Materials for the

Lectures

TRANSPORT PROCESSES I AND II

Winter semester 2014/15

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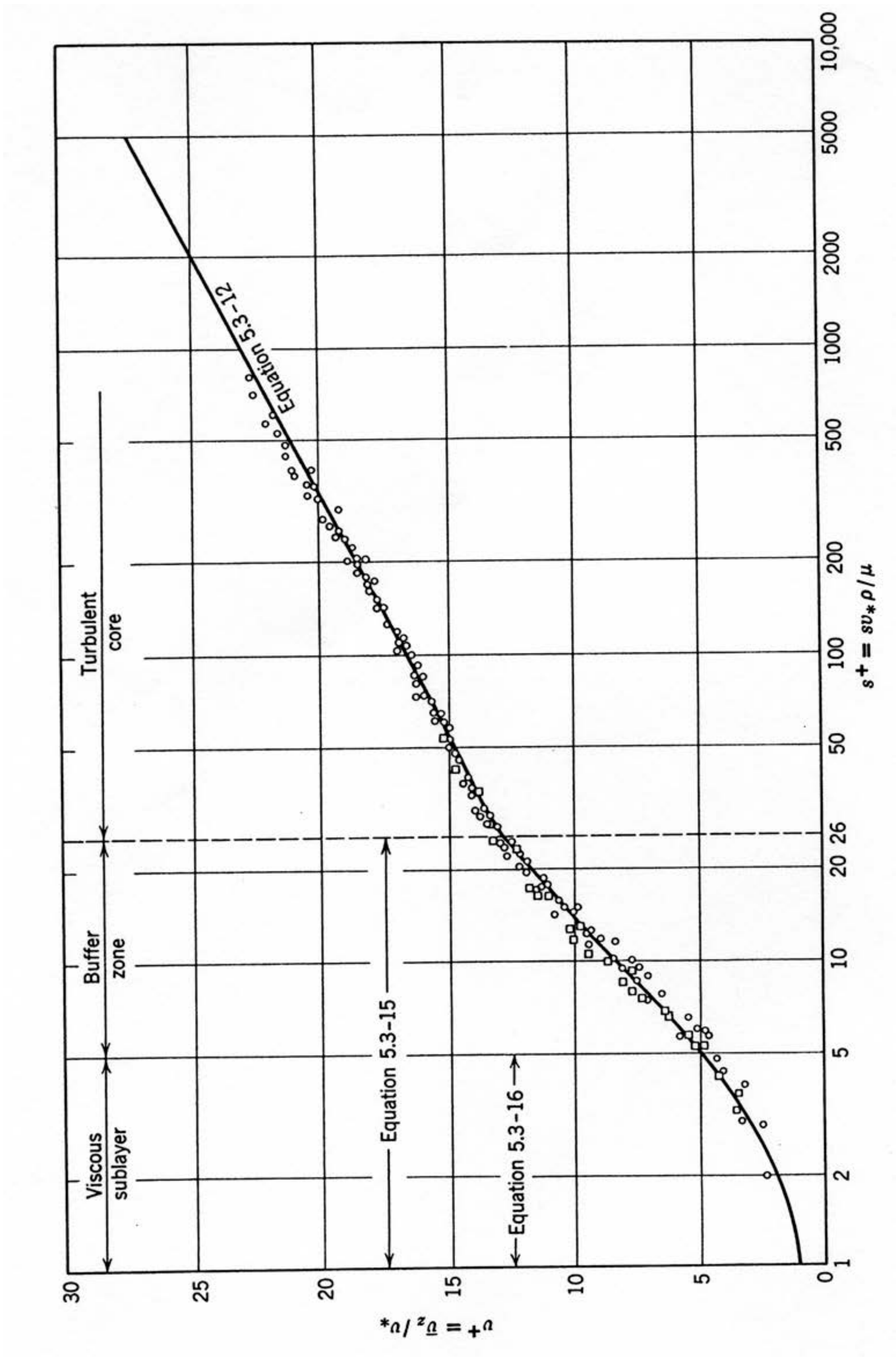
Turbulent near-wall velocity profile



