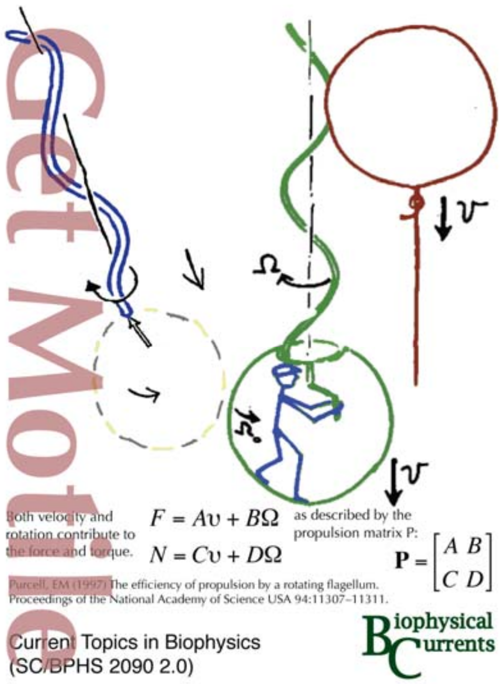
Exercise 2: propulsion in Stokes flows

Introduction

Propelling and swimming in Stokes flows (where Stokes equation is valid) have found a great interest in the last decades. “Natural” micro swimmers are found in biological processes, including reproduction (spermatozoa), infection (bacteria), and the marine life ecosystem (algae, plankton) ,… The development of artificial micro swimmers able to carry payloads and navigate autonomously is considered as a promising alternative to today´s drug delivery systems, lab-on-a-chip devices, and opens new approaches for performing microsurgery and micro/nano-fabrication. Work on artificial micro-swimmers is also performed in the military field aiming for the development of micro surveillance systems as well as micro-weapons.

In this exercise session, we will consider two kinds of motion under Stokes equation. The first part is dedicated to low Reynolds numbers and starts presenting the concept of resistance matrix which is applied on Purcell micro swimmer. The second part deals with lubrication theory. Lubrication flows can be described using Stokes equation due to their geometrical characteristic (parallel or quasi-parallel) but can correspond to “large” Reynolds numbers. We apply lubrication theory to snail locomotion.



Preliminary: Stokes equation

The Navier Stokes equation is recalled below. Under certain conditions, both the convective and unsteady parts of the Navier-Stokes equation can be neglected leading to the Stokes equation. Can you express these conditions? It may be useful to Introduce the Reynolds and Strouhal numbers -which compare respectively the convective term to the viscous one and the convective term to the unsteady term of the Navier Stokes equation.

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho\, \vec{v}.\overrightarrow{grad}\, \vec{v} = -\overrightarrow{grad}\, P + \eta\, \Delta \vec{v}$$

Preliminary: Movement of a solid body with Stokes flows

We consider a solid body immerged in a fluid. The solid body is subjected to a translational movement (no rotation) at a velocity **U**. We note v the velocity of the surrounding fluid. We consider no slip boundary condition at the surface of the body and v→0 for r→∞.

We note **T** the stress tensor. It can be shown that $T_{ij} = -p\, \delta_{ij} + 2\, \eta\, e_{ij}$ where η is the fluid viscosity (Newtonian case) and $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$. Stokes equation can be reformulated as div **T = 0** The force **F** exerted on a solid body of surface S by the stress field **T** is $\vec{F} = \oiint T\vec{n}dS$ or $F_i = \oiint T_{ij} n_j dS$. The pressure being isotropic, its contributions cancel each other and only the contribution of $2\, \eta\, e_{ij}$ remains for the force.

Considering no slip condition, we obtain that at the surface of the solid v=U

Stokes being linear, we further obtain that v must be proportional to U in all space

Finally, it leads to F proportional to v which is itself proportional to U → F is proportional to U!

This demonstration can be made on a more general way including rotational movement and torque. Doing so, we obtain:

$$F_i = -\eta\left(L A_{ij} U_j + L^2 B_{ij} \Omega_j\right)$$

$$G_i = -\eta\left(L^2 C_{ij} U_j + L^3 D_{ij} \Omega_j\right)$$

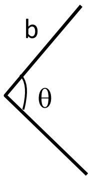
F_i and G_i are the component along i of the force and torque respectively, U_i is the velocity of the solid body along i axis and Ω_i is the angular velocity around the i axis. A and D are symmetric, $C_{ij} = B_{ji}$.

More generally, it writes $\begin{pmatrix} F \\ G \end{pmatrix} = -\eta \begin{pmatrix} L A & L^2 B \\ L^2 C & L^3 D \end{pmatrix} \begin{pmatrix} U \\ \Omega \end{pmatrix} = -R \begin{pmatrix} U \\ \Omega \end{pmatrix}$ with **R** the resistance matrix.

