Lecture 2: Hydrodynamics at milli-micrometer scale

Introduction

Flows at milli- and micro-meter scales are found in various fields, used for several processes and open up possibilities for new applications:

- **Injection - Engineering**

Engine, combustion, sprays, atomization, particle formation, lubrication, coating

- **Miniaturization - Microelectromechanical systems (MEMS)**

Chemical reactions-process engineering → decreasing set-up sizes allows increasing specific surfaces of exchanges → safer environment when dealing with exothermic reactions, smaller quantities, higher yields, better catalyses, may lead to better selectivity, faster…
Heat transfer is increased by decreasing length scales → exothermic reaction, cooling of microprocessors, controlled evaporation…
Lab on a chip → see diagnostic – biotechnology

- **Life science**

Network of natural micro canals, typical length scale of life science.

Atoms, molecules	*DNA*	*Proteins*	*Cells*	*Blood vessels*	*Organs*
10^{-10}m	$2.10^{-9} – 10^{-6}$ m	$10^{-9} – 10^{-6}$ m	$10^{-6} – 10^{-5}$ m	10^{-4} m	10^{-2} m

- **Diagnostic – Biotechnology – Sensor technology**

decreasing set-up sizes allows working with smaller quantities (body fluids – electrochemical sensors for blood analysis),
Lab on a chip → possibility for fast screening (phase diagrams water-surfactants, proteins crystallization), genomics with Polymerase Chain Reaction (PCR), …

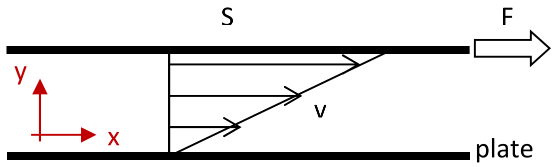
1) Concept of viscosity

1)1) Microscopic origin

At macroscopic level, viscosity can be perceived via the resistance a fluid opposes to any flow. Its physical origin is momentum transfers happening at the molecular scale due to thermal agitation. This thermal agitation cannot be directly modeled at the molecular scale using continuous approaches. Only its effects at a mesoscopic level are modeled: this is viscosity.

To remember → viscosity only in dynamic regime

1)2) Elementary manifestation



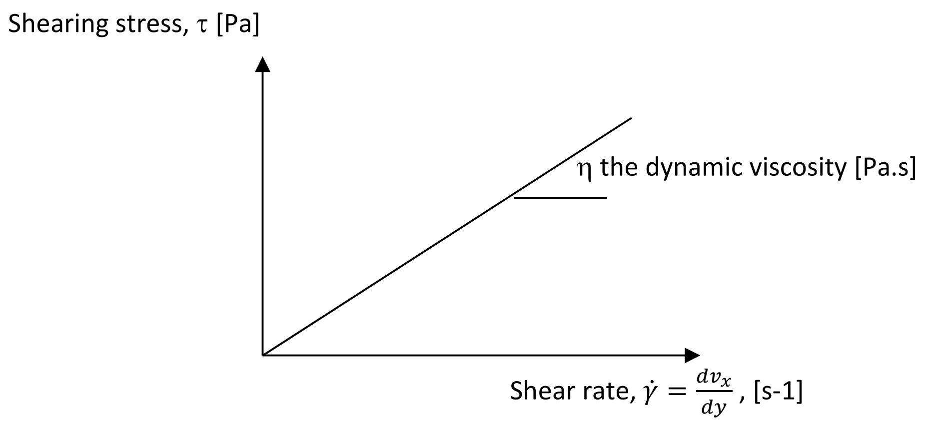
(Shear) strain: $\gamma = x/y$, (strain) shear rate $\dot{\gamma} = \frac{dv_x}{dy}$ [s^{-1}]

Stress: $\tau = F/S$ [Pa]

To generate a certain strain at the certain rate, one need to apply a certain force (stress as it scales with the surface of the plate) →mechanical work compensates viscous dissipation.

1)3) Newtonian fluids

For the Newtonian fluids: the stress τ is proportional to the shear rate $\dot{\gamma}$. The ratio between stress and shear rate is defined as the dynamic viscosity of the fluid: $\tau = \eta \dot{\gamma}$ with η the dynamic viscosity [Pa.s].