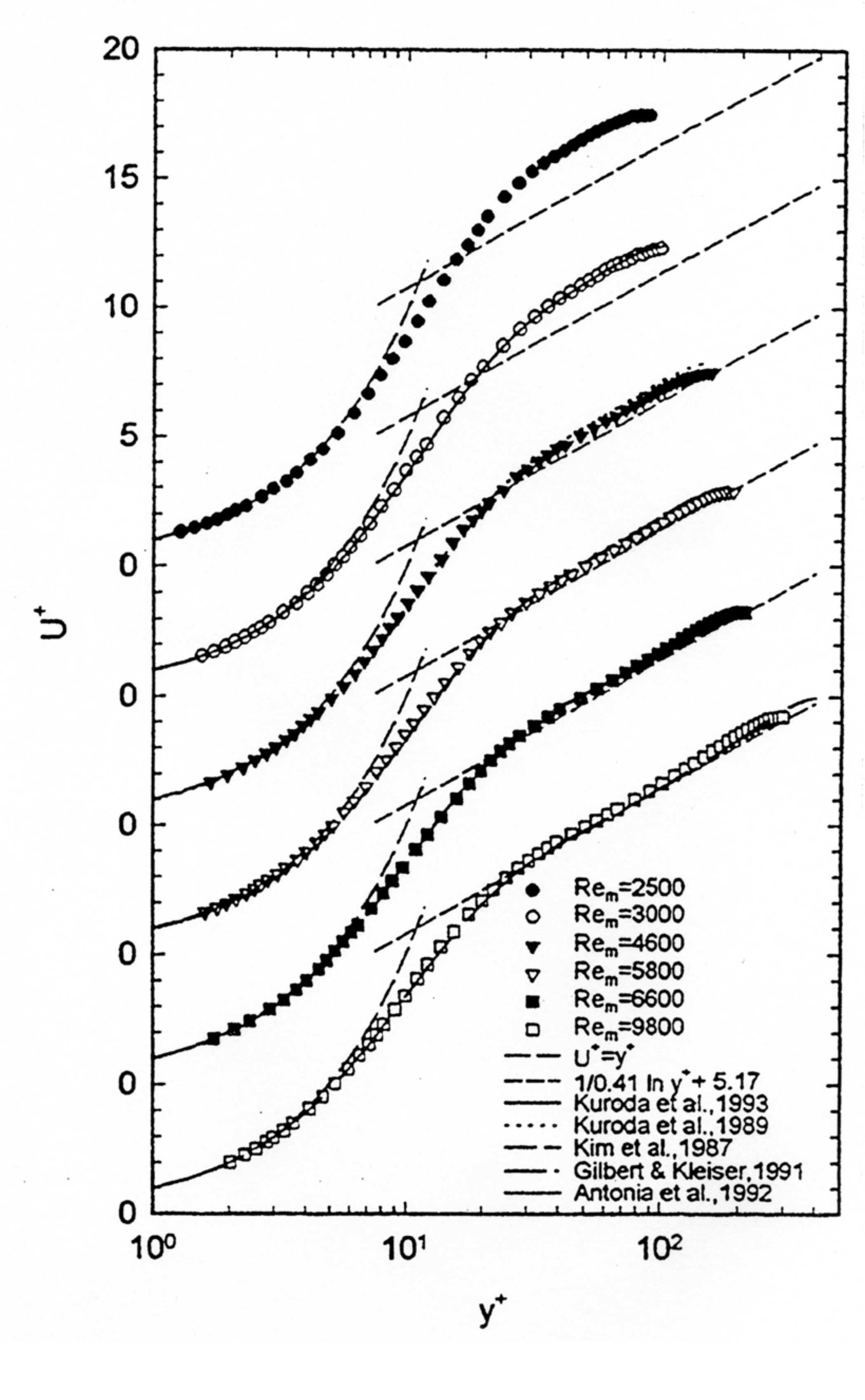
$$u^{+} = y^{+} \tag{5.3-16}$$

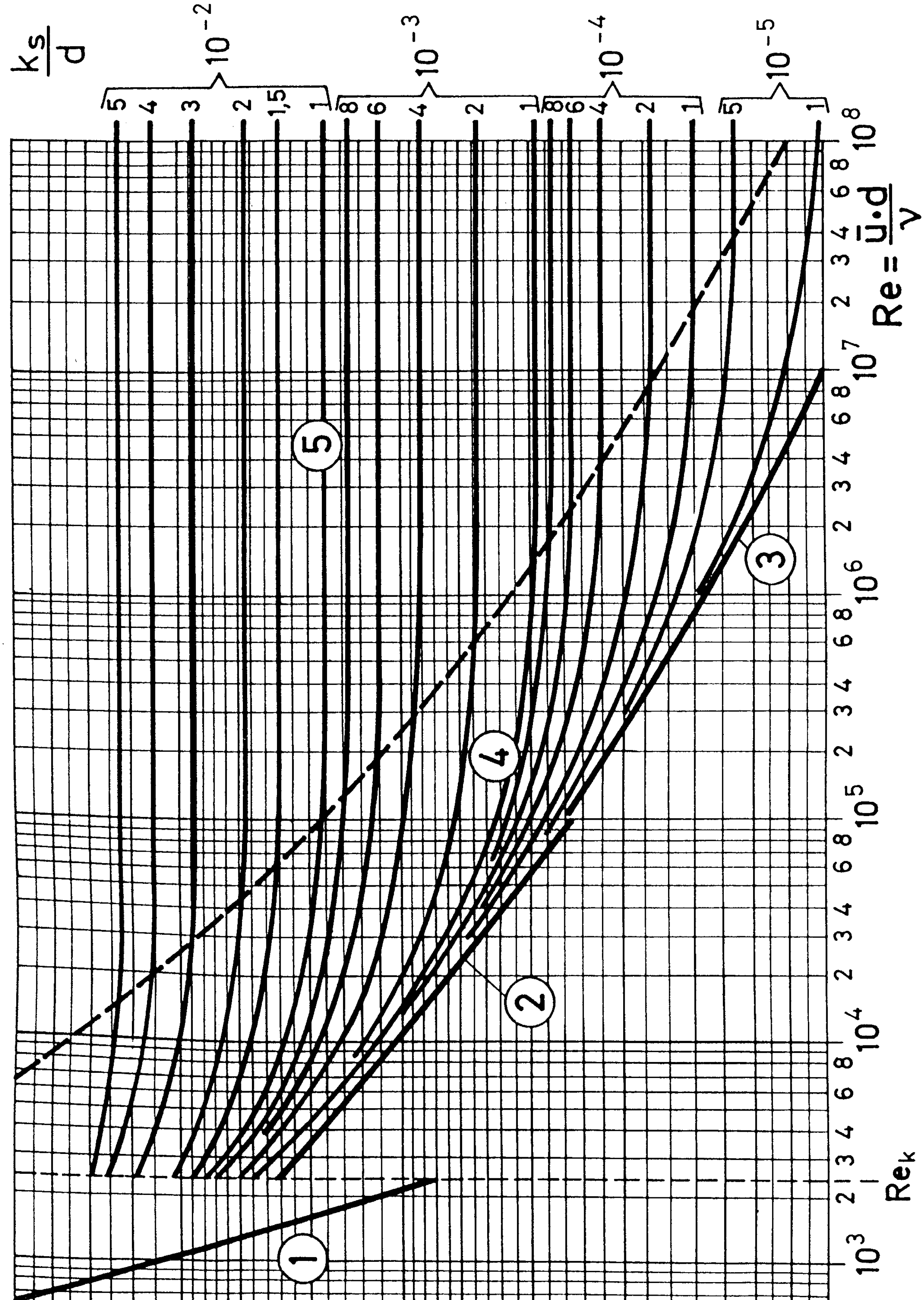
$$u^{+} = \int_{y^{+}=0}^{y^{+}} \frac{2}{1+\sqrt{1+4 l_m^{+2}}}\, dy^{+}, \qquad l_m^{+} = \kappa y^{+} \left\lfloor 1-\exp\left(-y^{+}/A^{+}\right) \right\rfloor \tag{5.3-15}$$

$$u^{+} = \frac{1}{\kappa} \ln y^{+} + B \tag{5.3-12}$$

Turbulent near-wall velocity profiles

Experimental data from Durst et al.: Methods to set up and investigate low Reynolds number, fully developed turbulent plane channel flows. J. Fluids Eng. 120, 496-503 (1998)





Moody diagram of the friction factor $\lambda=\lambda(Re,k_s/d)$ for straight round pipes

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Tensor Analytical Operations in Fluid Mechanics
Fluid Mechanics and Heat Transfer I, LV 321.100

- The tensor of viscous and/or elastic stresses τ reads

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

- The divergence of the stress tensor (line vector Nabla times the tensor) is a vector

$$\left[\vec{\nabla}\cdot\tau\right] = \left(\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z\right) \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{pmatrix}$$

- The inner product of the stress tensor with the velocity vector (column vector)

$$\left[\tau\cdot\vec{\mathrm{v}}\right] = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \cdot \begin{pmatrix} u \\ \mathrm{v} \\ w \end{pmatrix} = \begin{pmatrix} u\tau_{xx} + \mathrm{v}\tau_{xy} + w\tau_{xz} \\ u\tau_{yx} + \mathrm{v}\tau_{yy} + w\tau_{yz} \\ u\tau_{zx} + \mathrm{v}\tau_{zy} + w\tau_{zz} \end{pmatrix}$$

- The inner product of the velocity vector and the divergence of the stress tensor is clearly a scalar

$$\vec{\mathrm{v}}\cdot\left[\vec{\nabla}\cdot\tau\right] = u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \mathrm{v}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

- The divergence of the inner product of the stress tensor and the velocity vector is (of course) also a scalar

$$\vec{\nabla}\cdot\left[\tau\cdot\vec{\mathrm{v}}\right] = \frac{\partial}{\partial x}\left(u\tau_{xx} + \mathrm{v}\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + \mathrm{v}\tau_{yy} + w\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\tau_{zx} + \mathrm{v}\tau_{zy} + w\tau_{zz}\right)$$

- The difference of the two latter equations is denoted by the product with double points and represents the viscous dissipation function

$$\vec{\nabla}\cdot\left[\tau\cdot\vec{\mathrm{v}}\right] - \vec{\mathrm{v}}\cdot\left[\vec{\nabla}\cdot\tau\right] = \left(\tau : \vec{\nabla}\vec{\mathrm{v}}\right) = \Phi_\mu = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial \mathrm{v}}{\partial x} + \tau_{yy}\frac{\partial \mathrm{v}}{\partial y} + \tau_{zy}\frac{\partial \mathrm{v}}{\partial z}$$
$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

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THE EQUATION OF CONTINUITY IN SEVERAL COORDINATE SYSTEMS

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

Spherical coordinates (r, θ, φ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho u_\phi) = 0$$

THE EQUATION OF MOTION IN CARTESIAN COORDINATES (x, y, z)

x-component

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x$$

y-component

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y$$

z-component

$$\rho\left(\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

for a Newtonian fluid with constant ρ and μ:

x-component
$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)+\rho g_x$$

y-component
$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)+\rho g_y$$

z-component
$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z$$

THE EQUATION OF MOTION IN CYLINDRICAL COORDINATES

$$(r,\theta,z);\;\; x=r\cos\theta,\;\; y=r\sin\theta,\;\; z=z$$

r-component
$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r$$

θ-component
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta$$

z-component
$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

for a Newtonian fluid with constant ρ and μ:

r-component
$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

θ-component
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+\rho g_\theta$$

z-component
$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

THE EQUATION OF MOTION IN SPHERICAL COORDINATES

$$(r,\theta,\phi);\;\; x=r\sin\theta\cos\phi,\;\; y=r\sin\theta\sin\phi,\;\; z=r\cos\theta$$

r-component
$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\theta^2+u_\phi^2}{r}\right)$$
$$=-\frac{\partial p}{\partial r}+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta)\right.$$
$$\left.+\frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)+\rho g_r$$

θ-component
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\cot\theta}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi}\right.$$
$$\left.+\frac{\tau_{r\theta}}{r}-\frac{\cot\theta}{r}\tau_{\phi\phi}\right)+\rho g_\theta$$

$$\phi\text{-component} \qquad \rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi}{r}\cot\theta\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}\right) + \rho g_\phi$$

for a Newtonian fluid with constant ρ and μ:

$$r\text{-component} \qquad \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \phi^2}\right) + \rho g_r$$

$$\theta\text{-component} \qquad \rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta}{r} - \frac{u_\phi^2\cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(u_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right) + \rho g_\theta$$

$$\phi\text{-component} \qquad \rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + \frac{u_\theta u_\phi}{r}\cot\theta\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(u_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi}\right) + \rho g_\phi$$

COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS IN CARTESIAN COORDINATES (x, y, z)

$$\tau_{xx} = \mu\left[2\frac{\partial u_x}{\partial x} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\right]$$
$$\tau_{yy} = \mu\left[2\frac{\partial u_y}{\partial y} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\right]$$
$$\tau_{zz} = \mu\left[2\frac{\partial u_z}{\partial z} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\right]$$
$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]$$
$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right]$$
$$\tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right]$$

COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS IN CYLINDRICAL COORDINTES (r, θ, z)

$$\tau_{rr} = \mu\left[2\frac{\partial u_r}{\partial r} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\right]$$
$$\tau_{\theta\theta} = \mu\left[2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\right]$$
$$\tau_{zz} = \mu\left[2\frac{\partial u_z}{\partial z} - \frac{2}{3}(\vec{\nabla}\cdot\vec{u})\right]$$
$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$
$$\tau_{\theta z} = \tau_{z\theta} = \mu\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right]$$
$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right]$$

COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS IN SPHERICAL COORDINATES (r, θ, ϕ)

$$\tau_{rr} = \mu \left[2\frac{\partial u_r}{\partial r} - \frac{2}{3}(\vec{\nabla} \cdot \vec{u}) \right]$$

$$\tau_{\theta\theta} = \mu \left[2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \frac{2}{3}(\vec{\nabla} \cdot \vec{u}) \right]$$

$$\tau_{\phi\phi} = \mu \left[2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r} \right) - \frac{2}{3}(\vec{\nabla} \cdot \vec{u}) \right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r} \right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta} \right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} \right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu \left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r} \right) \right]$$

THE FUNCTION $(\tau : \nabla u) = \Phi_\mu$ FOR NEWTONIAN FLUIDS

Cartesian

$$\frac{1}{\mu}\Phi_\mu = 2\left[\left(\frac{\partial u_x}{\partial x} \right)^2 + \left(\frac{\partial u_y}{\partial y} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 \right] + \left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right]^2 + \left[\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right]^2 + \left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right]^2 - \frac{2}{3}\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right]^2$$

Cylindrical

$$\frac{1}{\mu}\Phi_\mu = 2\left[\left(\frac{\partial u_r}{\partial r} \right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right)^2 + \left(\frac{\partial u_z}{\partial z} \right)^2 \right] + \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r} \right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right]^2 + \left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right]^2 + \left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right]^2 - \frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right]^2$$

Spherical

$$
\begin{aligned}
\frac{1}{\mu}\Phi_\mu = 2\Bigg[&\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)^2 \\
&+ \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right)^2\Bigg] \\
&+ \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2 \\
&+ \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right]^2 \\
&+ \left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right]^2 \\
&- \frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]^2
\end{aligned}
$$

THE ENERGY EQUATION IN CARTESIAN COORDINATES

Total energy:

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{\vec{v}^2}{2}\right)\right]+\left(\vec{\nabla}\cdot\rho\vec{v}\left(e+\frac{\vec{v}^2}{2}\right)\right)=$$

$$=\rho(\vec{v}\cdot\vec{f^B})-(\vec{\nabla}\cdot p\vec{v})+\frac{\partial}{\partial x}(u\tau_{xx}+v\tau_{xy}+w\tau_{xz}+$$

$$+\frac{\partial}{\partial y}(u\tau_{yx}+v\tau_{yy}+w\tau_{yz})+\frac{\partial}{\partial z}(u\tau_{zx}+v\tau_{zy}+w\tau_{zz})-(\vec{\nabla}\cdot\vec{q})+\dot{q}_Q$$

vic.

$$\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho\left|\vec{v}\right|^2\right)+\left[\frac{\partial}{\partial x}u\left(\rho e+\frac{1}{2}\rho\left|\vec{v}\right|^2\right)+\frac{\partial}{\partial y}v\left(\rho e+\frac{1}{2}\rho\left|\vec{v}\right|^2\right)+\frac{\partial}{\partial z}w\left(\rho e+\frac{1}{2}\rho\left|\vec{v}\right|^2\right)\right]=$$

$$+\rho\left(uf_x^B+vf_y^B+wf_z^B\right)-\left(\frac{\partial pu}{\partial x}+\frac{\partial pv}{\partial y}+\frac{\partial pw}{\partial z}\right)$$

$$+\left[\frac{\partial}{\partial x}\left(u\tau_{xx}+v\tau_{xy}+w\tau_{xz}\right)+\frac{\partial}{\partial y}\left(u\tau_{yx}+v\tau_{yy}+w\tau_{yz}\right)+\frac{\partial}{\partial z}\left(u\tau_{zx}+v\tau_{zy}+w\tau_{zz}\right)\right]$$

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+\dot{q}_Q$$

Mechanical Energy:

$$\rho\left[\frac{\partial(\frac{1}{2}\left|\vec{v}\right|^2)}{\partial t}+u\frac{\partial(\frac{1}{2}\left|\vec{v}\right|^2)}{\partial x}+v\frac{\partial(\frac{1}{2}\left|\vec{v}\right|^2)}{\partial y}+w\frac{\partial(\frac{1}{2}\left|\vec{v}\right|^2)}{\partial z}\right]=\rho\frac{d}{dt}\left[\frac{1}{2}\left|\vec{v}\right|^2\right]=$$

$$-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+u\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)$$

$$+v\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+w\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+\rho\left(uf_x^B+vf_y^B+wf_z^B\right)$$

Thermal Energy:

$$\rho\left(\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}+w\frac{\partial e}{\partial z}\right)=-p\left(\vec{\nabla}\cdot\vec{v}\right)+\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+\tau_{zx}\frac{\partial u}{\partial z}$$

$$+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zy}\frac{\partial v}{\partial z}$$

$$+\tau_{xz}\frac{\partial w}{\partial x}+\tau_{yz}\frac{\partial w}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+\dot{q}_Q$$

where $\tau_{xx}\frac{\partial u}{\partial x}+\ldots=\Phi_\mu$ (viscous dissipation function)

Heat conduction equation:

$$\frac{\partial T}{\partial t} = a\Delta T + \frac{\dot{q}_Q}{\rho c}$$

Laplace operator in Cartesian coordinates:

$$\Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Laplace operator in cylindrical coordinates:

$$\Delta T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial x^2}$$

Laplace operator in spherical coordinates:

$$\Delta T = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rT) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2}$$

Energy equation in integral form:

$$\int_V \frac{\partial}{\partial t}\left[\rho\left(e+\frac{\vec{v}^2}{2}\right)\right]dV + \int_O \rho\left(e+\frac{\vec{v}^2}{2}\right)(\vec{v}\cdot\vec{n})dO =$$

$$= \int_V \left(\rho\vec{v}\cdot\vec{f}^B\right)dV - \int_O (p\,\vec{v}\cdot\vec{n})dO +$$

$$+ \int_O (\vec{v}\cdot\vec{\tau}_x)n_x\,dO + \int_O (\vec{v}\cdot\vec{\tau}_y)n_y\,dO + \int_O (\vec{v}\cdot\vec{\tau}_z)n_z\,dO +$$

$$- \int_O (\vec{q}\cdot\vec{n})dO + \int_V \dot{q}_Q dV$$

For solid bodies we have: $\vec{v} = 0\ ,\ \vec{\tau} = 0$

$$\Rightarrow \int_V \frac{\partial}{\partial t}[\rho\, e]dV = - \int_O (\vec{q}\cdot\vec{n})dO + \int_V \dot{q}_Q dV$$

Energy equation in differential form:

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{\vec{v}^2}{2}\right)\right] + \left(\vec{\nabla}\cdot\rho\vec{v}\left(e+\frac{\vec{v}^2}{2}\right)\right) =$$

$$= \rho\left(\vec{v}\cdot\vec{f}^B\right) - \left(\vec{\nabla}\cdot p\,\vec{v}\right) + \frac{\partial}{\partial x}\left(u\tau_{xx}+v\tau_{xy}+w\tau_{xz}\right) +$$

$$+ \frac{\partial}{\partial y}\left(u\tau_{yx}+v\tau_{yy}+w\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\tau_{zx}+v\tau_{zy}+w\tau_{zz}\right) - \left(\vec{\nabla}\cdot\vec{q}\right) + \dot{q}_Q$$

Convective Heat Transport:
Internal flows in pipes and channels

(1) General remarks

The marked difference from external flows around submerged bodies is that, in internal flow, the growth of the boundary layer thickness is limited due to the presence of the pipe or channel walls.

In both internal and external flow we distinguish between laminar and turbulent flow. In internal flow we additionally need to distinguish between the

- entrance region and
- developed flow.

Assuming incompressible flow, the fully developed flow is defined by the constant velocity profile in the downstream direction, which holds for flow with heat transfer also, provided that density changes due to varying temperature may be neglected. The latter is assumed in the following.

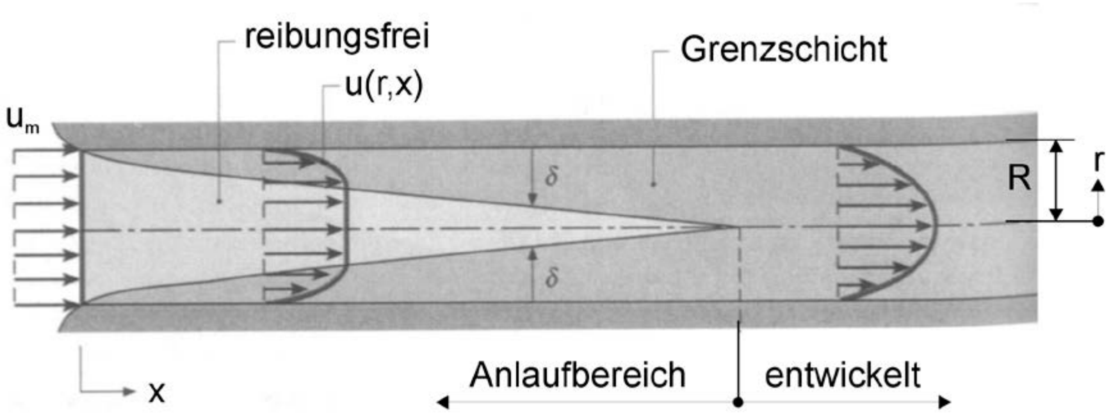
<u>This leads to the question if „thermally developed flow“ can exist at all.</u>

In the following we restrict our discussion to pipes with circular cross section. In many cases, the results may be applied to channels with non-circular cross section as well if the pipe diameter D is replaced by the hydraulic diameter D_h: D $\rightarrow D_h = 4$ A/U, where A is the flow cross section and U its wetted circumference.

a) Hydrodynamic entrance

We briefly repeat the hydrodynamic entrance flow, where the flow velocity profile remains constant from $x = x_E$ on: $\partial u/\partial x = 0$ and $v \equiv 0$.

