vectors derive as tangents to curves through a point p.



Canonical basis of differential one-forms

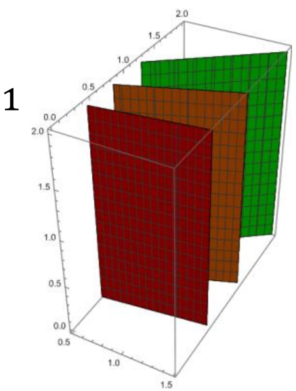
The coordinate functions $x^1(\boldsymbol{p})$, $x^2(\boldsymbol{p})$ and $x^3(\boldsymbol{p})$ are scalar functions of space. Their differentials $\mathrm{d}x^i, i = 1,2,3$ represent densities of isosurfaces.



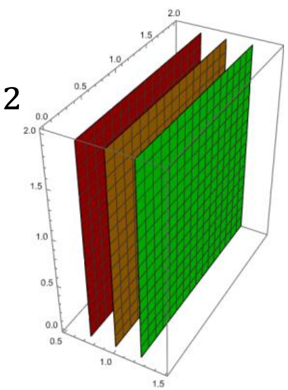
φ

φ^{-1}

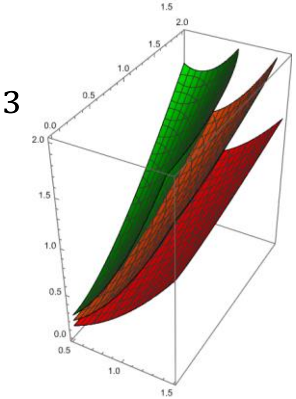
$\mathrm{d}x^1$



$\mathrm{d}x^2$

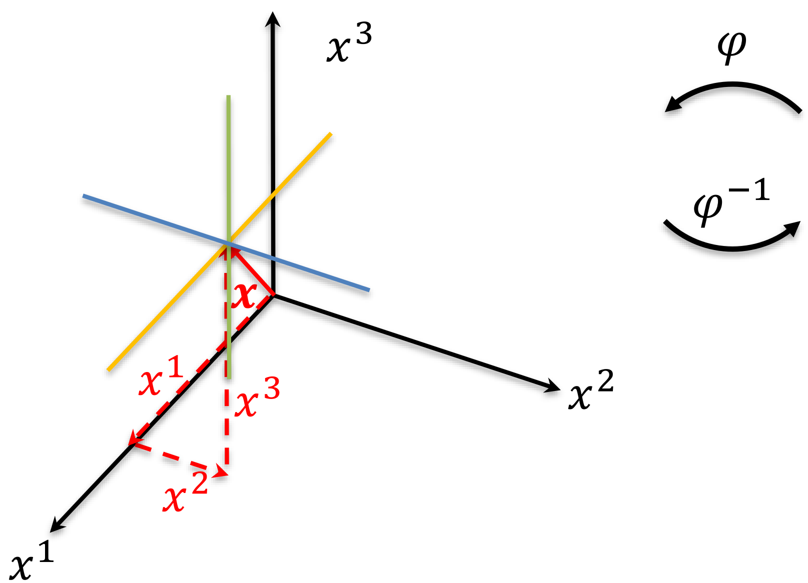


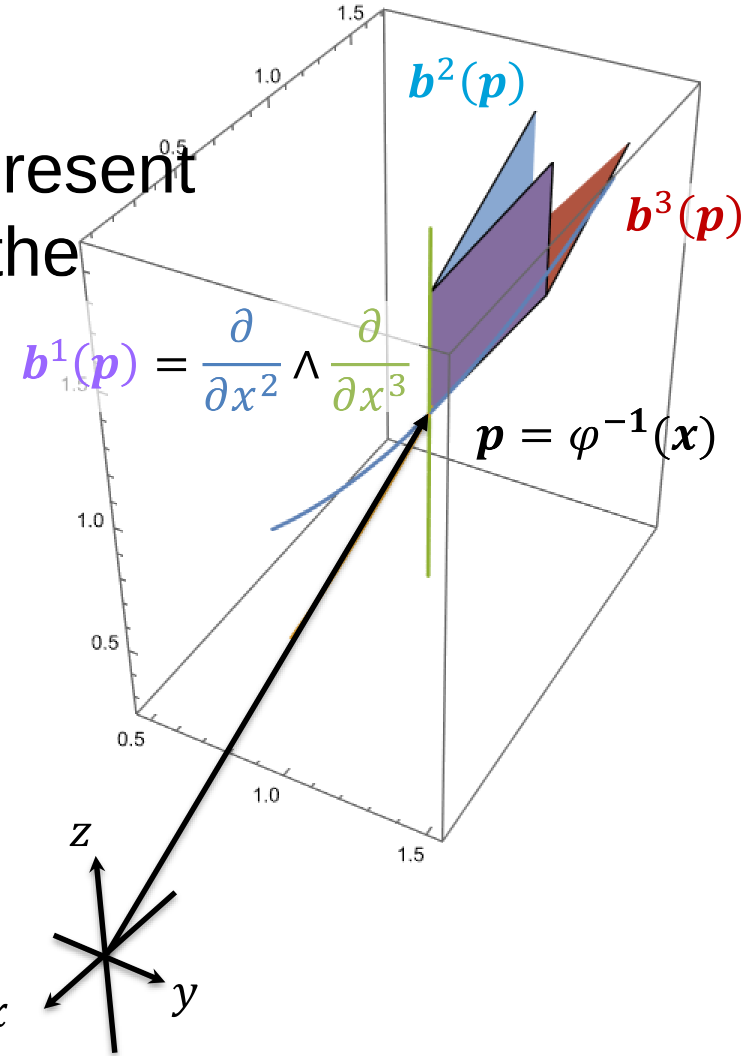
$\mathrm{d}x^3$



Canonical bi-vector basis

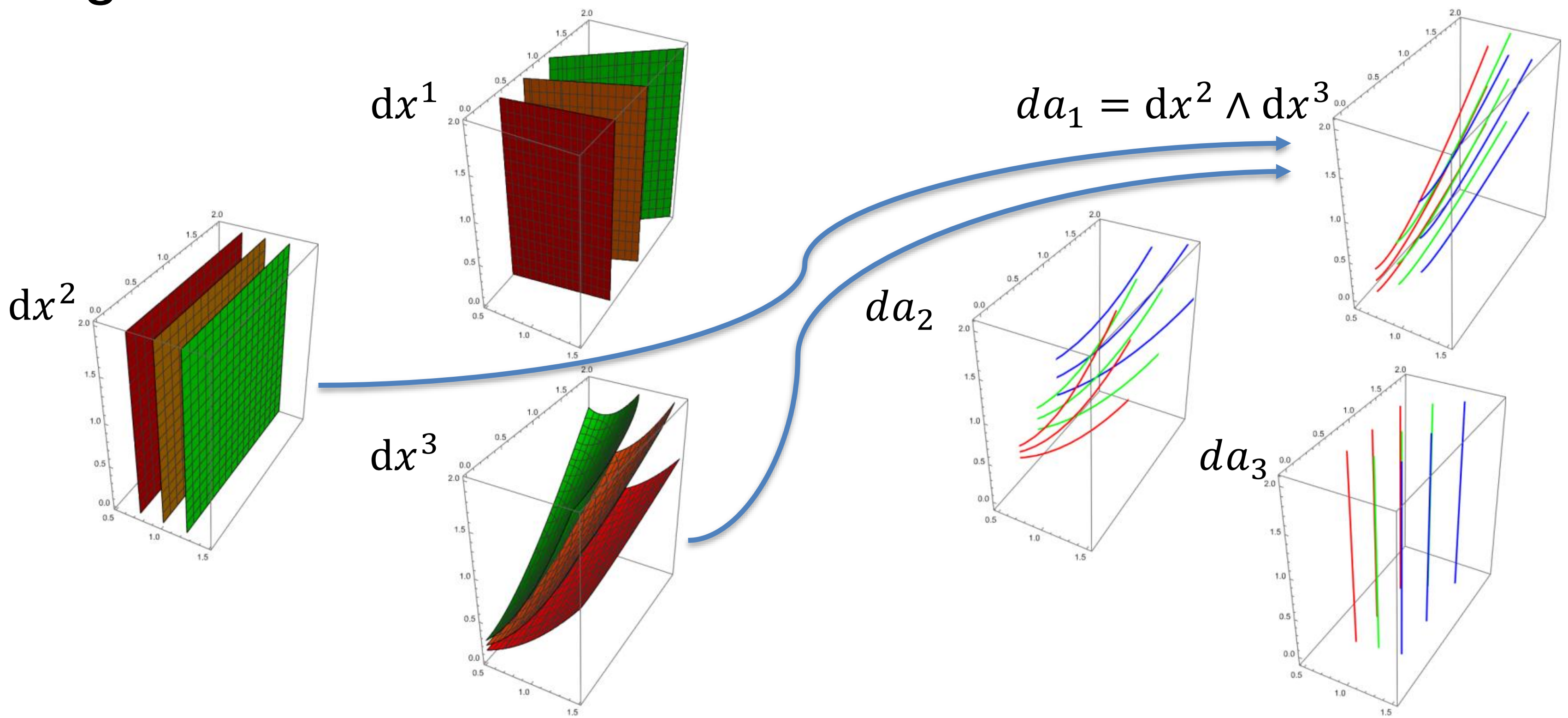
The local basis bi-vectors represent signed surfaces spanned by the tangent basis vectors p.





Canonical basis of differential two-forms

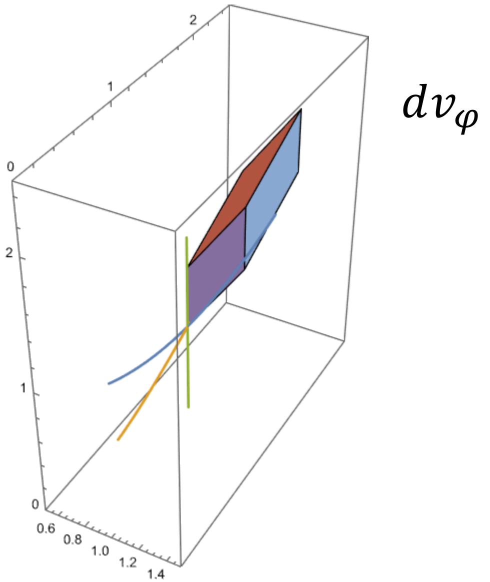
The basis of line densities $\mathrm{d}a_i, i = 1,2,3$ arise from signed intersections of the isosourfaces.

