Exercise 3: Mixing in Microsystems

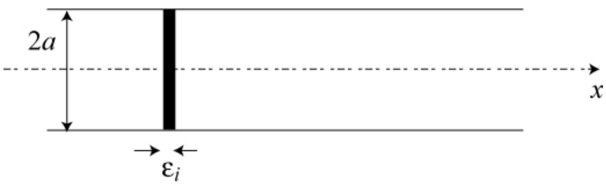
Introduction

In this exercise session, we are interested in the dispersion of a dye front advected by flow in a microcannal. Under specific conditions, the dye front spreads much faster than it would do it under simple molecular diffusion or under simple advection. This phenomenon called Taylor Aris dispersion is the basis under which some micromixers operate. The principle of such micromixers is described in the last part of this exercise.

Advection Diffusion equation

We consider a long cylindrical canal (axis Ox) of radius *a*. The canal is filled in with water in which a dye line of initial thickness ε_i and initial concentration C_i is dispersed. We note $\bar{C}$(x,t) the "mean" concentration defined as the concentration averaged for the section at position x and instant t. More generally, we note $\bar{Y}$ the mean value of Y averaged over the section.

A steady flow invariant for translation along the x axis is established within the cylindrical canal. The velocity of the fluid at position r (distance from the central axis) is note u(r).



1) Recall the advection-diffusion equation governing the dye concentration in the tube. By qualitatively analyzing the different terms of this equation, identify the various regimes of mixing.

Qualitative analysis of front dye dispersion

We note D the molecular diffusion coefficient of the dye within the water.

2) Describe qualitatively the evolution of the dye line thickness ε and its concentration *C* for a perfect fluid flowing with an average velocity $\overline{u}$. In this case, we assume $\varepsilon_i \ll a$.

For a real fluid, the velocity profile in the tube is parabolic (Hägen-Poiseuille flow). We note u_M the maximum velocity of the fluid.

3) Can you graphically describe the evolution of the dye line thickness ε and its concentration C neglecting molecular diffusion? What is the difference between the configurations where $\varepsilon_i << a$ and where $\varepsilon_i \sim a$ respectively?
4) Under which conditions is the mixing purely advective?

Aris-Taylor dispersion

In this part, we take into consideration both the hydrodynamic origin of dispersion and the molecular diffusion. We are interested in the temporal evolution of the dye concentration C within the canal (radius a).

The advection diffusion equation in the laboratory referential is:

$$\frac{\partial C}{\partial t} + u_r\left(1 - \frac{r^2}{a^2}\right)\frac{\partial C}{\partial x} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)\right)$$

The dye concentration can be written as the sum of the concentration averaged for the canal section $\bar{C}$(x,t) and the so called instantaneous concentration which we note C'(r, x, t). We notice that the average value of C' under the section is zero. Assuming the initial concentration of the line dye is axi symmetric and considering no flux through the wall ($\frac{\partial C}{\partial r} = 0$) it is possible to show that $\bar{C}$(x,t) and C'(r, x, t) are solutions of the two equations below:

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}\frac{\partial \overline{C}}{\partial x} + \overline{u\frac{\partial C'}{\partial x}} = D_\mu \frac{\partial^2 \overline{C}}{\partial x^2} \qquad (a)$$

$$\frac{\partial C'}{\partial t} + (u - \overline{u})\frac{\partial \overline{C}}{\partial x} + u\frac{\partial C'}{\partial x} - \overline{u\frac{\partial C'}{\partial x}} = D_\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C'}{\partial r}\right) + \frac{\partial^2 C'}{\partial x^2}\right) \qquad (b)$$

By considering a time scale of the order of a^2/D, it appears that the first, third, fourth and sixth terms of equation (b) can be neglected.

The simplified equation thus reads:

$$(u - \bar{u})\frac{\partial \bar{C}}{\partial x} = D_\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial C'}{\partial r}\right)$$

which can be solved leading to:

$$C' = \frac{u_M}{D_\mu}\frac{\partial \bar{C}}{\partial x}\left[\left(\frac{r^2}{8} - \frac{r^4}{16\,a^2}\right) - \frac{a^2}{24}\right]$$

Having explicited C', the third term of equation (a) can be calculated and is found to be equal to:

$$\overline{u\frac{\partial C'}{\partial x}} = -\frac{a^2}{192}\frac{{u_M}^2}{D_\mu}\frac{\partial^2 \bar{C}}{\partial x^2}$$

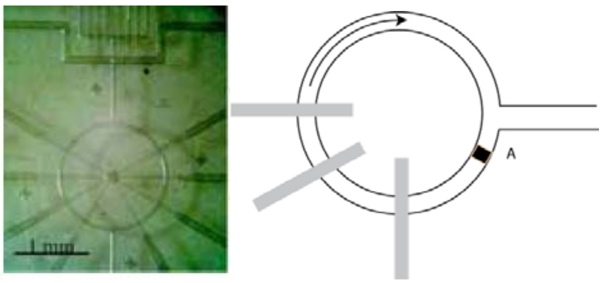
Using equation (a) and the expressions of C´ and $\overline{u\frac{\partial C'}{\partial x}}$, the expression of the effective diffusion coefficient D_{eff} of the Taylor Aris dispersion can be established.

$$D_{eff} = D\left(1 + \frac{a^2\bar{u}^2}{48\,D^2}\right) \quad \text{or} \quad D_{eff} = D\left(1 + \frac{\overline{Pe}^2}{48}\right)$$

5) What are the two regimes of Taylor Aris dispersion? Which criteria can be used to describe the transition between these two regimes? For a=100µm and D=100µm^2/s, calculate the flow rate Q which corresponds to this transition. Recall the basic assumption under which the Taylor-Aris dispersion equation is valid. To which condition on the canal length does it correspond?

Application: annular micromixer

We now consider the technical build-up of an annular micromixer operating under the Taylor-Aris dispersion principle. A picture of such a mixer and a typical scheme of it are presented below. The liquid flows in an annular canal which cross section is cylindrical under the effect of radial canals drawn in the form of grey bars. These canals are located on top of the annular canal where mixing actually takes place and are operated on a peristaltic pump mode. We note R the radius of the annular canal and a the radius of the cylindrical section.



6) Discuss the different regimes of mixing of this system as a function of the Péclet number. For each regime, give the period of time required to achieve mixing of the dye spot initially located in A. To how many loops does this correspond? To answer these questions, we notice that they are three different mixing regimes to consider.

Reference

[1] G.I. Taylor, *Dispersion of soluble matter in solvent flowing slowly through a tube* Proc. Roy. Soc. **A219**, 186-203 (1953)

[2] S. R. Quake and A. Scherer *From Micro- to Nanofabrication with Soft Materials* Science **290,** 1536-1539 (2000)