This flow field may be sketched as follows:



For developed laminar flow we have obtained (Hagen-Poiseuille flow):

$$\frac{u}{u_m} = 2\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad \text{and} \qquad \frac{u_m}{u_{max}} = \frac{1}{2}$$

u_m … volume flow rate equivalent mean velocity; u_{max} … maximum velocity on the pipe axis.

The length of the hydrodynamic entrance region is given (without derivation) by

$$\frac{x_E}{D} \approx 0.05 Re_D \quad \text{where} \quad Re_D = \frac{u_m D}{\nu}$$

For developed turbulent flow we have: $$\frac{u}{u_{max}} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} = \left(\frac{y}{R}\right)^{\frac{1}{7}}$$

where y is the distance from the wall, and, for developed flow, δ=R. We also do not derive the length of the entrance region in turbulent flow which may be estimated as

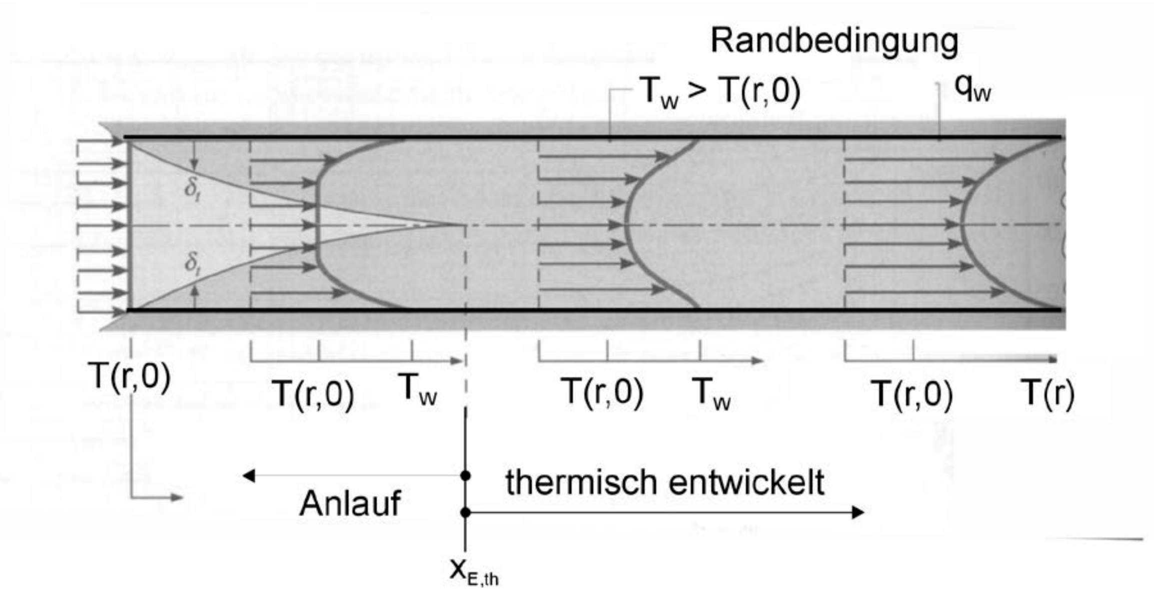
$$10 \leq \frac{x_E}{D} \leq 60 \, .$$

b) Thermal entrance

In the following derivation we assume that the fluid enters the pipe at uniform temperature T(r,0) = constant < T_W (pipe wall temperature). We distinguish two thermal boundary conditions for the flow, which are idealised, but nonetheless close to real situations:

- T_W = constant $\Rightarrow$ $q_W = q_W(x)$
- q_W = constant $\Rightarrow$ $T_W = T_W(x)$

In both cases the thermal boundary layer develops, and a state of „thermally developed flow" may be reached. This may be sketched as follows:



The resulting temperature profile depends on the thermal boundary condition. The temperature increase above the entrance level, however, will increase with the downstream position x in both cases. This means that the continuing heat transfer across the pipe walls changes the local fluid temperature continuously.

The thermal entrance length for laminar flow is given (without derivation) by

$$\frac{x_{E,\,th}}{D} \approx 0.05 \; Re_D \; Pr \quad \text{where} \quad Re_D = \frac{u_m D}{\nu},$$

This relation represents the dependency of the thermal boundary layer thickness on the Prandtl number in laminar flow.

- Pr = 1: $x_{E,\,th} = x_E$ Pr > 1: $x_{E,\,th} > x_E$ Pr < 1: $x_{E,\,th} < x_E$

Pr ~ 1 for most gases (~ 0.7 for air), Pr > 1 for many liquids,
Pr >> 1 e.g. for highly viscous oils, Pr << 1 for liquid metals with large thermal diffusivity.

(2) Definition of „mean quantities"

In internal flow, a well-defined velocity outside a boundary layer does not exist. It is conveniently replaced by the volume flow rate equivalent mean velocity:

$$u_m = \frac{\dot{V}}{A} = \frac{1}{A}\int_A u(r)\,dA$$

Equally important is the definition of a „mean fluid temperature“ T_m, which characterises the energy transport through a flow cross section.

The rate of enthalpy transport across a pipe cross section A is given as

$$\dot{H} = \int_A \rho u h(T)\,dA \quad \text{where} \quad T = T(x,r)$$

The mean temperature T_m is defined such that the product of mass flow rate and specific enthalpy at temperature T_m is the rate of enthalpy transport $\dot{H}$.

Mass flow rate: $\dot{m} = \int_A \rho u\,dA$, rate of enthalpy transport: $\dot{H} = \dot{m}h(T_m)$.

If $h = c_p T$ for ideal gases, and c_p = constant, and for incompressible flow, we obtain

$$T_m = \frac{1}{\dot{m}}\int_A \rho u T\,dA = \frac{\int_A \rho u T\,dA}{\int_A \rho u\,dA} = \frac{1}{u_m A}\int_A u T\,dA$$

This temperature is the „enthalpy transport rate equivalent mean fluid temperature“, and u_m is the volume flow rate equivalent mean flow velocity.

(3) Criterion for thermally developed flow

From the above discussion we conclude that, with continuing heat transfer to or from the fluid, the fluid temperature profile changes continuously, so that

$$\Rightarrow \quad \frac{\partial T(r,x)}{\partial x} \neq 0 \quad \text{and} \quad \frac{dT_m(x)}{dx} \neq 0$$

A detailed analysis of this situation showed that a „thermally developed“ state (denoted by subscript E) may be defined using a non-dimensional quantity. The following criterion proved reasonable as a criterion:

$$\frac{\partial}{\partial x}\left[\frac{T_W(x) - T(r,x)}{T_W(x) - T_m(x)}\right]_E = 0 \,. \qquad (\otimes)$$

In this relation, $T_W(x)$ is the pipe wall temperature, $T_m(x)$ the mean fluid temperature and $T(r,x)$ the local fluid temperature.

This state, characterised by a non-dimensional temperature profile which does not change with the x coordinate, may be reached with both above mentioned thermal boundary conditions, i.e. with T_W = constant and with q_W = constant.

(4) Conclusions from equation ($\otimes$)

If the non-dimensional temperature profile does not depend on x, we may conclude that the derivative w.r.t. the radial coordinate is also not a function of x, i.e.,

$\frac{\partial}{\partial r}[..] \neq f(x)$, since T_W and T_m are constants in this derivative. Especially at r=R we have:

$$\frac{\partial}{\partial r}\left[\frac{T_W - T}{T_W - T_m}\right]_{r=R} = \frac{-\left.\frac{\partial T}{\partial r}\right|_{r=R}}{T_W - T_m} \neq f(x)$$

For the wall heat flux we have

$$q_W = -\lambda \frac{\partial T}{\partial y} = \lambda \frac{\partial T}{\partial r} \ ,$$

which we may also formulate as

$$q_W = \alpha \left(T_W - T_m\right) \ .$$

Equating the two expressions yields

$$\Rightarrow \quad \frac{-\frac{q_W}{\lambda}}{\left(T_W - T_m\right)} = \frac{-\frac{\alpha}{\lambda}\left(T_W - T_m\right)}{\left(T_W - T_m\right)} \neq f(x)$$

From this we obtain the following conclusion, which is important for thermally developed flow:

For constant thermal conductivity (independent on x) the heat transfer coefficient α is constant, i.e.,

$$\alpha = \text{constant} \neq f(x) \quad \textbf{for thermally developed flow}$$

This holds both for T_W = constant and for q_w = constant. The value of the constant, however, is different in the two cases.