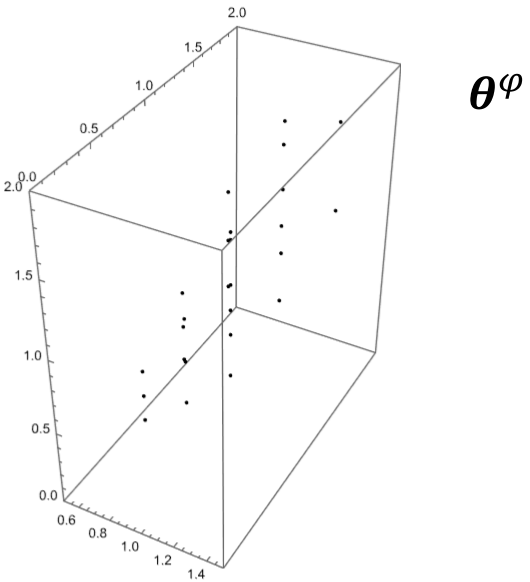


Basis of tri-vectors and three-forms

The basic tri-vector is the signed volume spanned by the base vectors.

The basic three-form arises from the signed intersection of the isosurfaces.





Mathematical basis definition

The „higher“ order basis elements derive from the elementary bases $\frac{\partial}{\partial x^i}$, $\mathrm{d}x^i$ as follows:

Bi-vector basis

$$\boldsymbol{b}^i = \varepsilon^{ijk} \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k},\ i = 1{,}2{,}3$$

Two-form basis

$$da_i = \varepsilon_{ijk} \mathrm{d}x^j \otimes \mathrm{d}x^k,\ i = 1{,}2{,}3$$

Basic tri-vector

$$\boldsymbol{\theta}^\varphi = \varepsilon^{ijk} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}.$$

Basic three-form

$$dv_\varphi = \varepsilon_{ijk} \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k.$$

Original Einstein summation convention applies!

