Analysis of vector fields via Helmholtz decomposition

Manfred Kaltenbacher
Stefan Schoder, Klaus Roppert

WISSEN
TECHNIK
LEIDENSCHAFT

manfred.kaltenbacher@tugraz.at

Content

- Helmholtz theorem
  - Unbounded domains
  - Bounded domains
- Maxwell's equations
  - Physics of the equations
  - Submodels and role of potentials
- Fluid dynamics
  - Models for aeroacoustics
  - Postprocessing of compressible flows

Helmholtz‘s decomposition: original

- **Theorem**

  Helmholtz's decomposition (original): Every vector field $\boldsymbol{u}$, $\mathcal{C}^1$ *smooth, on a simply connected domain* $\Omega \in \mathbb{R}^3$ *with the property* $\lim_{r\to\infty} \boldsymbol{u}(r) r = \boldsymbol{0}$ *of a radial coordinate* $r = ||\boldsymbol{x}||_2$ *with* $\boldsymbol{x} \in \Omega$ *, can be decomposed in* L_2*-orthogonal velocity field components*

  $$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_\mathrm{v} + \boldsymbol{u}_\mathrm{c} = \nabla \times \boldsymbol{A}_\mathrm{v}(\boldsymbol{x}) + \nabla \phi_\mathrm{c}(\boldsymbol{x}) \,,$$

  with the vector potential $\boldsymbol{A}_\mathrm{v}$ *satisfying* $\nabla \cdot \boldsymbol{A}_\mathrm{v} = 0$ *(toroidal component of the vector potential) and the scalar potential* ϕ_c.

- Scalar potential

$$\phi_\mathrm{c}(\boldsymbol{x}) = -\int_\Omega \frac{\nabla' \cdot \boldsymbol{u}}{4\pi ||\boldsymbol{x} - \boldsymbol{x}'||_2} \, \mathrm{d}\boldsymbol{x}'$$

- Vector potential

$$\boldsymbol{A}_\mathrm{v}(\boldsymbol{x}) = \int_\Omega \frac{\nabla' \times \boldsymbol{u}}{4\pi ||\boldsymbol{x} - \boldsymbol{x}'||_2} \, \mathrm{d}\boldsymbol{x}'$$

Helmholtz's decomposition: modified

- **Theorem**

  Helmholtz's decomposition (modified): Every square integrable vector field $\boldsymbol{u} \in [L_2(\Omega)]^3$, $\mathcal{C}^1$ *smooth, on a simply connected, Lipschitz domain* $\Omega \subseteq \mathbb{R}^3$, *has an* L_2*-orthogonal decomposition*

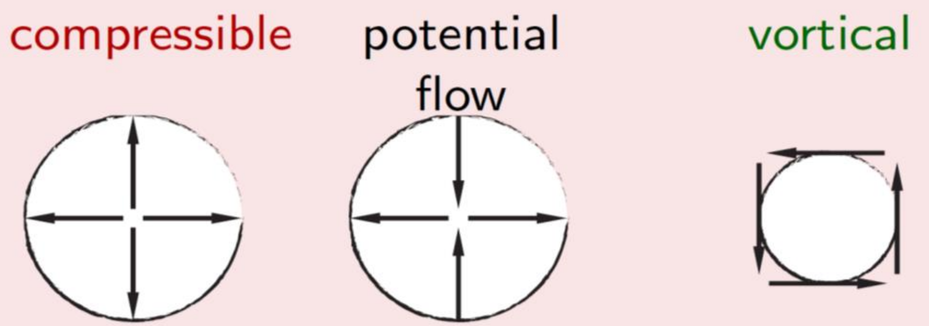
  $$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{v}} + \boldsymbol{u}_{\mathrm{c}} = \nabla \times \boldsymbol{A}_{\mathrm{v}} + \nabla \phi_{\mathrm{c}} \,,$$

  with the vector potential $\boldsymbol{A}_{\mathrm{v}} \in H(\mathrm{curl}, \Omega)$ *and the scalar potential* $\phi_{\mathrm{c}} \in H^1(\Omega)$.

- Decomposition on a restricted domain

What is the price of modifying the theorem w.r.t. the calculation?

- Third additive part: potential flow solution, boundary imposed solution



G. K. Batchelor. An Introduction to Fluid Dynamics. Cambridge Mathematical Library. Cambridge, University Press, 2000.

Helmholtz's decomposition: modified

- **Helmholtz decomposition**

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{c}} + \boldsymbol{u}_{\mathrm{h}} + \boldsymbol{u}_{\mathrm{v}} \\ &= \nabla \phi_{\mathrm{c}} + \boldsymbol{u}_{\mathrm{h}} + \nabla \times \boldsymbol{A}_{\mathrm{v}} \end{aligned}$$

Potential flow: Space of harmonic fields, curl and divergence free!

- Computation

Scalar potential

$$\begin{aligned} \boldsymbol{u} &= \nabla \phi_{\mathrm{c}} + \boldsymbol{u}_{\mathrm{h}} + \nabla \times \boldsymbol{A}_{\mathrm{v}} \quad | \nabla \cdot \\ \nabla \cdot \boldsymbol{u} &= \Delta \phi_{\mathrm{c}}^* \end{aligned}$$

- Scalar unknown → (one component):

$$\Delta \phi_{\mathrm{c}}^* = \nabla \cdot \boldsymbol{u}$$

Vector potential

$$\begin{aligned} \boldsymbol{u} &= \nabla \phi_{\mathrm{c}} + \boldsymbol{u}_{\mathrm{h}} + \nabla \times \boldsymbol{A}_{\mathrm{v}} \quad | \nabla \times \\ \nabla \times \boldsymbol{u} &= \nabla \times \nabla \times \boldsymbol{A}_{\mathrm{v}}^* \end{aligned}$$

- Vector unknown → (three components):

$$\nabla \times \nabla \times \boldsymbol{A}_{\mathrm{v}}^* = \nabla \times \boldsymbol{u}$$

Helmholtz‘s decomposition: modified

- **Why are the potentials stars? (Simple example)**
  - Velocity field: $\boldsymbol{u} = (u_0 + \sin(\pi x))\boldsymbol{e}_x + x\boldsymbol{e}_y$ on $\Omega = [0,2]^2$

- Scalar and vector potentials

Scalar potential

$$\Delta \phi_c^* = \nabla \cdot \boldsymbol{u} = \pi \cos(\pi x)$$

Homogeneous solution (*potential flow*):

$$\Delta \phi_h = 0$$

Scalar potential	Potential flow
$\phi_c^* = -\frac{1}{\pi}\cos(\pi x) + u_0 x + b$	$\phi_h = u_0 x + b$
$\boldsymbol{u}_c^* = (u_0 + \sin(\pi x))\boldsymbol{e}_x$	$\boldsymbol{u}_h = u_0 \boldsymbol{e}_x$

Vector potential

$$\nabla \times \nabla \times \boldsymbol{A}_v^* = \nabla \times \boldsymbol{u} = -\boldsymbol{e}_z$$

Homogeneous solution (*potential flow*):

$$\nabla \times \nabla \times \boldsymbol{A}_h = \boldsymbol{0}$$

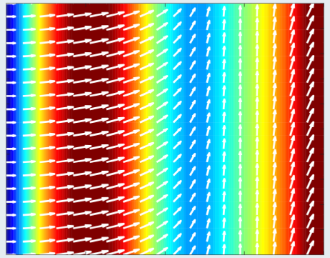
Vector potential	Potential flow
$A_{v,z}^* = u_0 y - \frac{1}{2}x^2 + b$	$A_{h,z} = u_0 y + b$
$\boldsymbol{u}_v^* = u_0 \boldsymbol{e}_x + x\boldsymbol{e}_y$	$\boldsymbol{u}_h = u_0 \boldsymbol{e}_x$

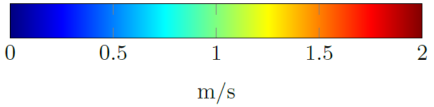
Helmholtz's decomposition: modified

- Vector field plots

Velocity field – example

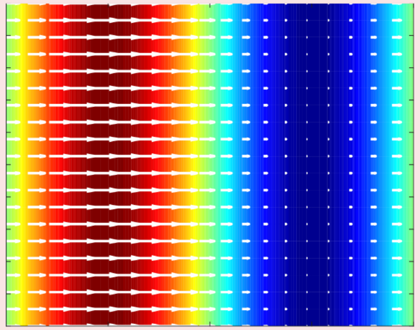
$$\boldsymbol{u} = (u_0 + \sin(\pi x))\boldsymbol{e}_x + x\boldsymbol{e}_y$$





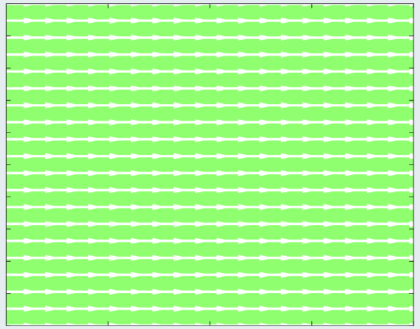
Scalar potential

$\boldsymbol{u}_{\mathrm{c}}^{*}$



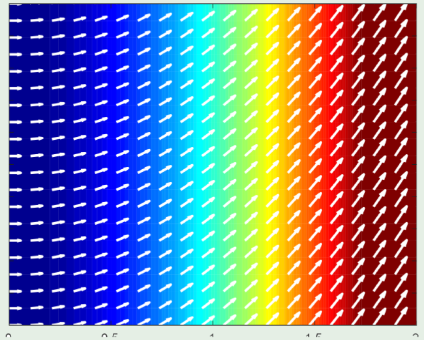
Potential flow

$\boldsymbol{u}_{\mathrm{h}}$



Vector potential

$\boldsymbol{u}_{\mathrm{v}}^{*}$



Helmholtz's decomposition: modified

- **Boundary conditions I – being orthogonal – partial integration**

$$(\boldsymbol{u}^{\mathrm{v}}, \boldsymbol{u}^{\mathrm{c}}) = \int_{\Omega} \boldsymbol{u}^{\mathrm{v}} \cdot \boldsymbol{u}^{\mathrm{c}} \mathrm{d}x = 0\,.$$

- To be forced

Scalar potential

$$(\boldsymbol{u}^{\mathrm{v}}, \boldsymbol{u}^{\mathrm{c}}) = \int_{\partial\Omega} \phi^{\mathrm{c}} (\boldsymbol{u} - \nabla \phi^{\mathrm{c}}) \cdot \boldsymbol{n} \mathrm{d}s = 0$$

→ Boundary condition

- Physics (*Low Mach number : Acoustics*)

Vector potential

$$(\boldsymbol{u}^{\mathrm{v}}, \boldsymbol{u}^{\mathrm{c}}) = \int_{\partial\Omega} \boldsymbol{A}^{\mathrm{v}} \cdot (\boldsymbol{u} - \nabla \times \boldsymbol{A}^{\mathrm{v}}) \times \boldsymbol{n} \mathrm{d}s = 0$$

→ Boundary condition

- Physics (*Vortical flow*)

Helmholtz's decomposition: modified

- **Boundary conditions II – regularity**
  - Is $\boldsymbol{u}_{\mathrm{c}} = \nabla\phi_{\mathrm{c}}(r,\theta)$ singular at the corner? It is possible.
  - Is $\boldsymbol{u}_{\mathrm{v}} = \nabla\times\boldsymbol{A}_{\mathrm{v}}(r,\theta)$ singular at the corner? No.