This criterion does not hold in the entrance region, where the heat transfer coefficient depends on x, i.e. $\alpha = \alpha(x)$. Around x=0, the thermal boundary layer thickness δ_t is small and, therefore, the heat transfer coefficient is high. The thickness δ_t increases with the downstream coordinate, and α decreases down to the value of the thermally developed state.

(5) Special conclusions from equation (⊗) for thermally developed flow

a) Wall heat flux q_W = constant

With α = constant, we conclude from $\frac{q_W}{\alpha} = T_W - T_m$ immediately that $\left(T_W - T_m\right) \neq f(x)$ and, furthermore

$$\left.\frac{dT_W}{dx}\right|_E = \left.\frac{dT_m}{dx}\right|_E \qquad \textbf{for } \mathbf{q_W} \textbf{ = constant}$$

b) Partial differentiation of equation (⊗) yields

$$\left.\frac{\partial T}{\partial x}\right|_E = \left.\frac{dT_W}{dx}\right|_E - \left(\frac{T_W - T}{T_W - T_m}\right)\left[\frac{dT_W}{dx} - \frac{dT_m}{dx}\right]_E$$

From this we may draw two different conclusions.

1) For q_W = constant, using a), we obtain immediately that [...] = 0 and furthermore that the axial temperature gradient does not depend on r:

$$\left.\frac{\partial T}{\partial x}\right|_E = \left(\left.\frac{dT_W}{dx}\right|_E\right) = \left.\frac{dT_m}{dx}\right|_E \neq g(r) \qquad \textbf{for } \mathbf{q_W} \textbf{ = constant}$$

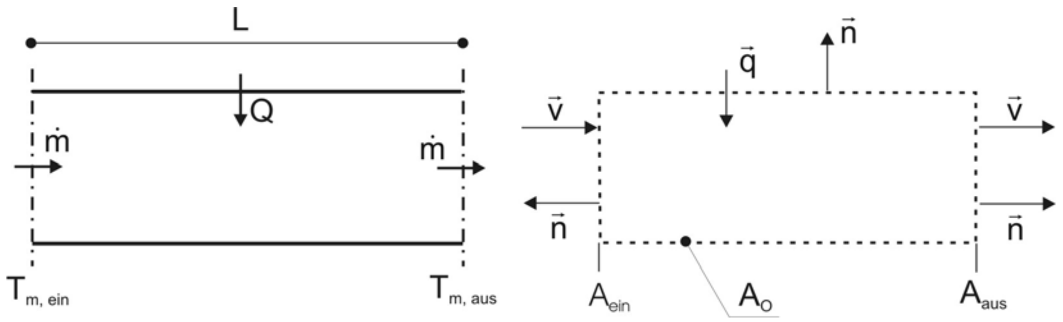
2) For T_W = constant we get $\frac{dT_W}{dx} = 0$ and we conclude that, in this case, the axial temperature gradient varies across the pipe cross section:

$$\left.\frac{\partial T}{\partial x}\right|_E = \frac{T_W - T}{T_W - T_m} \left.\frac{dT_m}{dx}\right|_E = h(r, x)$$

These discussions show that the enthalpy transport rate equivalent mean temperature T_m is a very important quantity for calculating internal flow with heat transfer.

(6) Global balances, energy balance

For this purpose we consider a pipe with constant cross section (diameter D) and length L. Heat is transferred across the pipe wall under the influence of convection. The kinetic energy of the flow is treated as constant, and heat conduction in the x direction of the flow is neglected because of its small influence on the balance.



The energy equation for the sketched control volume reads

$$\int_O \rho \left(e + \frac{\vec{v}^2}{2}\right)(\vec{v}\,\vec{n})\,dO = -\int_O p\,(\vec{v}\,\vec{n})\,dO - \int_O \vec{q}\,\vec{n}\,dO$$

where O denotes the total surface of the control volume. Special evaluation for the present case yields

$$-\int_{A_{in}} \rho \left(e + \frac{\vec{v}^2}{2}\right) v\,dO + \int_{A_{out}} \rho \left(e + \frac{\vec{v}^2}{2}\right) v\,dO = \int_{A_{in}} p\,v\,dO - \int_{A_{out}} p\,v\,dO - \int_{A_O} (-q)\,dO$$

Due to the equal kinetic energies at the entrance and exit, we may rewrite the equation and obtain

$$\int_{A_{out}} \rho\,v \left(e + \frac{p}{\rho}\right) dO - \int_{A_{in}} \rho\,v \left(e + \frac{p}{\rho}\right) dO = Q$$

where Q is the total rate of heat transferred across the control volume walls. Introducing the enthalpy $h = e + \frac{p}{\rho}$ we obtain

$$\dot{H}_{out} - \dot{H}_{in} = Q,$$

and using the earlier definition of the enthalpy transport rate $\dot{H} = \dot{m}\,h(T_m) = \dot{m}\,c_p\,T_m$ yields a very important relation between the change of mean fluid temperature between entrance and exit and the rate of heat transferred:

$$\boxed{Q = \dot{m}\,c_p\,\left(T_{m,out} - T_{m,in}\right)}$$

Except for the special assumptions used (ideal gas etc.), this relation is general and does not depend on the special boundary condition, which may be either q_W = constant or T_W = constant. There were also no restrictions about developed or developing flow.

The above calculation also did not use any restrictions about the pipe length – the relation may therefore be used between any two cross sections of the pipe. We may derive from it a balance for a pipe element with the differential length dx, which renders the formulation even more general:

$$dQ = \dot{m}\, c_p \left(\frac{dT_m}{dx} dx \right)$$

For dQ the relation $dQ = q_W\, U\, dx$ also holds, where U = π D is the circumference of the pipe cross section. For the heat flux we may substitute $q_W = \alpha\left(T_W - T_m\right)$ to obtain

$$dQ = \alpha\left(T_W - T_m\right) U\, dx$$

Equating the two relations for dQ yields

$$\frac{dT_m}{dx} = \frac{\alpha\, D\, \pi}{\dot{m}\, c_p}\left(T_W - T_m\right) \qquad (\otimes\otimes)$$

This relation is essential in the calculation of the mean temperature profile. The solution depends on the boundary condition. In general the pipe diameter D may be a function of the downstream coordinate x.

We may deduce the following behaviour from this equation:

$$T_W > T_m \quad \Rightarrow \quad \frac{dT_m}{dx} > 0: \quad \text{Heating}$$

$$T_W < T_m \quad \Rightarrow \quad \frac{dT_m}{dx} < 0: \quad \text{Cooling}$$

$$D = \text{constant} \quad \Rightarrow \quad \frac{D\,\pi}{\dot{m}\, c_p} = \text{constant}$$

In summary we conclude:

In the entrance region, the heat transfer coefficient α = $\alpha(x)$, for thermally developed flow α = constant. Independently on the special boundary condition, $T_m = T_m(x)$!

(7) Conclusions for q_W = constant

For this case we have $Q = q_W\, U\, L$, so that the difference between the entrance and exit fluid temperatures may be immediately calculated from the global energy balance.

From $q_W = \alpha\,\left(T_W - T_m\right) = \text{constant}$ with $\dot{m}\, c_p = \text{constant}$ we further conclude from equation ($\otimes\otimes$) that

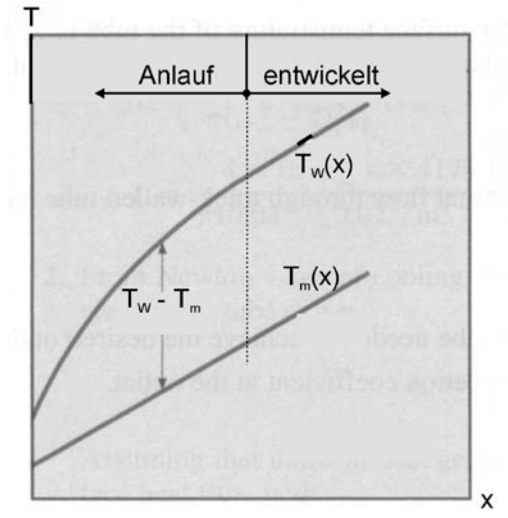
$$\frac{dT_m}{dx} = \frac{q_W\, D\, \pi}{\dot{m}\, c_p} \neq f(x)\,.$$

Integration between x = 0 and a variable position x down the pipe, using the entry boundary condition $T_m(x=0) = T_{m,in}$, we obtain

$$T_m(x) = T_{m,in} + \frac{q_W\, D\, \pi}{\dot{m}\, c_p}\, x \quad \textbf{for } \mathbf{q_W = constant}$$

This means that the mean fluid temperature varies linearly with the coordinate x, both in the entrance region and in thermally developed flow. For the entrance region we furthermore conclude from

$q_W = \alpha(x)\left(T_W - T_m\right)$ that $\left(T_W - T_m\right)$ increases with x, since $\alpha(x)$ decreases. The mean fluid temperature profile for q_w = constant may therefore be sketched as follows:



Note that, even in the more general case that $q_W = q_W(x)$ is variable, but a known function, and D = D(x), equation (⊗⊗) may be integrated.