General fields

Vector field

$$\boldsymbol{X} = X^i\big(\boldsymbol{x}(\boldsymbol{p})\big)\frac{\partial}{\partial x^i}(\boldsymbol{p}) = X^i\frac{\partial}{\partial x^i}$$

One-form field

$$\boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i$$

Bi-vector field

$$\boldsymbol{D} = D_i \boldsymbol{b}^i$$

Two-form field

$$\boldsymbol{\zeta} = \zeta^i da_i$$

Tri-vector field

$$\boldsymbol{T} = T\boldsymbol{\theta}^\varphi$$

Three-form field

$$\boldsymbol{\vartheta} = \vartheta dv_\varphi$$

Change of coordinate systems (i.e. of the chart map)

Let $\varphi: B \to \mathbb{R}^3$ be a chart map with coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$ and $\tilde{\varphi}: B \to \mathbb{R}^3$ an alternative chart map with coordinates $\tilde{\boldsymbol{x}} = (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$.

Functions f on B may then be expressed in dependence of either coordinates,

$$f(x^1, x^2, x^3) = f\left(\varphi^{-1}(\boldsymbol{x})\right) = f\left(\boldsymbol{p}(\boldsymbol{x})\right) = f\left(\tilde{\varphi}^{-1}(\tilde{\boldsymbol{x}})\right) = f(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3).$$

This likewise applies to the coordinate function themselves,

$$x^i(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3),$$

$$\tilde{x}^i(x^1, x^2, x^3).$$

Transformation behavior

The bases $\frac{\partial}{\partial x^i}$, $\mathrm{d}x^i$ are obtained from derivatives. The local matrix for the change of bases may be obtained from chain rule and the total differential

$$\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial}{\partial \tilde{x}^i}(\boldsymbol{p}) = \frac{\partial}{\partial \tilde{x}^i}\left(\boldsymbol{p}(x^1, x^2, x^3)\right) = \sum_{i=1}^{3} \frac{\partial}{\partial x^j}(\boldsymbol{p}) \frac{\partial x^j}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j},$$

$$\mathrm{d}\tilde{x}^i = \mathrm{d}\tilde{x}^i(x^1, x^2, x^3) = \frac{\partial \tilde{x}^i}{\partial x^j} \mathrm{d}x^j.$$

Analogously in inverse direction,

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j},$$

$$\mathrm{d}x^i = \mathrm{d}x^i(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial x^i}{\partial \tilde{x}^j} \mathrm{d}\tilde{x}^j.$$

Transformation of the coefficient functions

Objectivity requires, that vector fields and one-forms, represent objects which are independent of the coordinate system, i.e.

$$\boldsymbol{X} = X^i \frac{\partial}{\partial x^i} = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} \quad \text{and} \quad \boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i = \tilde{\alpha}_i \mathrm{d}\tilde{x}^i.$$

The coefficient functions thus need to transform inversely to their basis fields,

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j \qquad \Leftrightarrow X^i = \frac{\partial x^i}{\partial \tilde{x}^j} \tilde{X}^j$$

$$\tilde{\alpha}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \alpha_j \qquad \Leftrightarrow \alpha_i = \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\alpha}_j.$$

Transformation of bi-vector basis

The basis bi-vectors are defined as

$$\boldsymbol{b}^i = \varepsilon^{ijk} \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}.$$

For a different coordinate system we have

$$\begin{aligned}
\widetilde{\boldsymbol{b}}^i &= \varepsilon^{ijk} \frac{\partial}{\partial \tilde{x}^j} \otimes \frac{\partial}{\partial \tilde{x}^k} \\
&= \varepsilon^{ijk} \frac{\partial x^m}{\partial \tilde{x}^j} \frac{\partial x^n}{\partial \tilde{x}^k} \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^n} \qquad \left(\breve{A}^i_p \det \boldsymbol{A}\, \varepsilon^{pmn} = \varepsilon^{ijk} A^m_j A^n_k\right) \\
&= \frac{\partial \tilde{x}^i}{\partial x^l} \det \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}\widetilde{\boldsymbol{x}}}\, \varepsilon^{lmn} \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^n} \\
&= \det \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}\widetilde{\boldsymbol{x}}} \frac{\partial \tilde{x}^i}{\partial x^l} \boldsymbol{b}^l.
\end{aligned}$$

Summary of transformation behaviors

Object	$\boldsymbol{x}$	$\widetilde{\boldsymbol{x}}$
scalar	f	f
vector field	$\boldsymbol{X} = X^i \frac{\partial}{\partial x^i}$	$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^i$
one-form	$\boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i$	$\tilde{\alpha}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \alpha_j$
bi-vector	$\boldsymbol{D} = D_i \boldsymbol{b}^i$	$\widetilde{D}_i = \det \frac{\mathrm{D}\widetilde{\boldsymbol{x}}}{\mathrm{D}\boldsymbol{x}} \frac{\partial x^j}{\partial \tilde{x}^i} D_j$
two-form	$\boldsymbol{\zeta} = \zeta^i da_i$	$\tilde{\zeta}^i = \det \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}\widetilde{\boldsymbol{x}}} \frac{\partial \tilde{x}^i}{\partial x^j} \zeta^j$
tri-vector	$\boldsymbol{T} = T \boldsymbol{\theta}^{\varphi}$	$\tilde{T} = \det \frac{\mathrm{D}\widetilde{\boldsymbol{x}}}{\mathrm{D}\boldsymbol{x}} T$
three-form	$\boldsymbol{\vartheta} = \vartheta dv_{\varphi}$	$\tilde{\vartheta} = \det \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}\widetilde{\boldsymbol{x}}} \vartheta$

Remark

The different objects are principally different and without an additional structure attached to the space (or the material) they cannot be mapped into one-another.