- Scalar and vector potentialsComputation

Scalar potential

$$\Delta\phi(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

- Wall boundary (*non-penetrating*)

→ Singular if $\Theta > \pi$: $\nabla\phi(r,\theta)\to\infty$ for $k=1$ and $r\to 0$.

Vector potential

$$\nabla\times\nabla\times A_z(r,\theta) = 0$$

- Wall boundary (*no-slip*)
- Global incompressibility

→ *No singularity* in the physical space.

Maxwell‘s equations

- General form

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_\mathrm{f} + \frac{\partial \boldsymbol{D}}{\partial t} \quad \text{... Ampere with Maxwell correction}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{... Faraday}$$

$$\nabla \cdot \boldsymbol{D} = \rho_\mathrm{f} \quad \text{... Gauß}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{... introduced by Maxwell}$$

- Constitutive equations

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M}\right) = \boldsymbol{\mu} \boldsymbol{H}$$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \boldsymbol{\varepsilon} \boldsymbol{E}$$

$$\boldsymbol{J} = \gamma \boldsymbol{E}$$

$\boldsymbol{E}$ ... electric field intensity
$\boldsymbol{D}$ ... electric flux density
$\boldsymbol{H}$ ... magnetic field intensity
$\boldsymbol{B}$ ... magnetic flux density)
$\boldsymbol{J}_\mathrm{f}$ ... free current density
ρ_f ... free charge density
$\boldsymbol{M}$ ... magnetization
$\boldsymbol{P}$ ... electric polarization
$\boldsymbol{\mu}$ ... magnetic permeability
$\boldsymbol{\varepsilon}$ ... electric permittivity
γ ... electric conductivity

Maxwell‘s equations: Electro-quasistatic case

- Subset of Maxwell’s equation

$$\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{J}_\mathrm{f} + \frac{\partial \boldsymbol{D}}{\partial t} \\
\nabla \times \boldsymbol{E} &= 0 \quad \rightarrow \quad \text{fulfilled via } \boldsymbol{E} = -\nabla V \\
\nabla \cdot \boldsymbol{D} &= \rho_\mathrm{f} \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\boldsymbol{D} &= \varepsilon \boldsymbol{E} \\
\boldsymbol{J}_\mathrm{f} &= \gamma \boldsymbol{E}
\end{aligned}$$

- Applying the divergence to the first equation results in

$$\nabla \cdot \nabla \times \boldsymbol{H} = 0 = \nabla \cdot \boldsymbol{J}_\mathrm{f} + \frac{\partial}{\partial t} \left(\nabla \cdot \boldsymbol{D} \right)$$

$$\nabla \cdot \gamma \, \nabla V + \nabla \cdot \varepsilon \, \nabla \frac{\partial V}{\partial t} = 0$$

Maxwell's equations: Magneto-quasistatic case (Eddy current case)

- Subset of Maxwell's equations

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{J}_\mathrm{f} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \quad \rightarrow \quad \text{fulfilled via } \boldsymbol{B} = \nabla \times \boldsymbol{A} \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\boldsymbol{B} &= \mu \boldsymbol{H} \\
\boldsymbol{J}_\mathrm{f} &= \gamma \boldsymbol{E}
\end{aligned}
$$

- We perform the following Helmholtz decomposition
  - Electric field

$$\boldsymbol{E}_\mathrm{total} = \boldsymbol{E}_\mathrm{i} + \boldsymbol{E}_\mathrm{s}$$

$\boldsymbol{E}_\mathrm{i}$ ... irrotational (longititinal) part

$\boldsymbol{E}_\mathrm{s}$ ... solenoidal (vortical) part

  - Electric current

$$\boldsymbol{J}_\mathrm{f} = \boldsymbol{J}_\mathrm{p} + \gamma \big(\boldsymbol{E}_\mathrm{s} + \underbrace{\boldsymbol{v} \times \boldsymbol{B}}_{\text{motional emf term}} \big)$$

$\boldsymbol{J}_\mathrm{p}$ ... prescribed current density (e.g., current loaded coil)

$\boldsymbol{v}$ ... mechanical velocity

Maxwell‘s equations: Magneto-quasistatic case (Eddy current case)

- Substitution into the second Maxwell’s equation

$$\nabla \times \boldsymbol{E}_{\mathrm{total}} \quad = \quad \nabla \times \boldsymbol{E}_{\mathrm{s}} = -\frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{A}\right)$$

$$\nabla \times \left(\boldsymbol{E}_{\mathrm{s}} + \frac{\partial \boldsymbol{A}}{\partial t}\right) \quad = \quad 0 \quad \rightarrow \quad \boldsymbol{E}_{\mathrm{s}} = -\frac{\partial \boldsymbol{A}}{\partial t}$$

- Substitution into first Maxwell’s equation results in the final set of equations

$$\nabla \times \nu \nabla \times \boldsymbol{A} \quad = \quad \boldsymbol{J}_{\mathrm{p}} - \gamma \frac{\partial \boldsymbol{A}}{\partial t} + \gamma\left(\boldsymbol{v} \times \nabla \times \boldsymbol{A}\right)$$

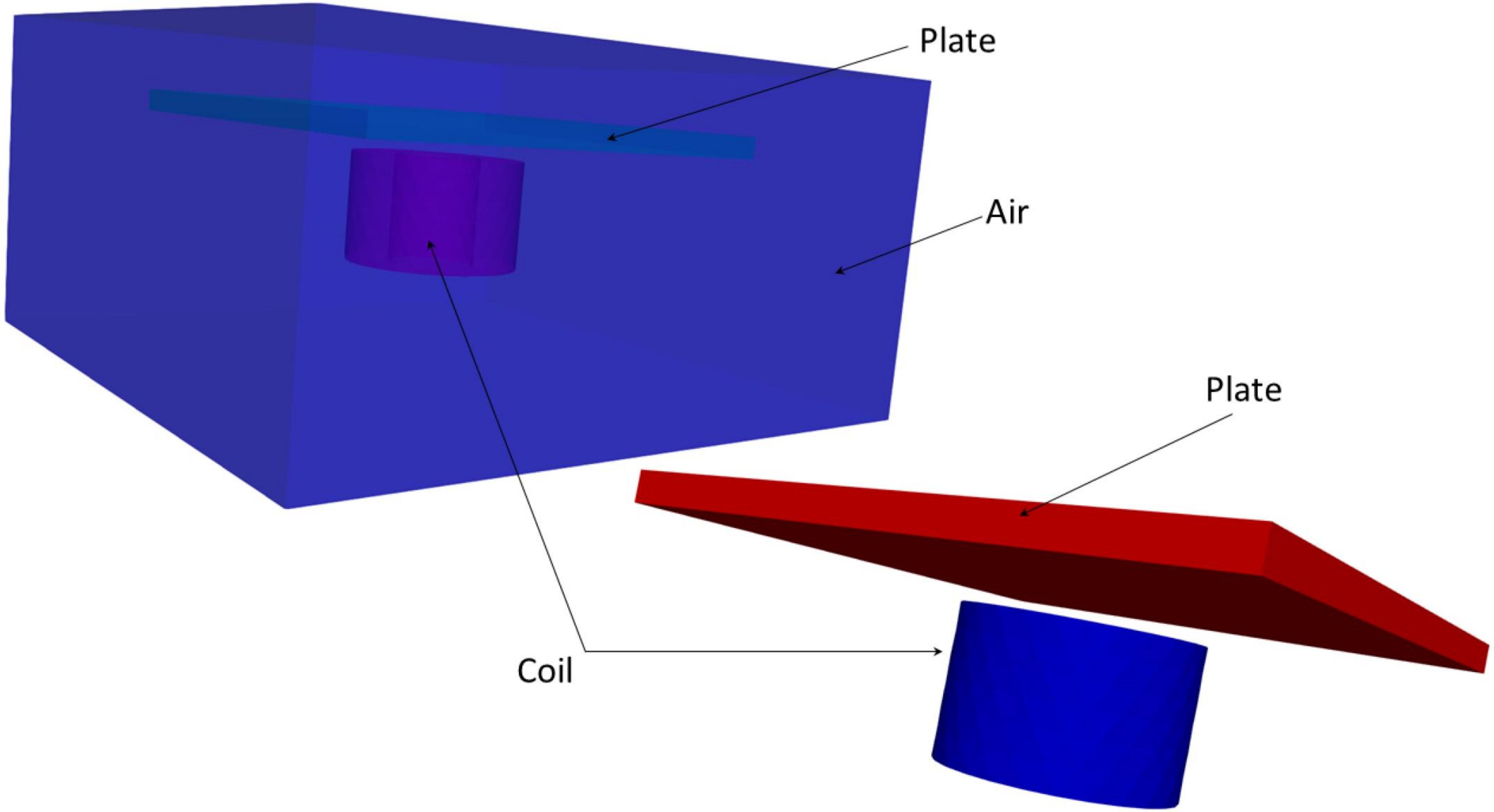
$$\nabla \cdot \boldsymbol{A} \quad = \quad 0$$

← Coulomb gauging: do we need it?