With $Q = \int_0^x q_W(x)U(x)\,dx$, the difference $\left(T_{m,out} - T_{m,in}\right)$ may also be calculated.

(8) Conclusions for T_W = constant

We start again from equation (⊗⊗) which we note down again here

$$\frac{dT_m}{dx} = \frac{\alpha D \pi}{\dot{m}\, c_p}\left(T_W - T_m\right)$$

With T_W = constant we may introduce conveniently a temperature difference $\Delta T(x) = T_W - T_m(x)$ to facilitate the calculation. This manipulation turns equation (⊗⊗) into the form

$$-\frac{d(\Delta T)}{dx} = \frac{U\alpha}{\dot{m}\, c_p}\Delta T$$

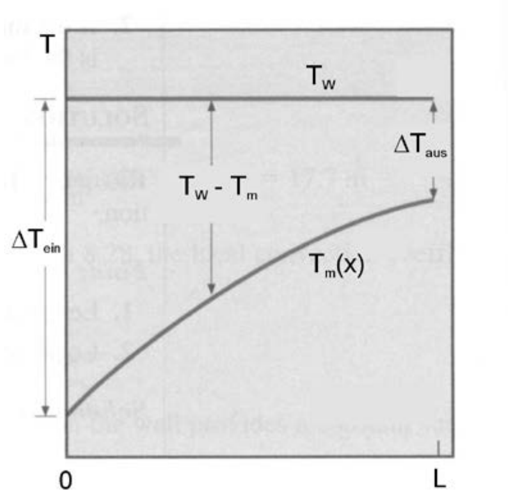
Separation of variables and integration yields

$$\int_{\Delta T_{in}}^{\Delta T_{out}} \frac{d(\Delta T)}{\Delta T} = -\frac{U}{\dot{m} c_p}\int_0^x \alpha(x)\,dx \quad \Rightarrow \quad \ln\frac{\Delta T_{out}}{\Delta T_{in}} = -\frac{U\,x}{\dot{m}\, c_p}\left[\frac{1}{x}\int_0^x \alpha(x)\,dx\right] \qquad (⊗⊗⊗)$$

where the expression in square brackets $\left[\frac{1}{x}\int_0^x \alpha(x)\,dx\right] = \bar{\alpha}_x$ is the mean heat transfer coefficient between x = 0 and any position x down the pipe. Rewriting the equation yields

$$\frac{\Delta T_{out}}{\Delta T_{in}} = \frac{T_W - T_m(x)}{T_W - T_{m,in}} = \exp\left(-\frac{U\,x}{\dot{m}\, c_p}\,\bar{\alpha}_x\right).$$

From this equation we see that the temperature difference $T_W - T_m(x)$ decreases exponentially with increasing coordinate x. This dependency is sketched in the following picture.



The exponential mean temperature profile makes the calculation of the rate of heat transferred a bit more complicated than before. The global energy balance between entry and exit reads in a rewritten form

$$Q = \dot{m}\ c_p\ \left[\left(T_W - T_{m,in}\right) - \left(T_W - T_{m,out}\right)\right] = \dot{m}\ c_p\ \left[\Delta T_{in} - \Delta T_{out}\right]$$

Expressing ($\dot{m}\,c_p$) with the help of equation (⊗⊗⊗) above we obtain

$$Q = U\ x\ \bar{\alpha}_x\ \frac{\Delta T_{out} - \Delta T_{in}}{\ln\dfrac{\Delta T_{out}}{\Delta T_{in}}}\ =\ A_O\ \bar{\alpha}_x\ \left(\Delta T_{log}\right)$$

where

A_O = Ux is the transfer surface and $\Delta T_{log} = \dfrac{\Delta T_{out} - \Delta T_{in}}{\ln\dfrac{\Delta T_{out}}{\Delta T_{in}}}$ the logarithmic mean temperature difference. This relation represents a heat transfer law relating the total rate of heat transferred to the mean fluid temperature difference between entry and exit.

Note: The two above mentioned thermal boundary conditions represent simplifications, but they may be related to practical situations in many cases.

- q_W = constant: electrically heated walls or constant heat load of the outer wall surface by radiation.
- T_W = constant: in many practical processes with phase transition (boiling, condensation).
- Both boundary conditions may be enforced by appropriate control of heating or cooling.

(9) Determination of heat transfer coefficients and Nusselt numbers

(a) Laminar flow in pipes of circular cross section, thermally fully developed.

For this case we know the velocity profile

$$\frac{u}{u_m} = 2\left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{where} \quad \frac{u_m}{u_{max}} = \frac{1}{2} \quad \text{and} \quad v = 0.$$

The energy equation for incompressible, steady flow and negligible viscous dissipation, using the boundary layer approximation $\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial r^2}$, reads

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

For fully developed flow with $v = 0$ we may solve the simplified equation

$$\boxed{u\frac{\partial T}{\partial x} = a\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]}$$

i) Wall heat flux q_W = constant

For this case we have shown above that

$$\left.\frac{\partial T}{\partial x}\right|_E = \left.\frac{dT_m}{dx}\right|_E,$$

and also that the mean fluid temperature varies linearly with the x coordinate. In this case the boundary layer approximation $\frac{\partial^2 T}{\partial x^2} = 0$ is exact. Substituting the velocity profile into the energy equation, we may integrate to obtain the temperature profile T(r,x). From this result and upon introduction of the mean fluid temperature T_m we may deduce by some manipulations the following important result:

$$\boxed{Nu = \frac{\alpha D}{\lambda} = 4.36}$$ for q_W **= constant** in a pipe with circular cross section

The Nusselt number formed with the pipe diameter as the length scale is constant for this case.

ii) Wall temperature T_W = constant

We start again from the energy equation with boundary layer approximation. Using the results derived above for T_W = constant we obtain an equation which cannot be solved in a closed form but requires an iterative method. The result for the Nusselt number is

$$\boxed{Nu = \frac{\alpha D}{\lambda} = 3.66}$$ for T_W **= constant** in a pipe with circular cross section

(b) Turbulent flow in pipes with circular cross section, thermally fully developed region

The two subsequent relations are valid for moderate temperature differences $(T_W - T_m)$ only, and the fluid material properties must be determined at the temperature T_m. Both relations are valid both for T_W = constant and for q_W = constant and were experimentally validated in the following ranges of the parameters:

$$0.7 \le Pr \le 160$$

$$Re_D = \frac{u_m D}{\nu} > 10000$$

$$\frac{L}{D} \ge 10$$

- Colburn equation:

$$\boxed{Nu = \frac{\alpha D}{\lambda} = 0.023\, Re_D^{4/5}\; Pr^{1/3}}$$

- Dittus-Boelter equation:

$$\boxed{Nu = \frac{\alpha D}{\lambda} = 0.023\, Re_D^{4/5}\; Pr^{n}}$$

where $n=0.4$ for heating $T_W > T_m$

and $n=0.3$ for cooling $T_W < T_m$.

Finally we note that both relations are very easy to apply, but they include errors in the rate of heat transfer of up to 25%. We do not present more complex relations here.

(c) Flow with heating in the entrance region

The calculation of heat transfer in the thermal entrance region and in a combined hydraulic and thermal entrance is more complex. Appropriate relations determining the Nusselt numbers may be found in the literature.

Physical properties of water at the pressure p = 1 bar

T [°C]	ρ [kg/m³]	c_p [J/kg K]	β [10^{-3}/K]	λ [W/m K]	μ [10^{-6} Pa s]	ν [10^{-6} m²/s]	a [10^{-6} m²/s]	Pr [-]
- 20	992.8	4375	- 0.7056	0.5118	4311.0	4.342	0.118	36.85
- 15	995.8	4312	- 0.4946	0.5259	3312.8	3.372	0.122	27.17
- 10	997.8	4269	- 0.3281	0.5388	2533.4	2.639	0.125	20.86
- 5	999.1	4238	- 0.1943	0.5508	2149.4	2.151	0.130	16.54
0	999.8	4217	- 0.0852	0.5620	1791.8	1.792	0.133	13.44
5	1000.0	4202	0.0055	0.5724	1519.6	1.520	0.136	11.16
10	999.8	4192	0.0823	0.5820	1307.6	1.308	0.139	9.42
15	999.2	4186	0.1486	0.5911	1139.0	1.140	0.141	8.07
20	998.3	4182	0.2067	0.5996	1002.6	1.004	0.144	6.99
25	997.2	4180	0.2586	0.6076	890.8	0.893	0.146	6.13
30	995.8	4178	0.3056	0.6151	797.7	0.801	0.148	5.42
35	994.1	4178	0.3488	0.6221	719.5	0.724	0.150	4.83
40	992.3	4179	0.3890	0.6287	653.1	0.658	0.152	4.34
45	990.3	4180	0.4267	0.6348	596.3	0.602	0.153	3.93
50	988.1	4181	0.4523	0.6405	547.1	0.554	0.155	3.57
55	985.7	4183	0.4963	0.6458	504.3	0.512	0.157	3.27
60	983.2	4185	0.5288	0.6507	465.8	0.475	0.158	3.00
65	980.5	4187	0.5590	0.6553	433.8	0.442	0.160	2.77
70	977.7	4190	0.5900	0.6595	404.5	0.414	0.161	2.57
75	974.7	4193	0.5190	0.6633	378.3	0.388	0.162	2.39
80	971.4	4196	0.6473	0.6668	355.0	0.365	0.164	2.23
85	968.5	4200	0.6748	0.6699	333.9	0.345	0.165	2.09
90	965.1	4205	0.7018	0.6728	315.0	0.326	0.166	1.97
95	961.7	4210	0.7284	0.6753	297.8	0.310	0.167	1.86
99.63 +)	958.4	4215	0.7527	0.6773	283.3	0.296	0.168	1.76