The most commonly used structure for such mappings is a (Riemannian) metric tensor, which determines lengths of and angles between vectors.

The metric tensor canonically determines measures for areas and volumes as well as distances between objects.

The metric tensor

The metric tensor is a purely covariant symmetric and positive definite tensor (bilinear form), i.e.,

$\boldsymbol{g}[\boldsymbol{v},\boldsymbol{w}] = \boldsymbol{g}[\boldsymbol{w},\boldsymbol{v}]$ and $\boldsymbol{g}[\boldsymbol{v},\boldsymbol{v}] > 0$ for $\boldsymbol{v} \neq \boldsymbol{0}$.

The metric is $\boldsymbol{g} = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ with the coefficients $\boldsymbol{g}\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right] = g_{ij}$.

The scalar product between two vectors is calculated as

$\boldsymbol{g}[\boldsymbol{v},\boldsymbol{w}] = g_{ij}v^i w^j$.

The according norm of a vector is

$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{g}(\boldsymbol{v},\boldsymbol{v})}$,

and the angle between two vectors is defined from

$$\cos\angle(\boldsymbol{v},\boldsymbol{w}) = \frac{\boldsymbol{g}[\boldsymbol{v},\boldsymbol{w}]}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|}.$$

Induced scalar products for other objects

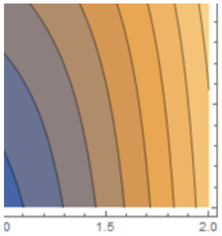
With g^{ij} denoting the coefficients of the invers matrix of g_{ij}, i.e. $g^{ik}g_{kj} = \delta^i_j$ the canonical metrics on the different objects are

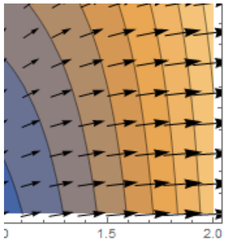
Object	g
scalar	1
vector field	$\boldsymbol{g} = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$
one-form	$\breve{\boldsymbol{g}} = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$
bi-vector	$\breve{\breve{\boldsymbol{g}}} = \det \boldsymbol{g}\, g^{ij} da_i \otimes da_j$
two-form	$\bar{\boldsymbol{g}} = \det \breve{\boldsymbol{g}}\, g_{ij} \boldsymbol{b}^i \otimes \boldsymbol{b}^j$
tri-vector	$\breve{\breve{\breve{\boldsymbol{g}}}} = \det \boldsymbol{g}\, dv_\varphi \otimes dv_\varphi$
three-form	$\bar{\bar{\boldsymbol{g}}} = \det \breve{\boldsymbol{g}}\, \boldsymbol{\theta}^\varphi \otimes \boldsymbol{\theta}^\varphi$

Switching between types

The metrics define canonical mappings between dual objects, e.g. mapping vectors to one-forms,

$\boldsymbol{v}^{\flat} = g_{ij} v^j \mathrm{d}x^i$, or vice versa $\boldsymbol{\alpha}^{\#} = g^{ij} \alpha_j \frac{\partial}{\partial x^i}$.





Or mapping bi-vectors to two-forms,

$\boldsymbol{D}^{\#} = \det \boldsymbol{g}\, g^{ij} D_j\, da_i$, or vice versa $\boldsymbol{\zeta}^{\flat} = \det \breve{\boldsymbol{g}}\, g_{ij} \zeta^j \boldsymbol{b}^i$.

A further canonical mapping (Hodge-star operator) maps between one- and two-forms,

$* \boldsymbol{\alpha} = \sqrt{\det \boldsymbol{g}}\, g^{ij} \alpha_j\, da_i$, or vice versa $* \boldsymbol{\zeta} = \sqrt{\det \breve{\boldsymbol{g}}}\, g_{ij} \zeta^j\, \mathrm{d}x^i$

Therefore, there are canonical mappings between all four types of vectorial objects.

Cartesian coordinates

In Cartesian coordinates, the basis vectors are orthonormal and thus the metric coefficients are

$g_{ij} = \delta_{ij}.$

Therefore

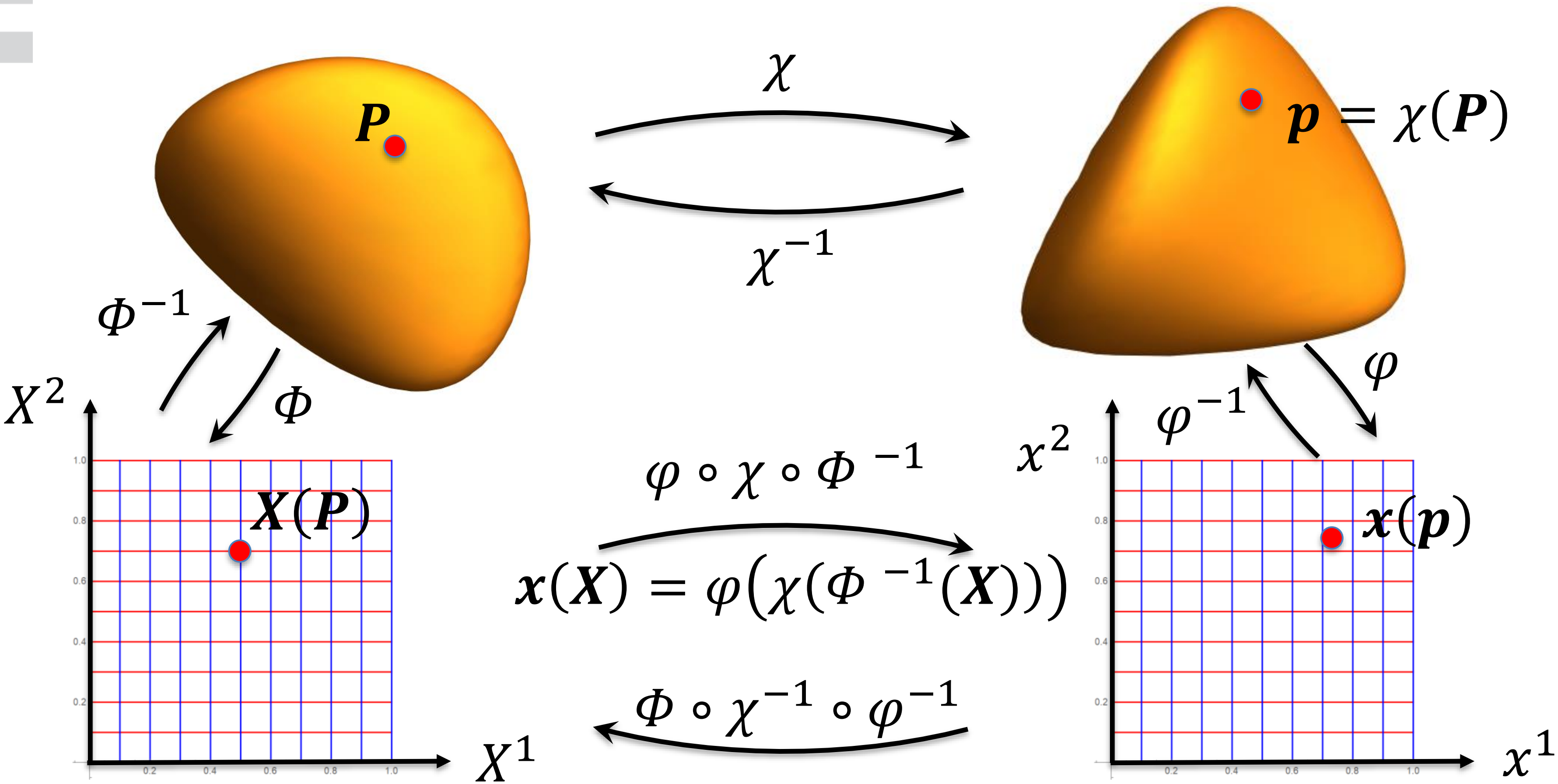
- all the above metrics are Kronecker symbols
- and all transformations leave the coefficients unchanged.