- Describes all resistive and inductive effects

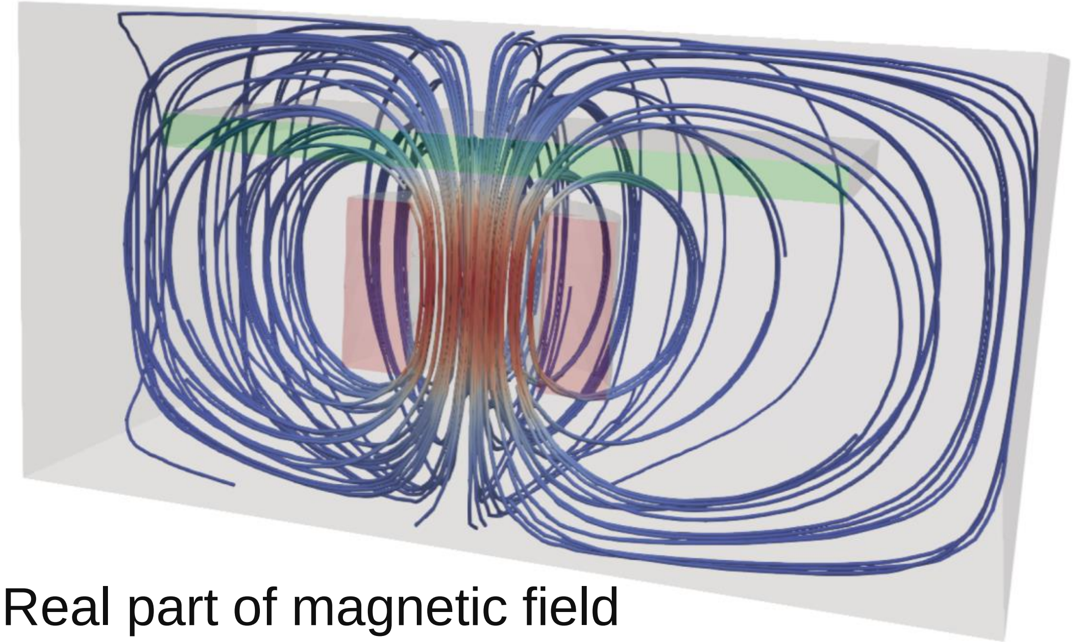
Maxwell‘s equations: Magneto-quasistatic case (Eddy current case)

- Example: conductive plate

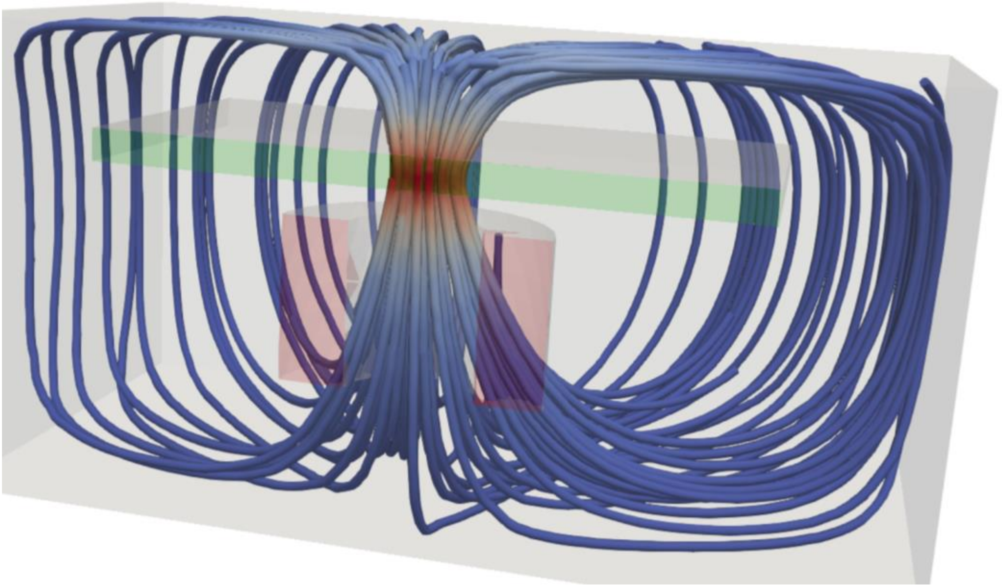


Maxwell's equations: Magneto-quasistatic case (Eddy current case)

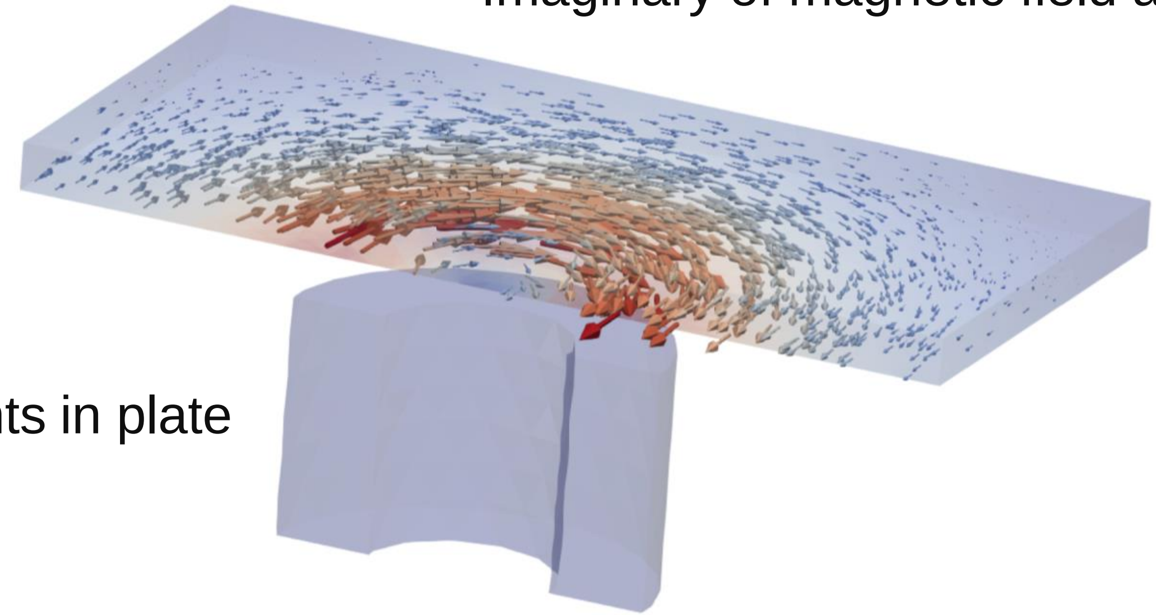
- Example: conductive plate



Real part of magnetic field as tubes



Imaginary of magnetic field as tubes



Eddy currents in plate

Maxwell‘s equations: Darwin‘s equations

- Subset of Maxwell’s equations

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{J}_\mathrm{f} + \frac{\partial \boldsymbol{D}}{\partial t} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \cdot \boldsymbol{D} &= \rho_\mathrm{f} \\
\nabla \cdot \boldsymbol{B} &= 0 \quad \rightarrow \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}\,; \quad \nabla \cdot \boldsymbol{A} = 0 \\
\boldsymbol{B} &= \mu \boldsymbol{H}\,; \quad \boldsymbol{J} = \gamma \boldsymbol{E} \\
\boldsymbol{D} &= \varepsilon \boldsymbol{E}
\end{aligned}
$$

- Helmholtz decomposition of electric field

$$
\boldsymbol{E} = \boldsymbol{E}_\mathrm{i} + \boldsymbol{E}_\mathrm{s} = -\nabla V + \boldsymbol{E}_\mathrm{s}
$$

i ... irrotational

s ... solenoidal

- Use of second Maxwell's equation

$$
\nabla \times \boldsymbol{E} = \nabla \times \boldsymbol{E}_\mathrm{s} = -\frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{A}\right) \quad \rightarrow \quad \boldsymbol{E}_\mathrm{s} = -\frac{\partial \boldsymbol{A}}{\partial t}
$$

Maxwell‘s equations: Darwin‘s equations

- Approximation: just irrotational part of displacement current density is used!

$$\begin{aligned}\nabla \times \nu \nabla \times \boldsymbol{A} &= \boldsymbol{J}_{\mathrm{p}} + \gamma \boldsymbol{E} + \frac{\partial}{\partial t}\left(-\varepsilon \nabla V - \varepsilon \frac{\partial \boldsymbol{A}}{\partial t}\right) \\ &= \boldsymbol{J}_{\mathrm{p}} - \gamma \nabla V - \gamma \frac{\partial \boldsymbol{A}}{\partial t} - \varepsilon \frac{\partial}{\partial t}(\nabla V) - \underbrace{\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2}}_{\approx 0}\end{aligned}$$

$$\nabla \cdot \varepsilon \boldsymbol{E} = -\nabla \cdot \varepsilon \nabla V - \nabla \cdot \varepsilon \frac{\partial \boldsymbol{A}}{\partial t} = \rho_{\mathrm{f}} \quad (*)$$

- Use conservation equation of charges (continuity equation)

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho_{\mathrm{f}}}{\partial t} = 0$$

and use it in $(*)$!