+) State of saturation

Physical properties of water at the pressure p = 5 bar								
T [°C]	ρ [kg/m^3]	c_p [J/kg K]	β [10^{-3}/K]	λ [W/m K]	μ [10^{-6} Pa s]	ν [10^{-6} m^2/s]	a [10^{-6} m^2/s]	Pr [-]
0	1000.0	4215	- 0.08376	0.5622	1791	1.79	0.133	13.4
25	997.3	4178	0.2590	0.6078	890.7	0.893	0.146	6.12
50	988.2	4180	0.4622	0. 6407	547.2	0.554	0.155	3.57
75	974.9	4192	0.6185	0.6635	378.4	0.388	0.162	2.39
100	958.3	4215	0.7539	0.6777	282.3	0.295	0.168	1.76
150	916.8	4310	1.024	0.6836	181.9	0.198	0.173	1.15

Physical properties of water at the pressure p = 10 bar								
T [°C]	ρ [kg/m^3]	c_p [J/kg K]	β [10^{-3}/K]	λ [W/m K]	μ [10^{-6} Pa s]	ν [10^{-6} m^2/s]	a [10^{-6} m^2/s]	Pr [-]
0	1000.3	4212	- 0.08199	0.5625	1790	1.79	0.134	13.4
25	997.6	4177	0.2595	0.6081	890.6	0.893	0.146	6.12
50	988.5	4179	0.4620	0.6410	547.2	0.554	0.155	3.57
75	975.1	4191	0.6179	0.6638	378.6	0.388	0.162	2.39
100	958.6	4214	0.7530	0.6780	282.4	0.295	0.168	1.76
150	917.1	4308	1.022	0.6839	182.0	0.198	0.173	1.15

Physical properties of water in the state of saturation (liquid)									
T [°C]	p [bar]	ρ [kg/m^3]	c_p [J/kg K]	β [10^{-3}/K]	λ [W/m K]	μ [10^{-6} Pa s]	ν [10^{-6} m^2/s]	a [10^{-6} m^2/s]	Pr [-]
0.01	0.00611	999.8	4217	- 0.0853	0.562	1791.4	1.792	0.1333	13.44
10	0.01227	999.7	4193	0.0821	0.582	1307.7	1.308	0.1388	9.42
20	0.02337	998.3	4182	0.2066	0.600	1002.7	1.004	0.1436	6.99
30	0.04242	995.7	4179	0.3056	0.615	797.7	0.801	0.1478	5.42
40	0.07375	992.2	4179	0.3890	0.629	653.1	0.658	0.1516	4.34
50	0.12335	988.0	4181	0.4624	0.640	547.1	0.554	0.1550	3.57
60	0.19919	983.1	4185	0.5288	0.651	466.8	0.475	0.1582	3.00
70	0.31151	977.7	4190	0.5900	0.659	404.4	0.414	0.1610	2.57
80	0.47359	971.6	4197	0.6473	0.667	355.0	0.365	0.1635	2.234
90	0.70108	965.1	4205	0.7019	0.673	315.0	0.326	0.1658	1.969
100	1.01325	958.1	4216	0.7547	0.677	282.2	0.294	0.1677	1.756
110	1.4326	950.7	4229	0.8068	0.681	254.9	0.268	0.1694	1.583
120	1.9854	942.8	4245	0.8590	0.683	232.1	0.246	0.1707	1.442
130	2.7012	934.6	4263	0.9121	0.684	212.7	0.228	0.1718	1.325

Physical properties of dry air at the pressure p = 1 bar

T [°C]	ρ [kg/m^3]	c_p [J/kg K]	β [10^{-3}/K]	λ [W/m K]	μ [10^{-6} Pa s]	ν [10^{-6} m^2/s]	a [10^{-6} m^2/s]	Pr [-]
- 40	1.4952	1006	4.304	0.02145	15.09	10.09	14.3	0.71
- 20	1.3765	1006	3.962	0.02301	16.15	11.73	16.6	0.71
0	1.2754	1006	3.671	0.02454	17.10	13.41	19.1	0.70
20	1.1881	1007	3.419	0.02603	17.98	15.13	21.8	0.70
40	1.1120	1008	3.200	0.02749	18.81	16.92	24.5	0.69
60	1.0452	1009	3.007	0.02894	19.73	18.88	27.4	0.69
80	0.9859	1010	2.836	0.03038	20.73	21.02	30.5	0.69
100	0.9329	1012	2.684	0.03181	21.60	23.15	33.7	0.69
120	0.8854	1014	2.547	0.03323	22.43	25.33	37.0	0.68
140	0.8425	1017	2.423	0.03466	23.19	27.53	40.5	0.68
160	0.8036	1020	2.311	0.03607	24.01	29.88	44.0	0.68
180	0.7681	1023	2.209	0.03749	24.91	32.43	47.7	0.68
200	0.7356	1026	2.115	0.03891	25.70	34.94	51.6	0.68
250	0.6653	1035	1.912	0.04243	27.40	41.18	61.6	0.67

T	Temperature in °C	β	Thermal expansion coefficient	a	Thermal diffusivity
p	Pressure	λ	Thermal conductivity	Pr	Prandtl number
ρ	Density	μ	Dynamic viscosity		
c_p	Specific heat capacity at p = constant	ν	Kinematic viscosity		

FICK's law (equimolar counter-diffusion in binary systems)

$$\boxed{\tilde{j}_A = -D_{AB}\frac{dc_A}{dz}} \quad \left[\frac{kmol}{m^2 \cdot s}\right]$$

$D_{AB}\ldots$ diffusion coefficient

Characterisation of mixtures with K components

molar quantities	$x_i = \dfrac{\frac{w_i}{M_{G,i}}}{\sum\limits_{j=1}^{K}\frac{w_j}{M_{G,j}}}$	$c_i = \dfrac{\rho_i}{M_{G,i}}$	$c_{ges} = \dfrac{\rho_{ges}}{M_{G,ges}}$
mass-based quantities	$w_i = \dfrac{x_i \cdot M_{G,i}}{\sum\limits_{j=1}^{K} x_j \cdot M_{G,j}}$	$\rho_i = c_i \cdot M_{G,i}$	$\rho_{ges} = c_{ges} \cdot M_{G,ges}$

molar quantities: $M_{G,ges} = \sum\limits_{i=1}^{K} x_i \cdot M_{G,i}$

mass-based quantities: $M_{G,ges} = \dfrac{1}{\sum\limits_{i=1}^{K}\frac{w_i}{M_{G,i}}}$

The balance equation

$$\boxed{\frac{\partial c_i}{\partial t} + \mathrm{div}\,\vec{n}_i = \dot{r}_i}$$

$\vec{n}_i = \tilde{j}_i + \tilde{v}c_i\ldots$ where $\tilde{j}_i$ is the diffusive molar flux

Chemical reaction rate

$$\boxed{\dot{r}_i = k_i \cdot c_i^n} \quad \left[\frac{kmol}{m^3 \cdot s}\right]$$

Equimolar diffusion

$$\boxed{dy_i = -\frac{\dot{n}_i}{D}\frac{dz}{c}} \quad \text{Cartesian one-dimensional}$$

One-sided diffusion - STEFAN diffusion

$$\boxed{\frac{dy_i}{1-y_i} = -\frac{\dot{n}}{D}\frac{dz}{c}} \quad \text{Cartesian one-dimensional}$$

Mass fluxes	**Molar fluxes**
$\vec{\dot{m}}_1 = \rho w_1 \vec{v} - \rho D \text{ grad } w_1$	$\vec{\dot{n}}_1 = c x_1 \vec{\bar{v}} - c D \text{ grad } x_1$
$\vec{\dot{m}}_2 = \rho w_2 \vec{v} - \rho D \text{ grad } w_2$	$\vec{\dot{n}}_2 = c x_2 \vec{\bar{v}} - c D \text{ grad } x_2$
$w_1 + w_2 = 1 \qquad \text{grad } w_1 + \text{ grad } w_2 = 0$	$x_1 + x_2 = 1 \qquad \text{grad } x_1 + \text{ grad } x_2 = 0$
$\text{grad} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$	
total: $\vec{\dot{m}} = \rho \vec{v}$	$\vec{\dot{n}} = c \vec{\bar{v}}$

One-dimensional differential component transport equation for binary mixtures in equimolar diffusion

$$\boxed{\frac{\partial y_1}{\partial t} = D \frac{\partial^2 y_1}{\partial z^2}} \qquad \text{2. FICK's law}$$

Diffusion coefficient for binary gas mixtures

FULLER/SCHETTLER/GIDDINGS developed the following empirical relation:

$$D = \frac{1,43 \cdot 10^{-7} \cdot T^{1,75}}{p\, M_{G,12}^{0,5} \left[(\sum v)_1^{1/3} + (\sum v)_2^{1/3} \right]^2} \qquad \left[\frac{m^2}{s} \right]$$

p .... Pressure $[bar]$
T ...... Temperature $[K]$
M_G .... Molar mass $[kg/kmol]$
v .... Atomic diffusion volume

$$M_{G,12} = \frac{2}{\frac{1}{M_{G,1}} + \frac{1}{M_{G,2}}}$$

Atomic diffusion volume v.