To remember → $\tau = \eta \dot{\gamma}$

Order of magnitude:

Liquid	*Glycerol*	*Ethanol*	*Water*	*Mercury*	*Blood*	*Honey*
η (mPa.s)	*$1.2\ 10^3$*	*~1*	*~1*	*1.5*	*3-4*	*$2.10^3 - 1.10^4$*

1)4) Remarks:

- There exist non-Newtonian fluids
- The study of viscosity → rheology

2) Hydrodynamics at milli-micro-meter scale

2)1) Equations of hydrodynamics

- **Continuity equation**

Mass conservation with ρ the fluid density and $\vec{v}$ its local velocity $\frac{\partial \rho}{\partial t} + \mathbf{div}(\overrightarrow{\rho v}) = 0$

Incompressibility → $\mathbf{div}\,\vec{v} = 0$

- **Navier Stokes equation (or momentum equation)**

Incompressibility and Newtonian fluid with a dynamic viscosity η [Pa.s]

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho\,(\vec{v}.\overrightarrow{grad})\vec{v} = \underbrace{-\overrightarrow{grad}\,P + \eta\,\Delta\,\vec{v}}_{\text{Forces on surface}} + \underbrace{\overrightarrow{f_{vol}}}_{\text{Forces in volume}}$$

If $\overrightarrow{f_{vol}} = \vec{0}$

$$\overbrace{\underbrace{\rho \frac{\partial \vec{v}}{\partial t}}_{\text{unsteady}} + \underbrace{\rho(\vec{v}.\overrightarrow{grad})\vec{v}}_{\text{convective}}}^{\text{Inertia}} = \underbrace{-\overrightarrow{grad}\,P}_{\text{pressure gradient}} + \underbrace{\eta\,\Delta\,\vec{v}}_{\text{viscosity}}$$

unsteady convective acceleration

Navier –Stokes equation

The transport of momentum ($\rho\,\vec{v}\,dV$) is caused by:

- convection $\rho(\vec{v}.\overrightarrow{grad})\vec{v}$
- diffusion (viscous origin) $\eta\,\Delta\,\vec{v}$

The relative importance of the two terms can be estimated and appears visible by making N-S dimensionless:

Length	r´=r/L
Velocity	v´= v/U
Time	t´= t /τ τ =ρL²/η (i) or τ = L/U (ii)
Pressure	P´ = PL/η U (i) or P´= P/ρ U² (ii)

$$\boldsymbol{Re}\left(\frac{\mathbf{1}}{\boldsymbol{St}}\frac{\boldsymbol{\partial \overrightarrow{v'}}}{\boldsymbol{\partial t'}} + \left(\overrightarrow{\boldsymbol{v'}}.\overrightarrow{\boldsymbol{grad'}}\right)\overrightarrow{\boldsymbol{v'}}\right) = -\overrightarrow{\boldsymbol{grad'}}\,\boldsymbol{P'} + \boldsymbol{\Delta'}\,\overrightarrow{\boldsymbol{v'}} \quad \text{(i)} \quad \text{Viscous regime}$$

$$\frac{\mathbf{1}}{\boldsymbol{St}}\frac{\boldsymbol{\partial \overrightarrow{v'}}}{\boldsymbol{\partial t'}} + \left(\overrightarrow{\boldsymbol{v'}}.\overrightarrow{\boldsymbol{grad'}}\right)\overrightarrow{\boldsymbol{v'}} = -\overrightarrow{\boldsymbol{grad'}}\,\boldsymbol{P'} + \frac{\mathbf{1}}{\boldsymbol{Re}}\boldsymbol{\Delta'}\,\overrightarrow{\boldsymbol{v'}} \quad \text{(ii)} \quad \text{Inertial regime}$$

Reynolds number $\boldsymbol{Re} = \frac{\rho L U}{\eta} = \frac{\rho L}{\nu}$ with $\boldsymbol{\nu}$ the kinematics viscosity; $\boldsymbol{St=\tau/(L/U)}$ Strouhal number and $\boldsymbol{\tau}$ is the typical time scale of acceleration/deceleration

Typical time for convection: L/U

Typical time from diffusion (viscosity): $\rho L^2/\eta$

$$\text{Re} = \frac{\text{diffusion time scale}}{\text{inertia time scale}}$$

2)2) Stokes equation

- **Neglecting convection:**

- Creeping flows (U<<1 → Re<<1)