Maxwell‘s equations: Darwin‘s equations

- We arrive at

$$\nabla\cdot\gamma\,\nabla V + \nabla\cdot\gamma\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\cdot\varepsilon\,\nabla\frac{\partial V}{\partial t} + \underbrace{\nabla\cdot\varepsilon\frac{\partial^2 \boldsymbol{A}}{\partial t^2}}_{\approx 0} = 0$$

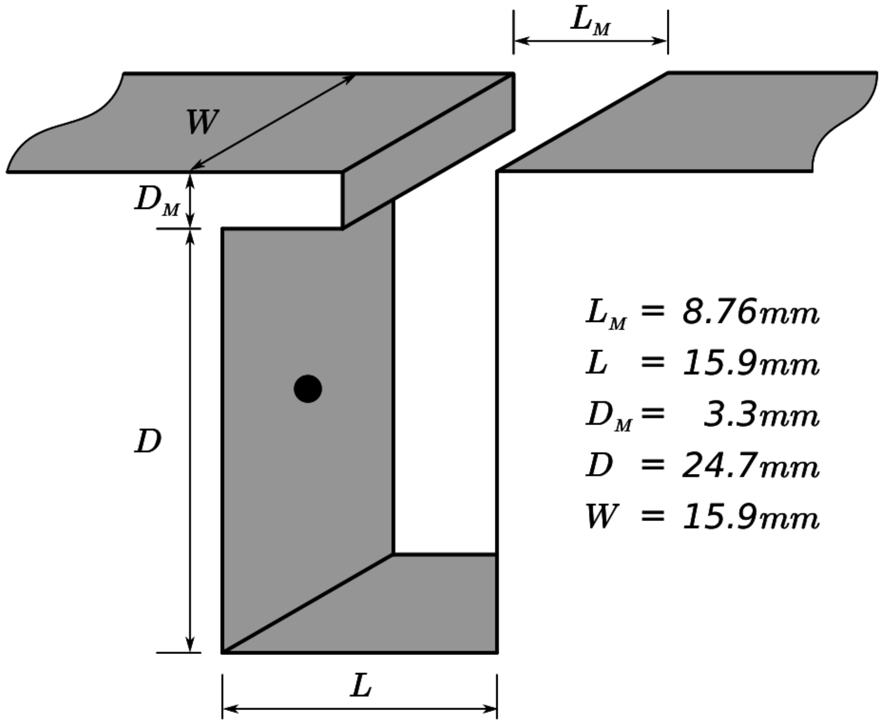
- Final set of equations

$$\begin{aligned}
\nabla\times\nu\,\nabla\times\boldsymbol{A} + \gamma\,\nabla V + \gamma\frac{\partial \boldsymbol{A}}{\partial t} + \varepsilon\frac{\partial}{\partial t}\left(\nabla V\right) &= \boldsymbol{J}_{\mathrm{p}}\\
\nabla\cdot\gamma\,\nabla V + \nabla\cdot\gamma\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\cdot\varepsilon\,\nabla\frac{\partial V}{\partial t} &= 0\\
\nabla\cdot\boldsymbol{A} &= 0
\end{aligned}$$

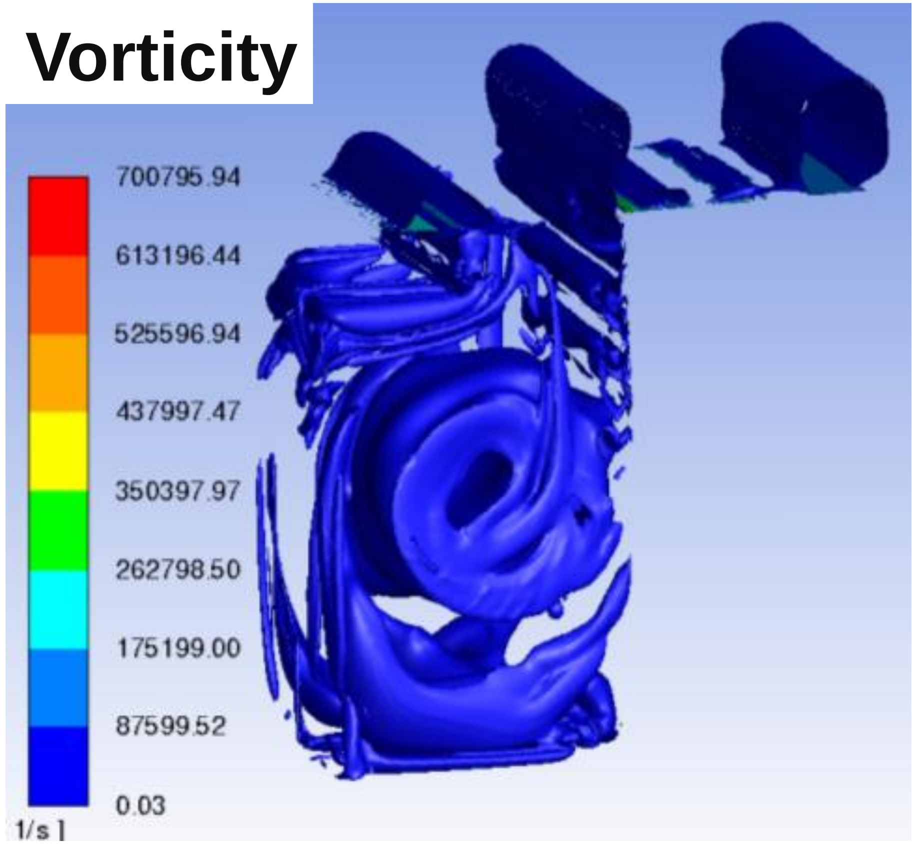
Coulomb gauging

Allows to model resistive, inductive and capacitive effects; no wave effects are considered!

- Cavity with a lip

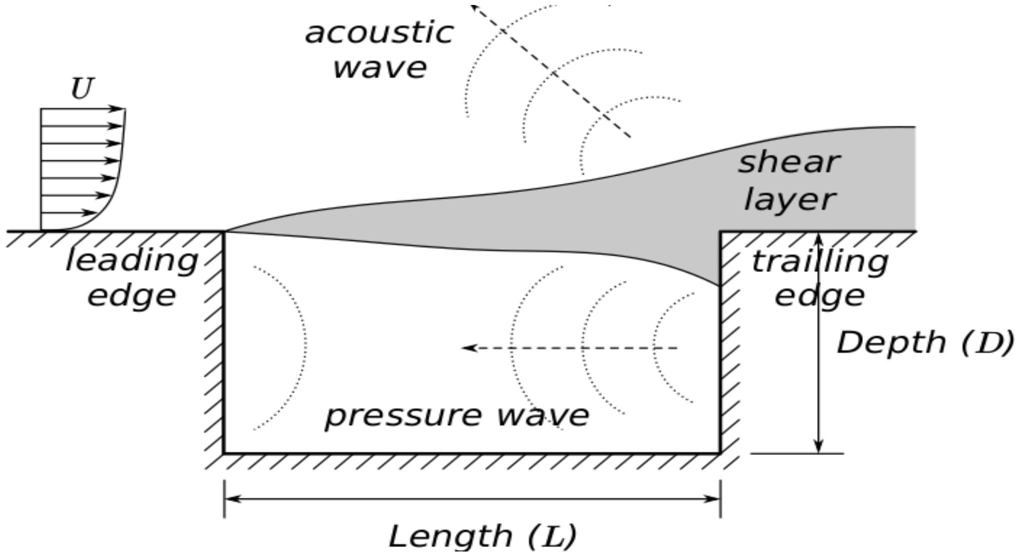


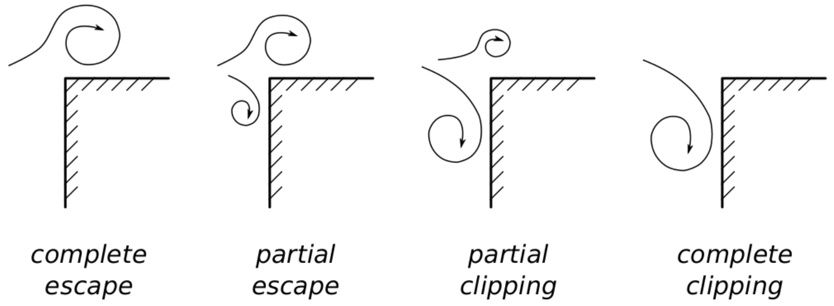
Third Computational Aeroacoustics (CAA) Workshop on Benchmark Problems – Category 6., NASA, vol. CP-2000-209790, Cleveland, Ohio, Nov. 1999.



Aeroacoustics

- Shear layer modes (Rossiter modes)
  - Boundary layer separates at the leading edge and forms a shear layer
  - At appropriate flow conditions, the shear layer gets unstable and Kelvin-Helmholtz instabilities are convected towards the trailing edge
  - Interaction with the trailing edge generates acoustic waves, which generate new instabilities at the leading edge.





$$St_n = \frac{n - \alpha}{U_\infty / U_c + \mathrm{Ma}}$$

St	...	Strouhal number
n	...	Mode number
α	...	Phase lag
U_∞	...	Free stream velocity
U_c	...	Convective velocity

Aeroacoustics: Fundamental Equations

- Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

- Conservation of momentum

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f}$$

- Conservation of energy

$$\frac{\partial \rho e_{\mathrm{T}}}{\partial t} + \nabla \cdot (\rho e_{\mathrm{T}} \boldsymbol{v}) + \nabla \cdot (p \boldsymbol{v}) = q_{\mathrm{h}} - \nabla \cdot \boldsymbol{q}_{\mathrm{T}} + \nabla \cdot ([\boldsymbol{\tau}] \cdot \boldsymbol{v}) + \boldsymbol{f} \cdot \boldsymbol{v}\,; \quad e_{\mathrm{T}} = e + v^2/2$$

$\boldsymbol{v}$ ... velocity
ρ ... density
p ... pressure
$[\boldsymbol{\tau}]$ ... viscous tensor
$\boldsymbol{f}$ ... body force
e ... inner energy
q_{h} ... heat source
$\boldsymbol{q}_{\mathrm{T}}$ ... heat flux

- Constitutive equation
  - Viscous tensor: Newtonian fluid

$$[\boldsymbol{\tau}] = \mu\left(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}\right) + \left((\lambda - \frac{2\mu}{3}\right) \nabla \cdot \boldsymbol{v}\, [\boldsymbol{I}]$$

  - Pressure – density relation

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \mathrm{d}s \quad \xrightarrow{\mathrm{d}s = 0} \quad c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

... insentropic speed of sound

μ ... dynamic viscosity
λ ... bulk viscosity
$[\boldsymbol{I}]$ ... unit tensor
s ... specific entropy