It therefore may seem futile to distinguish between these vectorial objects.

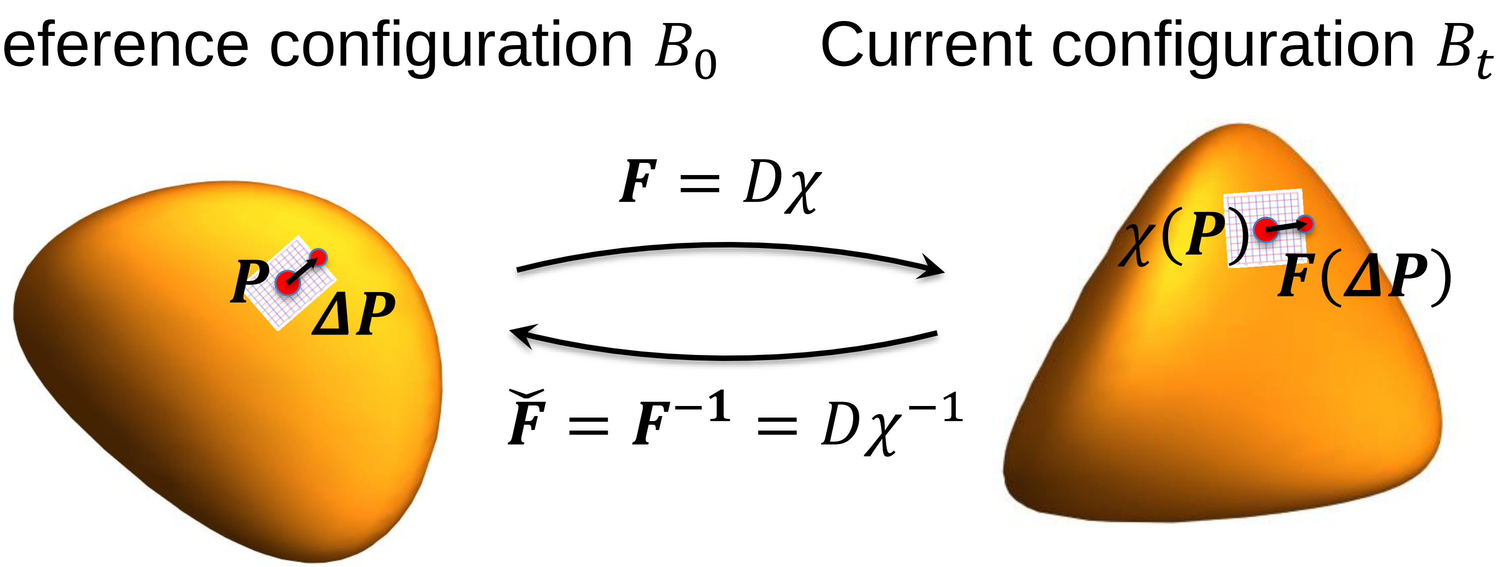
However, upon deformation (even in rate form) these different characteristics actually matter.

Deformation map and local coordinates

Reference configuration B_0 Current configuration B_t



Deformation gradient



The deformation gradient $\boldsymbol{F} = D\chi$ is the derivative of the deformation map. As a local linearization of χ the deformation gradient maps tangent vectors $\boldsymbol{\Delta P}$ at $\boldsymbol{P}$ to tangent vectors at $\boldsymbol{p}$

$$\chi(\boldsymbol{P} + \boldsymbol{\Delta P}) \approx \chi(\boldsymbol{P}) + \boldsymbol{F}(\boldsymbol{\Delta P})$$

"push forward"

Push-forward of base vectors

For the local base vectors we have

$$\left(\frac{\partial}{\partial X^I}\right)_* = \left(\frac{\partial \boldsymbol{P}}{\partial X^I}\right)_* := \frac{\partial \chi(\boldsymbol{P})}{\partial X^I} = \frac{\partial \boldsymbol{p}}{\partial x^i}\frac{\partial x^i}{\partial X^I} = \frac{\partial x^i}{\partial X^I}\frac{\partial}{\partial x^i} = F^i_I \frac{\partial}{\partial x^i},$$

where $\boldsymbol{F}$ is expressed in local coordinates by the coefficients

$$F^i_I = \frac{\partial x^i}{\partial X^I}.$$

Push-forward of tangent vectors

For a vector $\boldsymbol{V} = V^I \frac{\partial}{\partial X^I}$ we have

$$\boldsymbol{V}_* = \left(V^I \frac{\partial}{\partial X^I}\right)_* = V^I \left(\frac{\partial}{\partial X^I}\right)_* = V^I F^i_I \frac{\partial}{\partial x^i}.$$

With respect to the basis $\frac{\partial}{\partial x^i}$ on the current configuration, $\boldsymbol{V}_*$ has the coefficient functions $(\boldsymbol{V}_*)^i = F^i_I V^I$.