- Parallel or quasi parallel (only one component of $\overrightarrow{\boldsymbol{v}} \neq \mathbf{0}$)

$$\frac{\boldsymbol{\partial \overrightarrow{v'}}}{\boldsymbol{\partial t'}} = -\overrightarrow{\boldsymbol{grad'}}\,\boldsymbol{P'} + \boldsymbol{\Delta'}\,\overrightarrow{\boldsymbol{v'}}$$

- **Quasi-steady:**

Time scale on which velocity variations establish >> Time scale on which momentum get transported by viscosity (diffusion) $\rho L^2/\eta$.

$$\vec{0} = -\overrightarrow{\boldsymbol{grad'}}\,\boldsymbol{P'} + \boldsymbol{\Delta'}\,\overrightarrow{\boldsymbol{v'}}$$

$$\overrightarrow{\boldsymbol{grad}}\,\boldsymbol{P} = \boldsymbol{\eta}\,\boldsymbol{\Delta}\,\vec{\boldsymbol{v}} \quad \textbf{Stokes equation}$$

To remember → *Stokes equation writes* $\overrightarrow{grad}\,P = \eta\,\Delta\,\vec{v}$ *and is valid for:*

- *Fluid is incompressible div* $\vec{v}$ *= 0* *(often true in general)*
- *Convection can be neglected (parallel flows / creeping flows)* *(often true in microfluidics)*
- *Quasi-steady flows T>>* $\rho L^2/\eta$ *(often true in microfluidics)*

- **Properties of Stokes equation**

- Linearity –> superposition of solutions

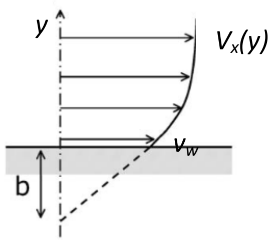
- Reversibility (t → -t)

- Unique solution fulfilling boundary conditions

2)3) Boundary conditions

Boundary liquid/solid. Classical kinematics condition is $v(z=0)=v_{wall}$

In this case Hägen-Poiseuille flow in a cylinder of radius R gives a flow rate: $Q_{no\ slip} = \frac{\pi R^4}{8\eta}\frac{\Delta P}{L}$



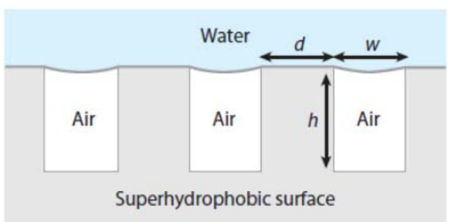
Typical flow field at the wall in slipping conditions.

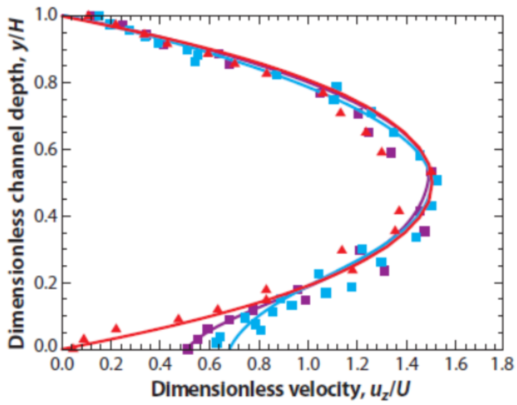
The slip length is defined as the extrapolated distance relative to the wall where the tangential velocity component vanishes.

$v_w = b\left.\frac{\partial v_x}{\partial y}\right)_w$ and flow rate is increased to $Q_{slip} = \frac{\pi R^3}{8\eta}\frac{\Delta P}{L}(R+4b)$

Order of magnitude: 1nm <b<1µm

Physical origin(s): roughness, turbulences, presence of gas, air entrapment at superhydrophobic surfaces (application→ increased Q),… not yet fully understood





Velocity profiles measured through micro-particle image velocimetry (µ-PIV) for the flow through an H = 85 µm tall microchannel past a series of superhydrophobic surfaces containing w = 30 µm wide microridges spaced d = 30 µm (*red and purple symbols*) and d = 60 µm (*blue symbols*) apart. The data include the velocity profile for a vertical slice taken above the center of the microridge (*red triangles*), above the center of the 30 µm shear-free interface (*purple squares*), above the center of the 60 µm shear-free interface (*blue squares*), and the corresponding predictions of the computational fluid dynamics simulations (*lines*). Figure reprinted