Aeroacoustics: Linear Wave equation

- Perturbation ansatz

$$p = p_0 + p_{\mathrm{a}}\,; \quad \rho = \rho_0 + \rho_{\mathrm{a}}\,; \quad \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_{\mathrm{a}}\,; \qquad ||p_{\mathrm{a}}|| \ll ||p_0||\,;\ ||\rho_{\mathrm{a}}|| \ll ||\rho_0||\,;\ \boldsymbol{v}_0 = \boldsymbol{0}$$

- Linearized mass and momentum equations for acoustics

$$\begin{aligned} \frac{\partial \rho_{\mathrm{a}}}{\partial t} + \rho_0 \nabla \boldsymbol{v}_{\mathrm{a}} &= 0 \\ \rho_0 \frac{\partial \boldsymbol{v}_{\mathrm{a}}}{\partial t} + \nabla p_{\mathrm{a}} &= \boldsymbol{0} \end{aligned}$$

$$\xrightarrow{p_{\mathrm{a}} = c_0^2 \rho_{\mathrm{a}}}$$

$$\begin{aligned} \frac{1}{\rho_0 c_0^2} \frac{\partial p_{\mathrm{a}}}{\partial t} + \nabla \boldsymbol{v}_{\mathrm{a}} &= 0 \quad (I) \\ \frac{\partial \boldsymbol{v}_{\mathrm{a}}}{\partial t} + \frac{1}{\rho_0} \nabla p_{\mathrm{a}} &= \boldsymbol{0} \quad (II) \end{aligned}$$

- Acoustic wave equation:

$$\frac{\partial}{\partial t}(I) - \nabla \cdot \big(II\big) \longrightarrow \frac{1}{\rho_0 c_0^2} \frac{\partial^2 p_{\mathrm{a}}}{\partial t^2} - \nabla \cdot \frac{1}{\rho_0} \nabla p_{\mathrm{a}} = 0$$

ρ_0 ... no function of space

$$\frac{1}{c_0^2} \frac{\partial^2 p_{\mathrm{a}}}{\partial t^2} - \nabla \cdot \nabla p_{\mathrm{a}} = 0$$

Aeroacoustics: Formulation

- General acoustic formulation

$$\Box p' = \mathbf{RHS}(p, \boldsymbol{u}, \rho, \ldots)$$

wave operator

p'	...	radiating pressure
p	...	pressure
$\boldsymbol{v}$	...	velocity
ρ	...	density

- Lighthill's inhomogeneous wave equation

$$\frac{\partial^2(\rho - \rho_0)}{\partial t^2} - c_0^2 \nabla \cdot \nabla(\rho - \rho_0) = \nabla \cdot \nabla \cdot [\boldsymbol{T}]$$

$$[\boldsymbol{T}] = \rho \boldsymbol{u}\boldsymbol{u} + \Big((p - p_0) - c_0^2(\rho - \rho_0)\Big)[\boldsymbol{I}] - [\boldsymbol{\tau}]$$

c_0	...	speed of sound
ρ_0	...	mean density
p_0	...	mean pressure
$[\boldsymbol{I}]$	...	unit tensor
$[\boldsymbol{\tau}]$	...	viscous tensor

The right hand side of Lighthill's inhomogeneous wave equation contains not only source terms, but also nonlinear and interaction terms between the sound and flow field, which includes effects such as convection and refraction of the sound by the flow.

The whole set of compressible flow dynamics equations have to be solved in order to calculate the right hand side of Lighthill's wave equation .

- Perform a compressible flow simulation on a restricted computational domain
- Split overall flow quantities into a base flow (non-radiating) and remaining components (acoustic, radiating fluctuations)[1]

$$\star = \tilde{\star} + \star'$$

$$\Box p' = \mathbf{RHS}(\tilde{p}, \tilde{\boldsymbol{u}}, \tilde{\rho}, p', \boldsymbol{u}', \rho', \ldots)$$

$$\rho \boldsymbol{u}\boldsymbol{u} \rightarrow \underbrace{\tilde{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}}_{\text{source term}} + \underbrace{\tilde{\rho}\tilde{\boldsymbol{u}}\boldsymbol{u}'}_{\text{interaction term}} + \underbrace{\tilde{\rho}\boldsymbol{u}'\boldsymbol{u}'}_{\text{nonlinear term}} + \ldots$$

- Reformulation of vortex sound model (low Mach number)[2]

$$\frac{1}{c^2}\frac{D^2}{Dt^2}H - \nabla\cdot\nabla H = \nabla\cdot(\boldsymbol{\omega}\times\tilde{\boldsymbol{u}}) = \nabla\cdot\boldsymbol{L}(\tilde{\boldsymbol{u}})$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{\boldsymbol{u}}\cdot\nabla$$

H: total enthalpy

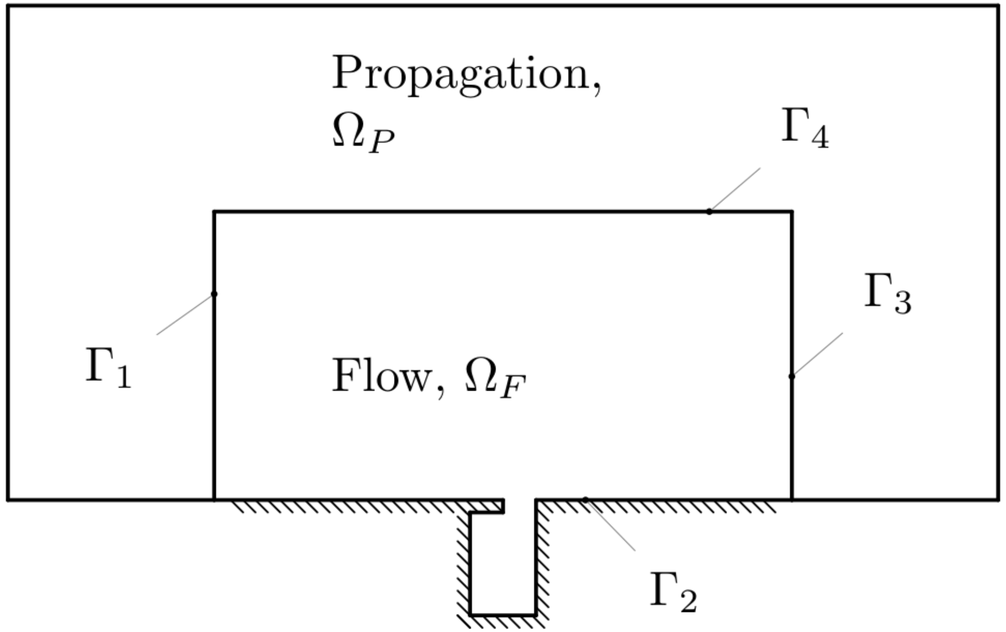
$$\nabla\times\boldsymbol{u} = \nabla\times\tilde{\boldsymbol{u}}$$

$\boldsymbol{L}(\tilde{\boldsymbol{u}})$: Lamb vector

[1]M. E. Goldstein. A generalized acoustic analogy. Journal of Fluid Mechanics, 2003.

[2]M. S. Howe. Theory of Vortex Sound. Cambridge Texts in Applied Mathematics, 2003

- Practical situation



Γ_1 - Inlet : $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{in}}$
Γ_2 - Wall : $\boldsymbol{u} = 0$
Γ_3 - Outlet : $p = p_{\mathrm{ref}}$
Γ_4 - Top : $\boldsymbol{u} = \mathbf{0}$

Helmholtz decomposition on a restricted domain

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{v}} + \boldsymbol{u}^{\mathrm{c}} + \boldsymbol{u}^{\mathrm{h}} = \nabla \times \boldsymbol{A}^{\mathrm{v}} + \nabla \phi^{\mathrm{c}} + \boldsymbol{u}^{\mathrm{h}}$$

Harmonic part (potential flow): the boundary condition dictate the harmonic part as an external influence!

- Extraction of base flow
  - Relation

$$\boldsymbol{u} = \underbrace{\nabla \times \boldsymbol{A}^{*,\mathrm{v}}}_{\boldsymbol{u}^{\mathrm{v}} + \boldsymbol{u}^{\mathrm{h}}} + \nabla \phi^{\mathrm{c}}$$

apply $\nabla \times$

$$\nabla \times \nabla \times \boldsymbol{A}^{*,\mathrm{v}} = \nabla \times \boldsymbol{u} = \boldsymbol{\omega}$$

vorticity

  - Orthogonal decomposition

$$\int_\Omega \boldsymbol{u}^{\mathrm{c}} \cdot \nabla \times \boldsymbol{A}^{*,\mathrm{v}} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \underbrace{\nabla \times \boldsymbol{u}^{\mathrm{c}}}_{=0} \cdot \boldsymbol{A}^{*,\mathrm{v}} \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{A}^{*,\mathrm{v}} \cdot \underbrace{(\boldsymbol{u} - \nabla \times \boldsymbol{A}^{*,\mathrm{v}})}_{=\boldsymbol{u}^{\mathrm{c}}} \times \boldsymbol{n} \, \mathrm{d}\boldsymbol{s}$$

$$= \int_{\partial\Omega} \boldsymbol{A}^{*,\mathrm{v}} \cdot (\boldsymbol{u} - \nabla \times \boldsymbol{A}^{*,\mathrm{v}}) \times \boldsymbol{n} \, \mathrm{d}\boldsymbol{s} \overset{!}{=} 0$$

Aeroacoustics: Formulation

- Weak formulation (with mass regularization):
  Find $\boldsymbol{A}^{*,\mathrm{v}} \in \boldsymbol{V} = \{\boldsymbol{\psi} \in (L_2)^3 \,|\, \nabla \times \boldsymbol{\psi} \in (L_2)^3 \,,\, \boldsymbol{n} \times \boldsymbol{\psi} = \boldsymbol{\psi}_\mathrm{e} \,,\; [\boldsymbol{n} \times \boldsymbol{\psi}] = 0\}$ such that

$$\int_\Omega \nabla \times \boldsymbol{A}' \cdot \nabla \times \boldsymbol{A}^{*,\mathrm{v}} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \varepsilon_\mathrm{reg} \boldsymbol{A}' \cdot \boldsymbol{A}^{*,\mathrm{v}} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{A}' \cdot \boldsymbol{\omega} \,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_\mathrm{N}} \boldsymbol{A}' \cdot (\boldsymbol{u} \times \boldsymbol{n}) \,\mathrm{d}\boldsymbol{s}$$