Remark: This is an actual transformation, mapping one object into another and not just a change of basis as regarded before.

Push-forward of basis differentials

The push-forward is obtained by a total differential

$(\mathrm{d}X^I)_* = \mathrm{d}X^I(\boldsymbol{p}) = \frac{\partial X^I}{\partial x^i}\mathrm{d}x^i(\boldsymbol{p}) = \check{F}_i^I \mathrm{d}x^i$

For the coefficients of one-forms $\boldsymbol{\alpha} = \alpha_I dX^I$ holds $(\alpha_*)_i = \check{F}_i^I \alpha_I$.

We see that also upon deformation, one-form-coefficients transform inversely to those of vector fields.

Summary push-forward

Object	$\boldsymbol{X}$	$\boldsymbol{x}$
scalar	f	$f_* = f$
vector field	$\boldsymbol{V} = V^I \dfrac{\partial}{\partial X^I}$	$(\boldsymbol{V}_*)^i = F^i_I V^I$
one-form	$\boldsymbol{\alpha} = \alpha_I \mathrm{d}X^I$	$(\boldsymbol{\alpha}_*)_i = \check{F}^I_i \alpha_I$
bi-vector	$\boldsymbol{D} = D_I \boldsymbol{B}^I$	$(\boldsymbol{D}_*)_i = J\check{F}^I_i D_I$
two-form	$\boldsymbol{Z} = Z^I dA_I$	$(\boldsymbol{Z}_*)^i = J^{-1} F^i_I Z^I$
tri-vector	$\boldsymbol{T} = T_0 \boldsymbol{\Theta}^{\Phi}$	$T = JT_0$
three-form	$\boldsymbol{\rho} = \rho_0 dV_{\Phi}$	$\rho = J^{-1}\rho_0$

Where $J = \det \boldsymbol{F}$

Summary pull-back

Object	$\boldsymbol{X}$	$\boldsymbol{x}$
scalar	$f^* = f$	f
vector field	$(\boldsymbol{v}^*)^I = \check{F}_i^I v^i$	$\boldsymbol{v} = v^i \frac{\partial}{\partial x^i}$
one-form	$(\boldsymbol{\alpha}^*)_I = F_I^i \alpha_i$	$\boldsymbol{\alpha} = \alpha_i \mathrm{d}x^i$
bi-vector	$(\boldsymbol{d}^*)_I = J^{-1} F_I^i d_i$	$\boldsymbol{d} = d_i \boldsymbol{b}^i$
two-form	$(\boldsymbol{\zeta}^*)^I = J \check{F}_i^I \zeta^i$	$\boldsymbol{\zeta} = \zeta^i da_i$
tri-vector	$T_0 = J^{-1} T$	$\boldsymbol{T} = T \boldsymbol{\Theta}^\varphi$
three-form	$\rho_0 = J\rho$	$\boldsymbol{\rho} = \rho dv_\varphi$