First part: Purcell swimmer

1) Scallop theorem

Let consider a scallop made of two arms of length b connected by a join.

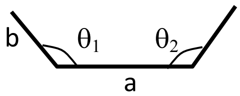


1) How many degrees of freedom does the scallop possess? As the scallop should swim for a certain time period, we will consider a periodic way of swimming. Can you propose the simplest sequence of states which could lead to swimming?

2) We note θ_{close} and θ_{open} the two extreme states the scallop can take. In which direction does the scallop swim if Re>>1? If Re<<1?

1) Purcell swimmer

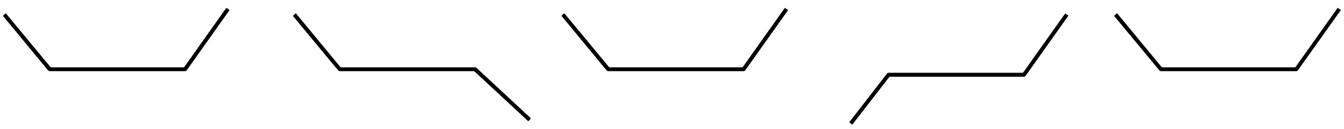
We now consider a swimmer made of a central body of length a with two arms of length b. The so-called Purcell swimmer has two joints with respective angles θ_1 and θ_2 has depicted on the figure below.



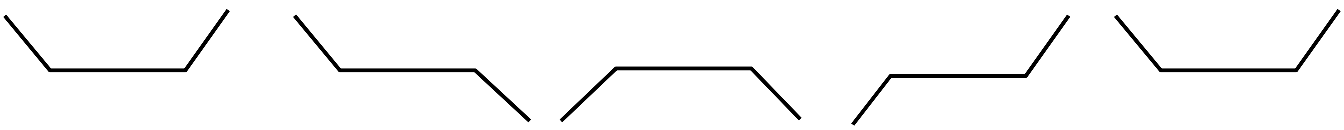
We propose two different sequences of movements which are represented below.

3) Using mirror/time reverse transformation for each step, find which one of these two sequences could lead to a net propulsion. Can you explain why? Find this result again by drawing the phase diagram of each sequence.

- Sequence 1:



- Sequence 2:



4) We note Δx and Δy the elementary displacement achieved in the first step of the sequence 2 shown above. Can you deduce the direction of the propulsion and its amplitude for one cycle of swimming?

5) The experimental results obtained by Brian Chan are reproduced below. Comment the dependence of the net propulsion with the ratio a/*l* where *l* is the total length of the swimmer i.e. *l*=2a+b?

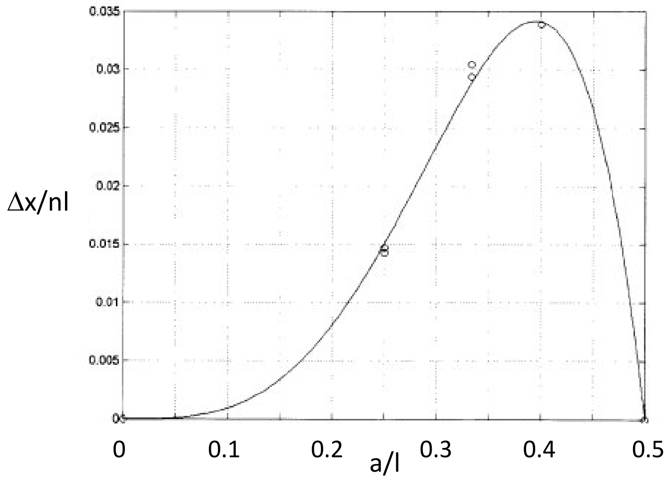


Figure 3-5: Net translation per cycle as a function of arm length, normalized to total body length (l = 2a + b). Note the zero points at a/l = 0 and a/l = 0.5 The curve $y = 1.95x^{3.3}(1 - 4x^2)$ is included merely as a likely interpolation and was not derived from any theory.

Reference

1) E.M. Purcell, *Life at low Reynolds number*, course at Harvard Univ. ,Cambridge, USA, (1976)
2) L.E. Becker, S.A. Koehler and H.A. Stone, *On self propulsion of micro machines at low Reynolds number: Purcell´s three link swimmer*, J. Fluid. Mech., **490**, pp15-35, (2003)
3) Brian Chan, Master of Science at the MIT, *Propulsion for locomotion at low Reynolds number*, (2004)