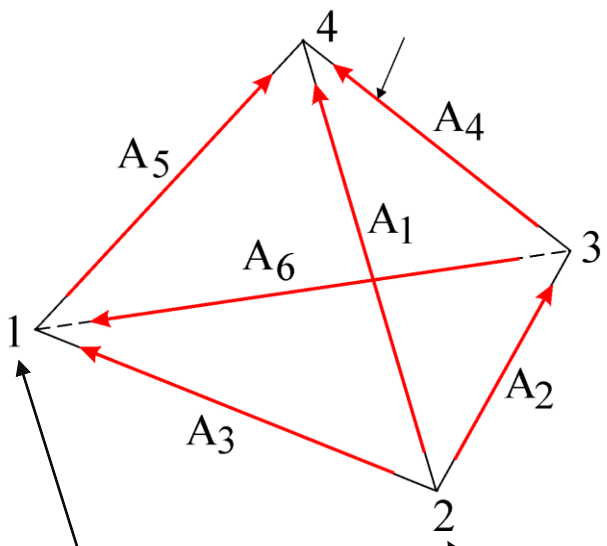
$\varepsilon_\mathrm{reg} \ll 1$

for all $\boldsymbol{A}' \in \boldsymbol{W} = \{\boldsymbol{\psi} \in (L_2)^3 \,|\, \nabla \times \boldsymbol{\psi} \in (L_2)^3 \,,\, \boldsymbol{n} \times \boldsymbol{\psi} = \boldsymbol{0} \,,\, [\boldsymbol{n} \times \boldsymbol{\psi}] = 0\}$

- Boundary conditions
  - Wall
    $$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$$
  - Inlet / outlet
    $$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{u}_\mathrm{inlet/outlet} \times \boldsymbol{n}$$
  - Top:
    $$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{u}_\mathrm{Top} \times \boldsymbol{n}$$

- Finite Element Discretization: properties of functional space, namely $[\boldsymbol{n} \times \boldsymbol{\psi}] = 0$, has to be preserved on the discrete level

Needs edge finite elements (Nedelec elements)!



Lagrangian shape function for node 1

Lagrangian shape function for node 2

$$\boldsymbol{A}^{*,\mathrm{v}} \approx \sum_{i=1}^{n_\mathrm{e}} \boldsymbol{E}_i \, A_i$$

A_i ... vector potential along edge i

FE-basis functions (vectors)

n_e ... number of edges

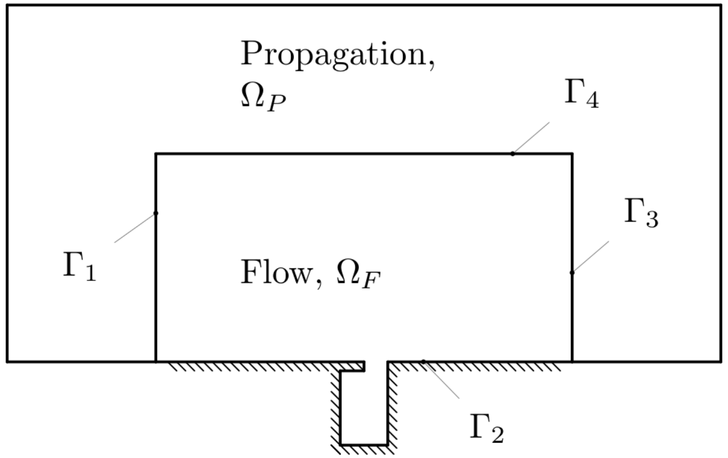
$$\boldsymbol{E}_3 = N_1 \nabla N_2 - N_2 \nabla N_1$$

Again local support

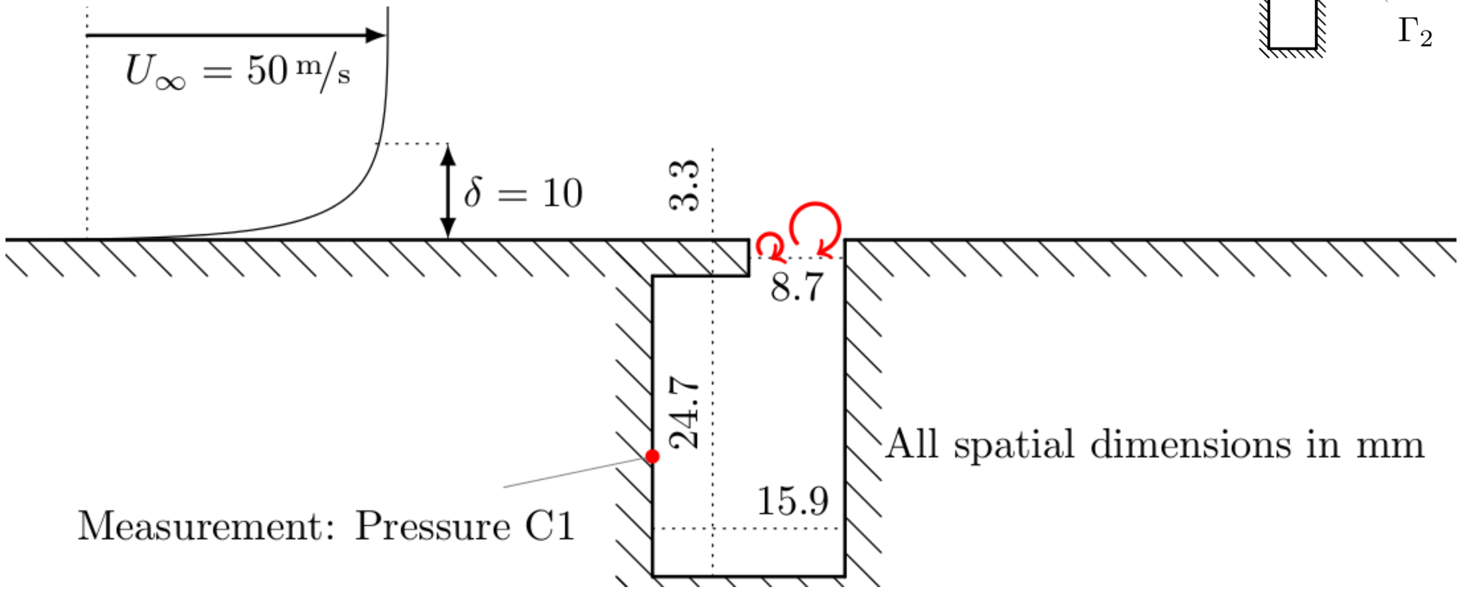
Aeroacoustics: Cavity with a lip

- Unsteady, compressible, laminar, 2D flow
  - Boundary conditions

$$
\begin{array}{lcl}
\Gamma_1 \text{ - Inlet} & : & \boldsymbol{u} = \boldsymbol{u}_{\text{in}} \\
\Gamma_2 \text{ - Wall} & : & \boldsymbol{u} = 0 \\
\Gamma_3 \text{ - Outlet} & : & p = p_{\text{ref}} \\
\Gamma_4 \text{ - Top} & : & \boldsymbol{u} = \boldsymbol{0}
\end{array}
$$



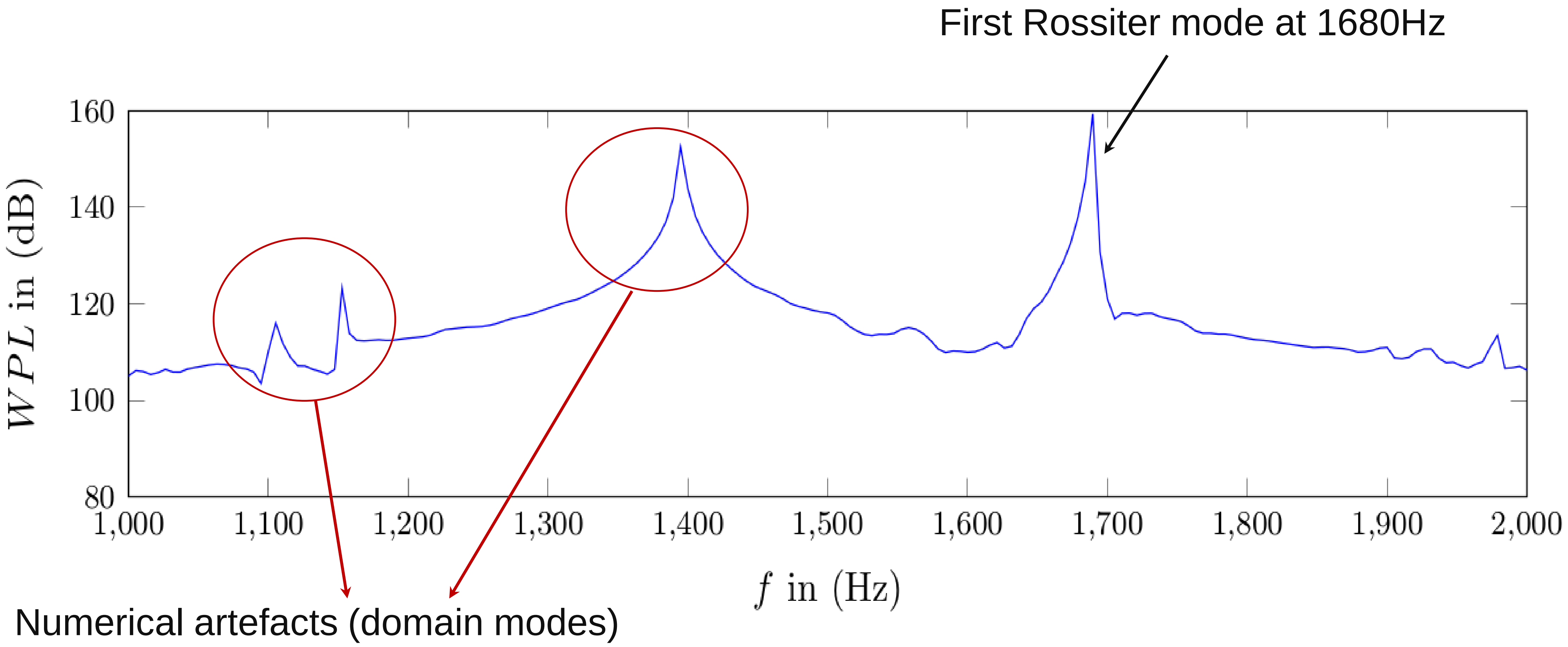
  - Details



B. Henderson. Automobile noise involving feedback- sound generation by low speed cavity flows. Technical report, In: Third Computational Aeroacoustic(CAA) Workshop on Benchmark Problems, 2000.