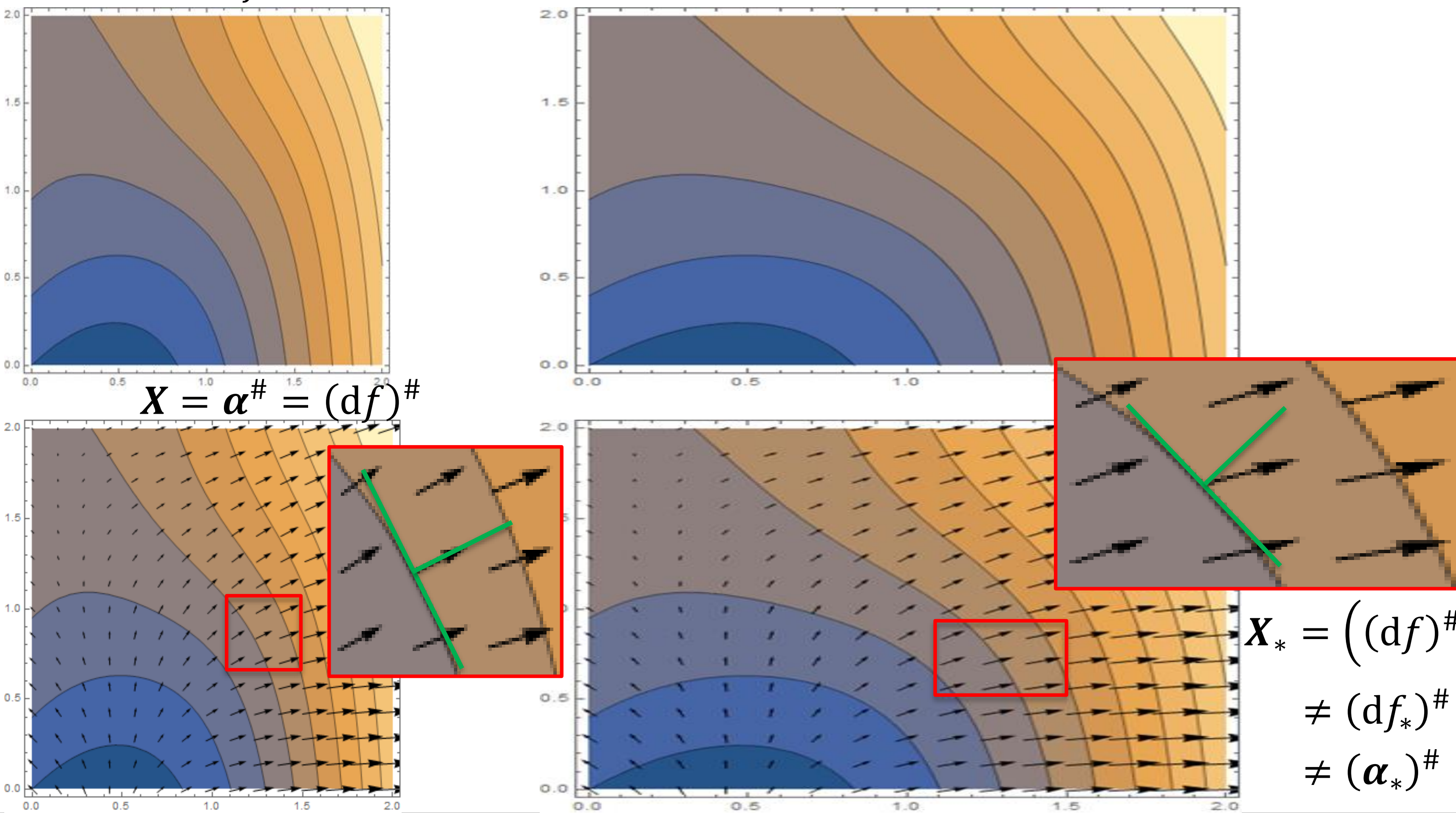
Where $J = \det \boldsymbol{F}$

Transformation between objects does not commute with deformation

$\boldsymbol{\alpha} = \mathrm{d}f$

$\boldsymbol{\alpha}_* = (\mathrm{d}f)_* = \mathrm{d}f_*$

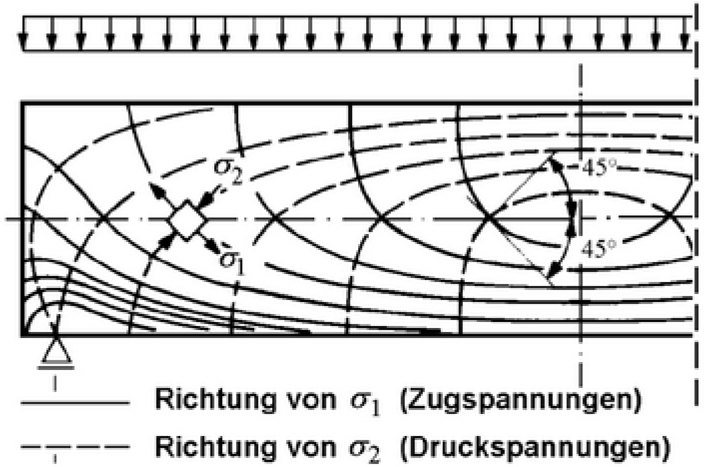
$\boldsymbol{X} = \boldsymbol{\alpha}^\# = (\mathrm{d}f)^\#$



$$\boldsymbol{X}_* = \left((\mathrm{d}f)^\#\right)_* \neq (\mathrm{d}f_*)^\# \neq (\boldsymbol{\alpha}_*)^\#$$

Cauchy stress tensors

https://baustatik-wiki.fiw.hs-wismar.de/



Arguably, the nature of the (Cauchy) stress tensor, assigning a force (one-form) to surface-elements, is a one-form-valued two-form,

$$\bar{\boldsymbol{\sigma}} = \sigma_i^j \mathrm{d}x^i \otimes da_j .$$

However, commonly we work with contravariant stress tensors

$$\boldsymbol{\sigma} = \sigma^{ij} \frac{\partial}{\partial x^i} \otimes da_j .$$

The coefficients are related by $\sigma^{ij} = \breve{g}^{ik}\sigma_k^j$ and $\sigma_i^j = g_{ik}\sigma^{kj}$.

In Cartesian coordinates the coefficients remain unchanged!

Pull-back to reference configurations

These seemingly „equal“ objects on the current configurations map to different stress tensors on the reference configuration:

The contravariant tensor maps to the 2nd Piola-Kirchhoff-Tensor

$$\boldsymbol{S} = J\check{F}_i^I \check{F}_j^J \sigma^{ij} \frac{\partial}{\partial X^I} \otimes dA_J,$$

while the mixed-variant tensor maps to the Mandel stress tensor

$$\boldsymbol{M} = JF_I^i \check{F}_j^J \sigma_i^j \mathrm{d}X^I \otimes dA_J.$$

Note that

$$M_I^J = JF_I^i \check{F}_j^J g_{ik} \sigma^{kj} = JF_I^i \check{F}_j^J g_{il} F_K^l \check{F}_k^K \sigma^{kj} = g_{il} F_I^i F_K^l J \check{F}_k^K \check{F}_j^J \sigma^{kj} = C_{IK} S^{KJ},$$

with the right Cauchy-Green-Tensor $C_{IJ} = g_{ij} F_I^i F_J^j$.

Conjugate variance in Hyperelasticity

In Hyperelasticity we postulate the existence of an energy density $\boldsymbol{\psi} = \psi(\boldsymbol{\epsilon}) dV_\Phi$ solely depending on a strain or deformation tensor $\boldsymbol{\epsilon}$.

The conjugate stress $\boldsymbol{\Sigma}$ tensor to $\boldsymbol{\epsilon}$ is obtained as

$$\boldsymbol{\Sigma} = \frac{\mathrm{d}\psi(\boldsymbol{\epsilon})}{\mathrm{d}\boldsymbol{\epsilon}}.$$

The stress tensor is a **globally** dual object to $\boldsymbol{\epsilon}$ – i.e. it has opposite index positions and upon contracting all indices it must result a three-form coefficient.

Therefore, the conjugate stress to C_{IJ} is contravariant with one upper index being a two-form index.

Hyperelasticity and Hypoelasticity

Hyperelasticity may be *equivalenty* formulated based on different strain or deformation measures, usually in conjunction with the conjugate stress measure.

While the variance-types matter for the resulting equations, the physics remains the same.

This is different in *hypoelasticity*, where the constitutive laws connect rates of strains with stresses or rates of stresses, e.g.

$\overset{?}{\sigma}^{ij} = c^{ijkl} d_{kl}$, where $\boldsymbol{d}$ is the rate of deformation tensor.