Ou & Rothstein, Phys. Fluids, 17:103606 (2005)

2)3) Flows in micro-canals

Steady flows with without slip at the wall → Hägen-Poiseuille flow

Geometry	Velocity [m/s]	Flow rate [m3]
Between 2 plates	$v_x = \frac{a^2}{8\eta}\frac{\Delta P}{L}\left(1 - \frac{4y^2}{a^2}\right)$	$Q = \frac{a^3}{12\eta}\frac{\Delta P}{L}$
In a cylinder	$v_r = \frac{R^2}{4\eta}\frac{\Delta P}{L}\left(1 - \frac{r^2}{R^2}\right)$	$Q = \frac{\pi R^4}{8\eta}\frac{\Delta P}{L}$

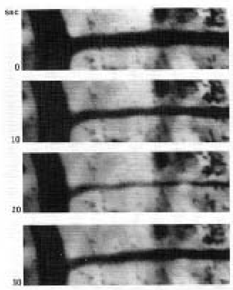
- **Pressure effect**

Heart : over-pressure of ~15 kPa, power of ~ 2 W and yield of ~ 15 %.

Sheep heart surrounded by the lungs.

- **Section effect**

Strong dependency of the flow rate with the section S ($Q \sim S^2$).



Vasomotion on an artériole of 100 µm

2)5) Analogy with electricity

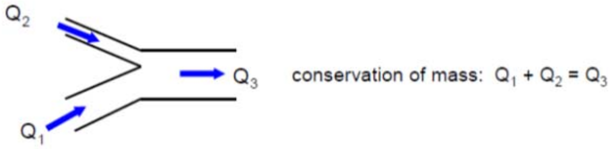
Pressure – Electrical voltage

Flow rate – Electrical current intensity

→Hydrodynamic resistance: $\Delta P = R_h Q$

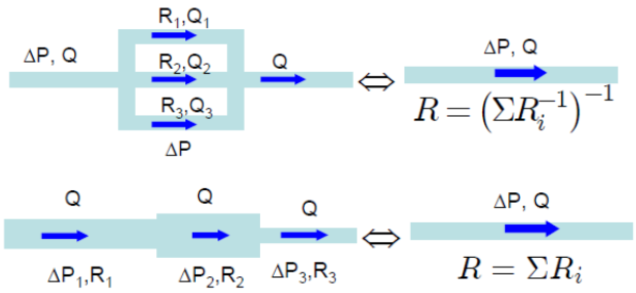
Laws for nodes and branches –similar to the ones of electrokinetics

- For a node: $\sum_i Q_i = 0$



Salmon, 2010

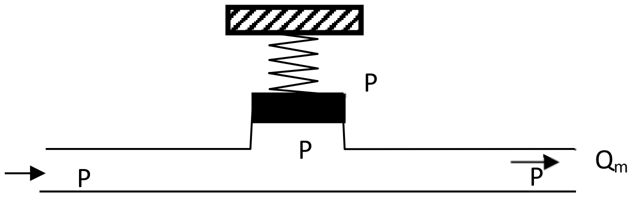
- For a branch: “Kirchoff law” - R_1 and R_2 in series are equivalent to $R_{eq}=R_1+R_2$; R_1 and R_2 in parallel are equivalent to $R_{eq}=R_1R_2/(R_1+R_2)$



Salmon, 2010

For a cylindrical canal (radius R): $R_h = \dfrac{8\eta L}{\pi R^4}$

→ Hydrodynamic capacitance: $Q = C\dfrac{d\Delta P}{dt}$



For a rigid canal C→∞ ; practically not the case for most of the canals

It is possible to show that in the system like above: $Q = R_h\,\Delta P + C\,\frac{d\Delta P}{dt}$ with $C = \frac{2\rho}{k}$ and k is spring stiffness.

This type of calculation is useful because it allows the determination of the response time constant of a microfluidics circuits including various valves and/or plastic tubing. In the example we consider here, the time constant is RC.

By controlling the flow rate (syringe pump for example), microfluidics circuits can need several minutes/hours to reach a steady state. In nature, this effect is used to smooth the blood flux (difference systole-diastole and compliance of the vessels).

To reach an immediate steady state, we will always prefer to work controlling the pressure difference.