Aeroacoustics: Cavity with a lip

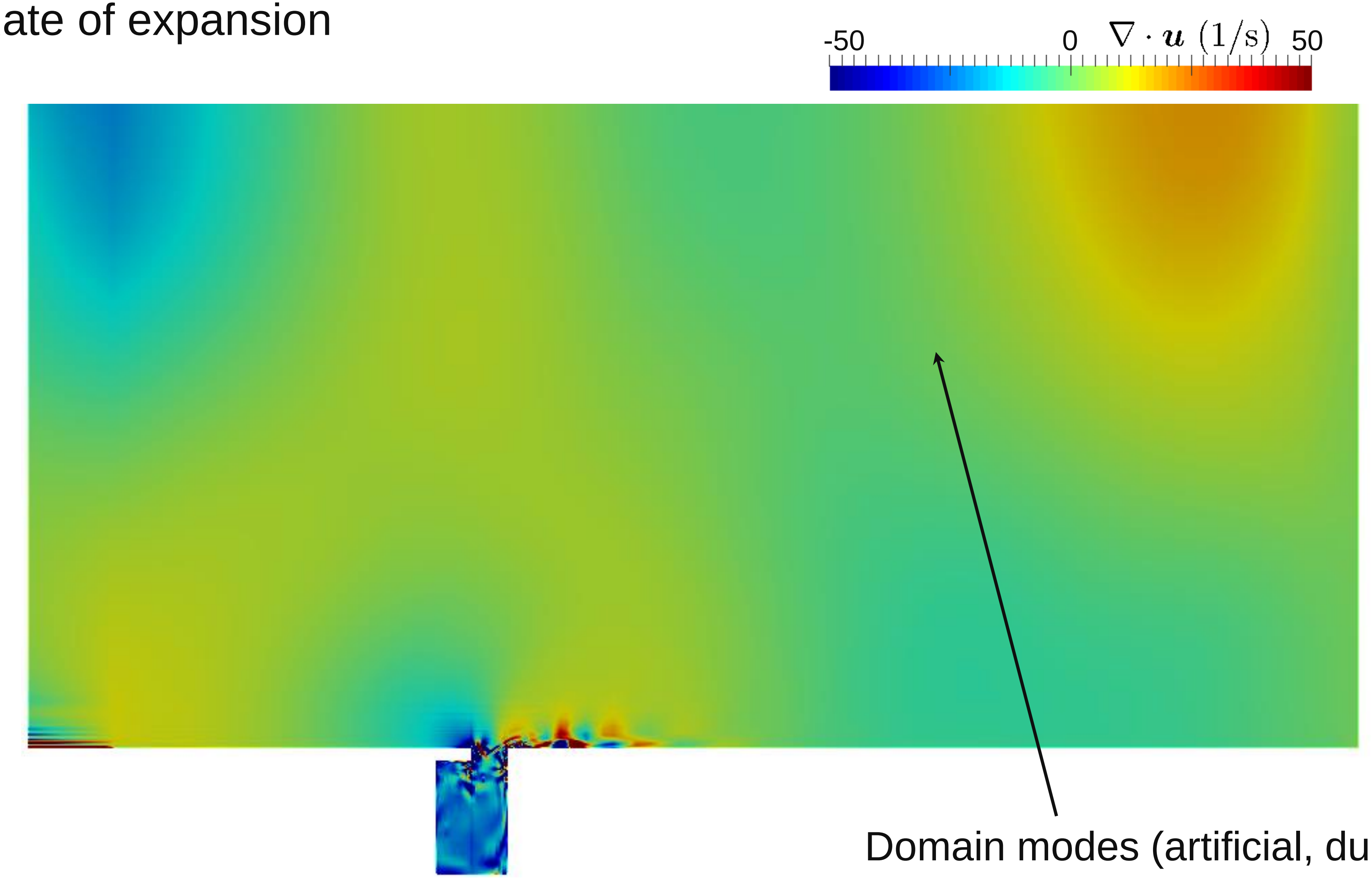
- Computed wall pressure spectrum at point C1 in dB (ref. 20µPa)



T. Seitz. Experimentelle Untersuchungen der Schallabstrahlung bei der Überströmung einer Kavität. Master’s thesis, FAU Erlangen, Germany, 2005.

Aeroacoustics: Cavity with a lip

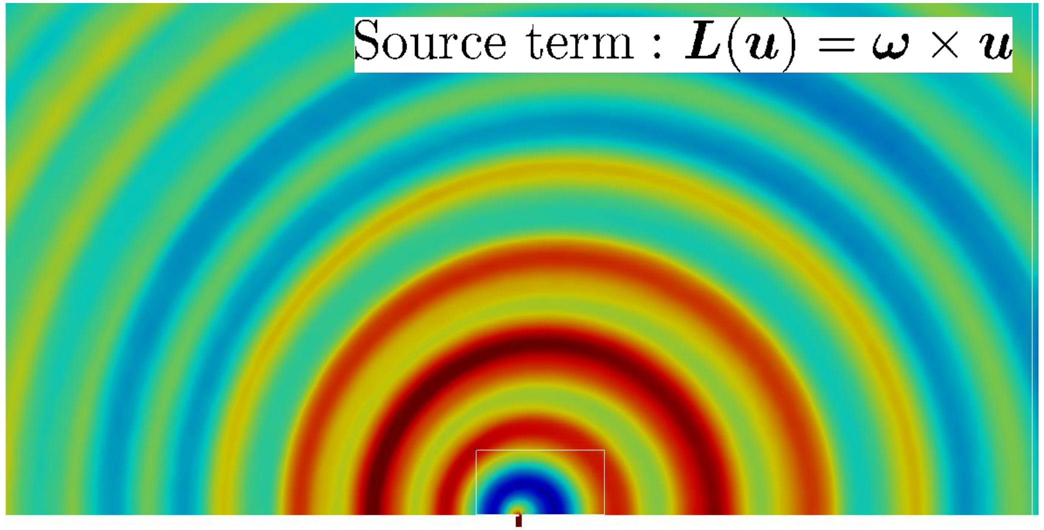
- Rate of expansion

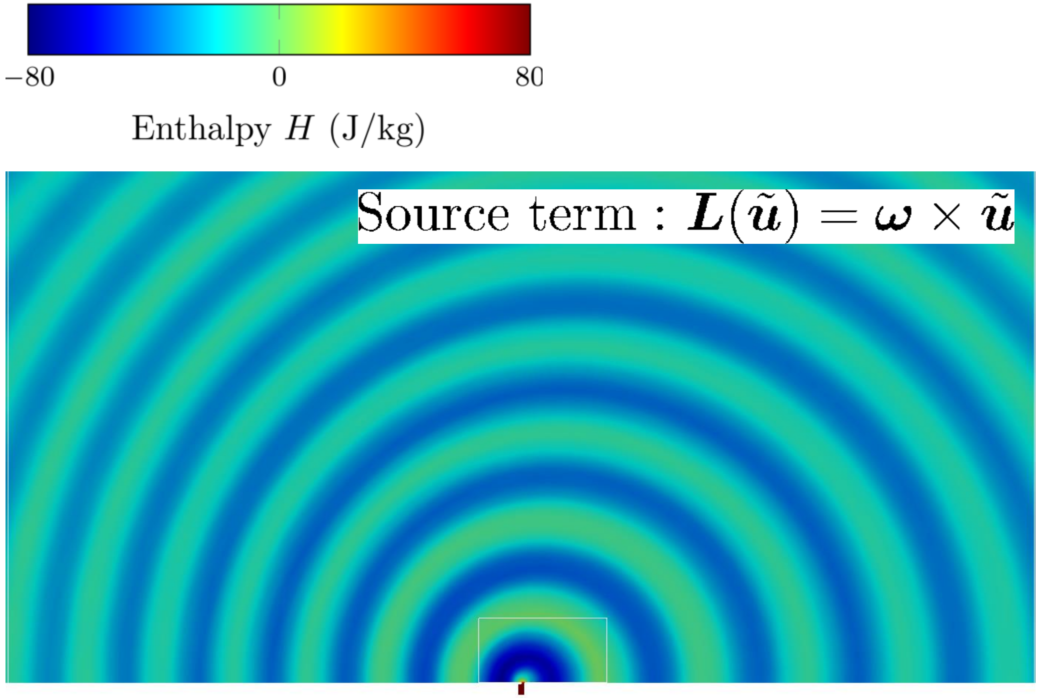


Domain modes (artificial, due to none wave-absorbing boundary conditions!