Objective rates

When connecting rates of changes of quantities on the current configuration, there arises a difficulty in that the total rate of change of a tensor fails to be a tensor in that it shows the wrong transformation behavior upon changing the coordinate system.

One therefore introduced so-called objective rates, which transform like tensors,

$$\overset{\circ}{\sigma}^{ij} = c^{ijkl} d_{kl}$$

some objective rate

When variance changes physics

However, even hypoelastic laws based on objective rates suffer from *history dependence* of the results and are no longer used much in solid mechanics.

This is different in fluid mechanics, where a Lagrangian description is not an option.

However, in descriptions based on objective rates, the „choice“ of variance of stress and strain actually yields different constitutive equations.

Objective rates in fluid mechanics

Maybe the most famous occurrence of objective rates in fluid dynamics are the models A and B devised by Oldroyd for describing visco-elastic fluids,

Oldroyd A: $\boldsymbol{\sigma} + \tau_1 \overset{\Delta}{\boldsymbol{\sigma}} = 2\mu^*(\boldsymbol{d} + \tau_2 \overset{\Delta}{\boldsymbol{d}})$ „lower convected rate“

Oldroyd B: $\boldsymbol{\sigma} + \tau_1 \overset{\nabla}{\boldsymbol{\sigma}} = 2\mu^*(\boldsymbol{d} + \tau_2 \overset{\nabla}{\boldsymbol{d}})$ „upper convected rate“

Lie-derivative

Many objective rates are so-called Lie-derivatives.
In non-linear solid mechanics the Lie-derivative of tensorial objects $\boldsymbol{T}$ when the body moves with velocity $\boldsymbol{v}$ may be defined via pull-back χ^* and push-forward χ^* as

$$L_{\boldsymbol{v}}\boldsymbol{T} = \chi_*\left(\dot{\overline{\chi^*(\boldsymbol{T})}}\right).$$

For instance, the rate of deformation tensor $\boldsymbol{d} = \frac{1}{2}\chi_*(\dot{\boldsymbol{C}})$ is up to the factor ½ the Lie-derivative of the metric

$\boldsymbol{d} = \frac{1}{2}\chi_*(\dot{\boldsymbol{C}}) = \frac{1}{2}\chi_*\left(\dot{\overline{\chi^*(\boldsymbol{g})}}\right) = \frac{1}{2}L_{\boldsymbol{v}}\boldsymbol{g}$, since $\boldsymbol{C} = \chi^*(\boldsymbol{g})$.

The difference btw. upper and lower convected rates

„Extra“ terms occuring in Lie-derivatives of objects with upper indices follow from

$$L_{\boldsymbol{v}} \frac{\partial}{\partial x^i} = -\frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial x^j},$$

while in the case of lower indices we have

$$\mathcal{L}_{\boldsymbol{v}} \mathrm{d}x^i = \frac{\partial v^i}{\partial x^j} \mathrm{d}x^j.$$

For example, the coefficients of the rate-of-deformation tensor are in standard coordinates

$$\boldsymbol{d}_{ij} = \frac{1}{2}\left(\frac{\partial v^k}{\partial x^i} g_{kj} + \frac{\partial v^k}{\partial x^j} g_{ki}\right).$$

The Truesdell rate

The Lie-derivative of the contravariant stress tensor,

$$\left(\mathcal{L}_{\boldsymbol{v}}\boldsymbol{\sigma}(\boldsymbol{x})\right)^{ij} = \dot{\sigma}^{ij} - \frac{\partial v^i}{\partial x^k}\sigma^{kj} - \frac{\partial v^i}{\partial x^k}\sigma^{il} + \frac{\partial v^k}{\partial x^k}\sigma^{ij},$$

is known as the Truesdell rate.

Why the variance matters in rate form

The variance matters, when the change of type is done by raising or lowering indices with the standard metric.

If for simplicity the rates of two vectors are connected through

$L_{\boldsymbol{v}}\boldsymbol{X} - \mu L_{\boldsymbol{v}}\boldsymbol{Y} = \boldsymbol{0}$ (1), i.e. with „upper convected rates“,

how does this relation differ from

$L_{\boldsymbol{v}}\boldsymbol{X}^{\flat} - \mu L_{\boldsymbol{v}}\boldsymbol{Y}^{\flat} = \boldsymbol{0}$ (2), i.e., with „lower convected rates“?

We have

$$L_{\boldsymbol{v}}\boldsymbol{Y}^{\flat} = L_{\boldsymbol{v}}(\boldsymbol{g}\boldsymbol{Y}) = L_{\boldsymbol{v}}\boldsymbol{g}\,\boldsymbol{Y} + \boldsymbol{g}L_{\boldsymbol{v}}\boldsymbol{Y} = 2\boldsymbol{d}\boldsymbol{Y} + \boldsymbol{g}L_{\boldsymbol{v}}\boldsymbol{Y},$$

$$L_{\boldsymbol{v}}\boldsymbol{X}^{\flat} = L_{\boldsymbol{v}}(\boldsymbol{g}\boldsymbol{X}) = L_{\boldsymbol{v}}\boldsymbol{g}\,\boldsymbol{X} + \boldsymbol{g}L_{\boldsymbol{v}}\boldsymbol{X} = 2\boldsymbol{d}\boldsymbol{X} + \boldsymbol{g}L_{\boldsymbol{v}}\boldsymbol{X}.$$

Accordingly, we obtain from (1)

$$L_{\boldsymbol{v}}\boldsymbol{X}^{\flat} - \mu L_{\boldsymbol{v}}\boldsymbol{Y}^{\flat} = 2\boldsymbol{d}(\boldsymbol{X} - \mu\boldsymbol{Y}).$$

Summary

The variance of physical fields is inherent to the fields‘ physical/geometrical identity.

Changing the variance

- changes the physical identity of a field,
- does not commute with push-forwards and pull-backs,
- does not commute with Lie-derivatives.

Hyperelasticity may be formulated equivalently based on different variances.

In elastic fluids, the right variance may depend on shape and characterisitc of elastic particles.