Aeroacoustics: Cavity with a lip

- Field of total enthalpy fluctuation





- Comparison of acoustic pressure in the far field at 1st shear layer mode

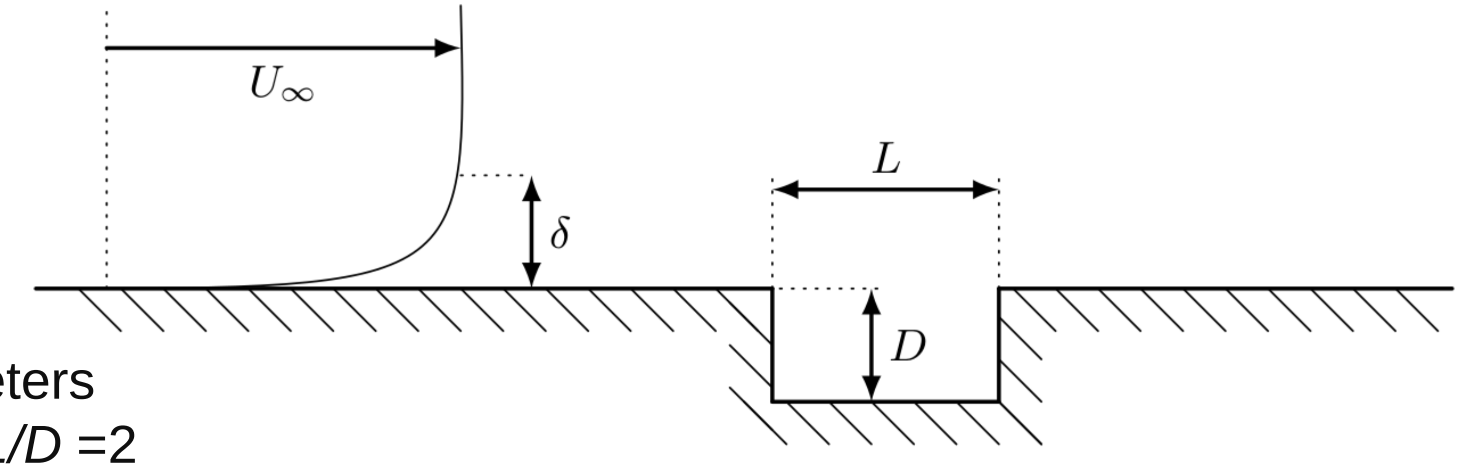
	$f_{\mathrm{s1}}/\mathrm{Hz}$	$\mathrm{SPL}_{\mathrm{s1}}/\mathrm{dB}$
Experiment	1650	30
Simulation $\boldsymbol{L}(\boldsymbol{u}) = \boldsymbol{\omega} \times \boldsymbol{u}$	1660	52
Simulation $\boldsymbol{L}(\tilde{\boldsymbol{u}}) = \boldsymbol{\omega} \times \tilde{\boldsymbol{u}}$	1660	34

$$p_{\mathrm{a}} = \rho_0 H$$

T. Seitz. Experimentelle Untersuchungen der Schallabstrahlung bei der Überströmung einer Kavität. Master's thesis, FAU Erlangen, Germany, 2005.

Aeroacoustics: Cavity at high Mach number

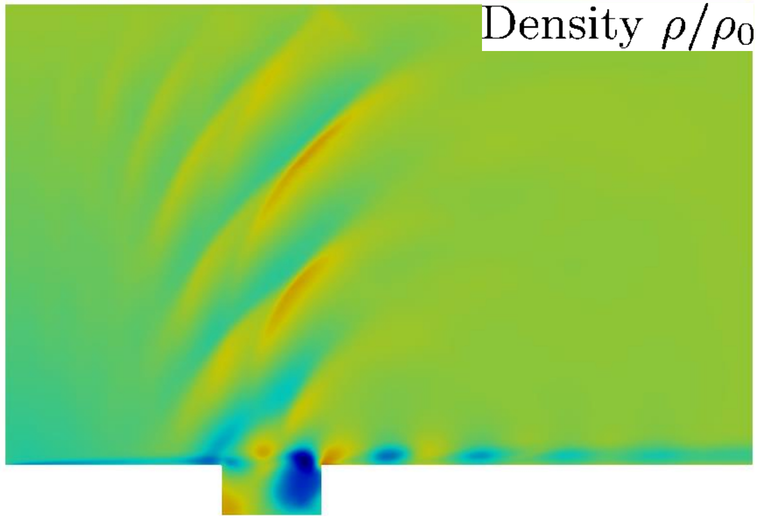
- Helmholtz decomposition applied cavity with to Mach 0.8 (Colonius, 1999)
  - Setup

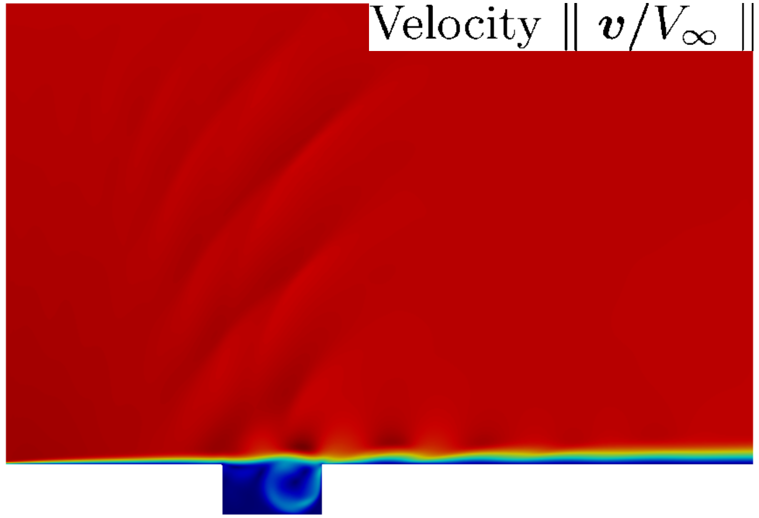


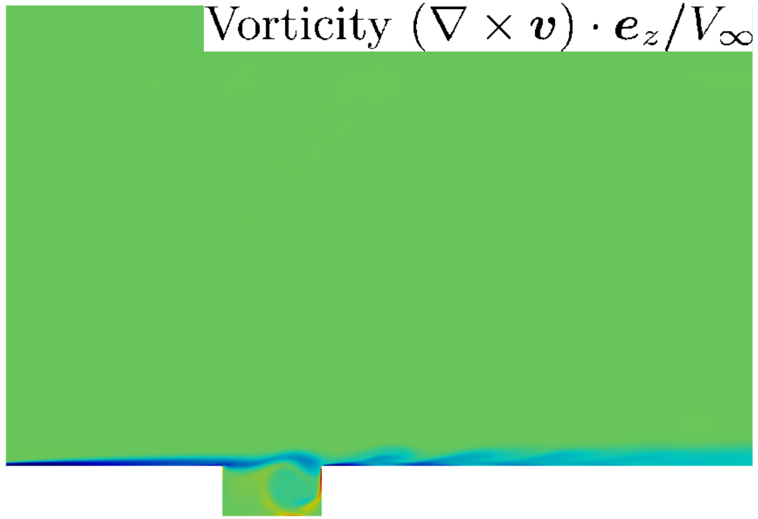
  - Parameters
    - L/D =2
    - Ma = 0.8
    - L/δ = 52.8
    - Re_δ = 56.8
  - Solver
    - Flexi, DG solver (https://www.flexi-project.org/)
    - Higher order basis functions
    - Compressible, direct numerical simulation
    - Prescribed boundary layer

Aeroacoustics: Cavity at high Mach number

- Results of the direct numerical simulation



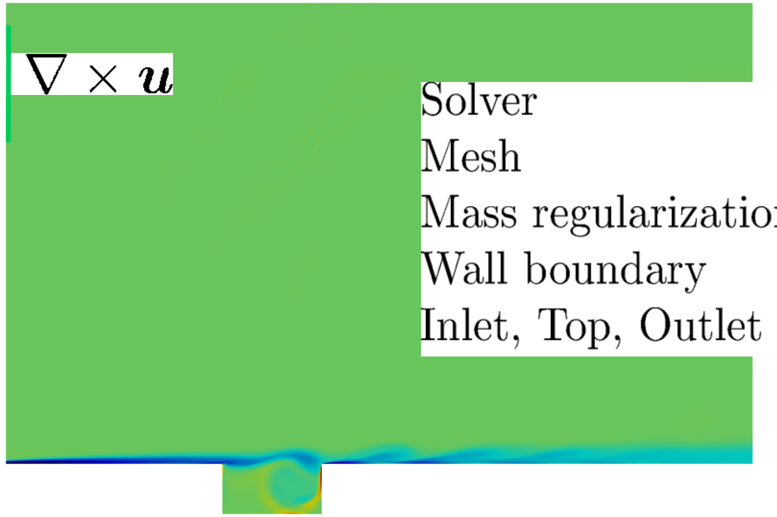




Aeroacoustics: Cavity at high Mach number

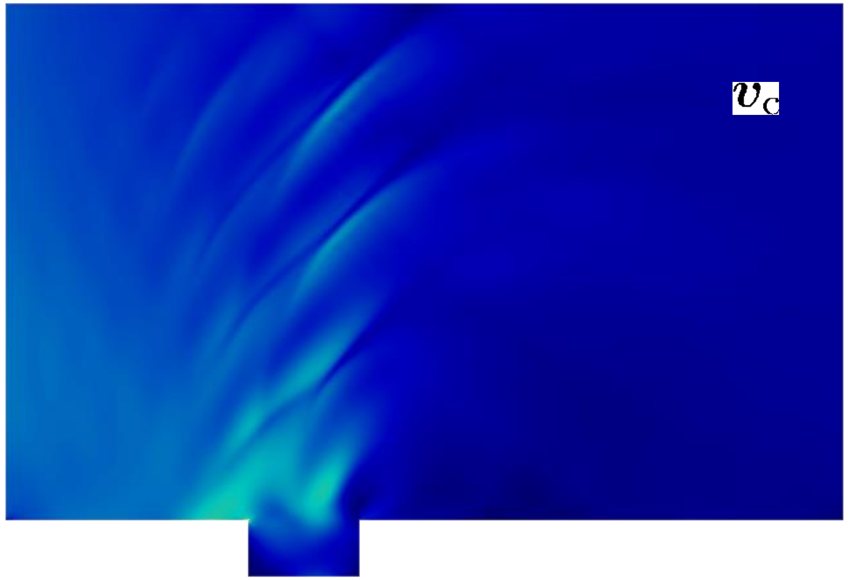
- Helmholtz decomposition (post-processing to compressible CFD)

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{u}$$



Solver	openCFS
Mesh	*Structured*
Mass regularization	*Active*
Wall boundary	$\boldsymbol{u} \times \boldsymbol{n} = 0$
Inlet, Top, Outlet boundary	$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{u}_{\mathrm{inlet/top/outlet}} \times \boldsymbol{n}$

- Results



Thank you

for your attention!