Atomic and structural increase of diffusion volume			
C	15.9	F	14.7
H	2.31	Cl	21.0
O	6.11	Br	21.9
N	4.54	I	29.8
Aromatic ring	−18.3	S	22.9
Heterocyclic ring	−18.3		
Diffusion volume of simple atoms, molecules and mixtures			
He	2.67	CO	18.0
Ne	5.98	CO_2	26.9
Ar	16.2	N_2O	35.9
Kr	24.5	NH_3	20.7
Xe	32.7	H_2O	13.1
H_2	6.12	SF_6	71.3
D_2	6.84	Cl_2	38.4
N_2	18.5	Br_2	69.0
O_2	16.3	SO_2	41.8
Luft	19.7		

Non-dimensional mass transfer numbers

Mass transfer across a flat plate in laminar parallel flow

$$Sh_x = 0.332\sqrt{Re_x} \cdot Sc^{1/3}$$

for

$$Re < 10^5 \qquad \text{and} \qquad 0.6 < Sc < 2000$$

Mass transfer in turbulent flow

Turbulent flow along a flat plate (Zhukauskas)

$$Sh_x = 0.0296\ Re_x^{0.8} \cdot Sc^{0.43}$$

$$Sh = 0.037\ Re^{0.8} \cdot Sc^{0.43}$$

for $Re > 10^5$ and $Sc < 380$

Mass transfer across a flat plate in laminar or turbulent flow

In most practical cases, a laminar boundary layer is not formed even at moderate Reynolds numbers due bluff plate tips and the turbulence level of the incoming flow. Krischer and Kast represented experimental data including both the laminar and the turbulent flow regimes by a mean curve.
The result allows the mean Sherwood number for the plate length L to be calculated for a wide range of Schmidt numbers using the relation

$$Sh = \sqrt{Sh_{lam}^2 + Sh_{turb}^2} \qquad \text{for} \quad 10 \leq Re \leq 10^7 \quad \text{and} \quad 0.7 \leq Sc \leq 70,000$$

The laminar Sh number is given by Pohlhausen's relation

$$Sh_{lam} = 0.664\sqrt{Re_L} \cdot \sqrt[3]{Sc}$$

and the turbulent Sherwood number follows from the more recent relation by Petukhov and Popov

$$Sh_{turb} = \frac{0.037 \cdot Re^{0.8} \cdot Sc}{1 + 2.443 \cdot Re^{-0.1}\left(Sc^{2/3} - 1\right)}$$

The characteristic numbers in these relations are defined as

$$Re_L = \frac{u_\infty L}{\nu} \qquad Sc = \frac{\nu}{D} \qquad Sh = \frac{\beta L}{D}$$

The global mean rate of mass transfer

Empirical and semi-empirical relations for mass transfer on spheres

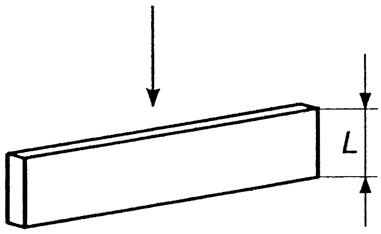
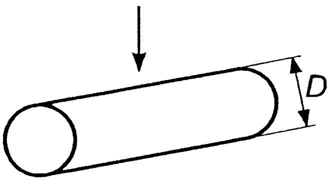
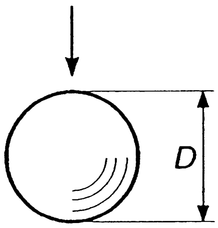
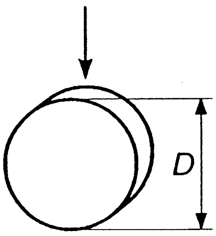
	Re	**Sc**	**Sh**
Garner/Suckling	$100 \div 700$	$1100 \div 2200$	$2 + 0.95\sqrt{Re}Sc^{1/3}$
Frössling	> 100	≤ 1000	$2 + 0.552\sqrt{Re}Sc^{1/3}$
Steinberger/Treybal	$10 \div 17 \cdot 10^3$	$1 \div 70000$	$2 + 0.347Re^{0.62}Sc^{0.31}$
Rowe u.a.	$25 \div 1150$	1220	$0.79\sqrt{Re}Sc^{1/3}$

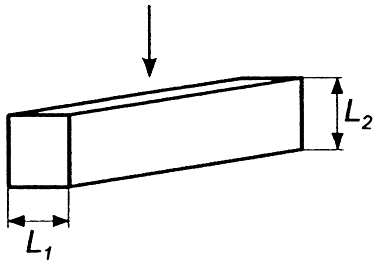
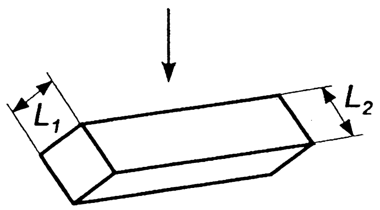
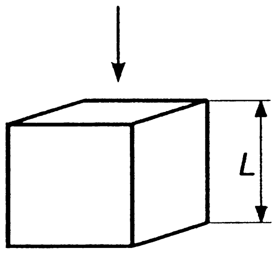
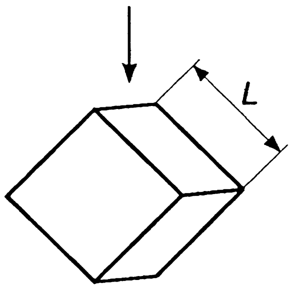
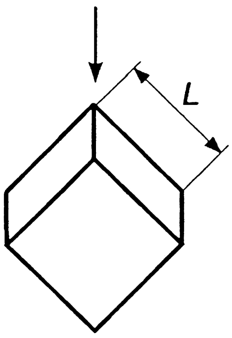
Mass transfer on submerged individual bodies of various shape

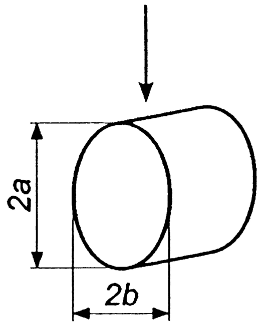
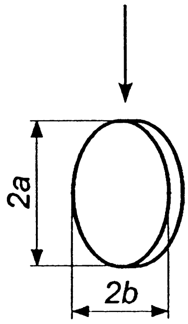
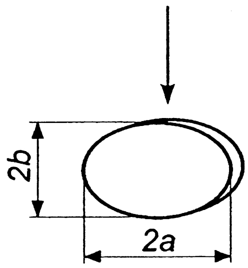
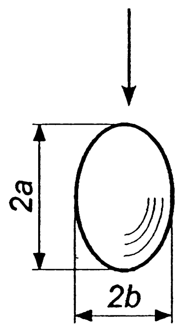
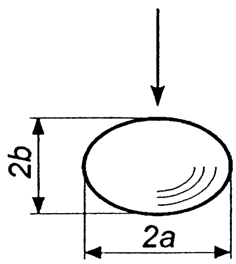
$$Sh = Sh_{min} + \sqrt{Sh_{lam}^2 + Sh_{turb}^2}$$

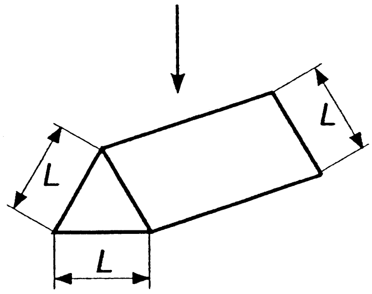
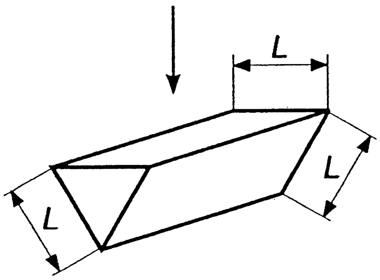
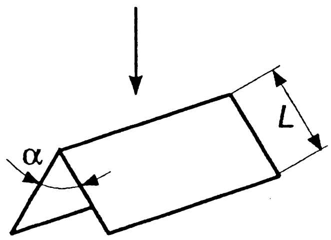
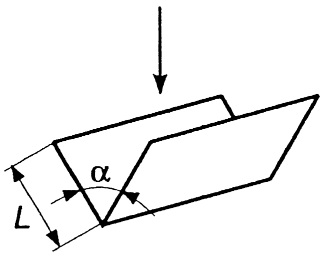
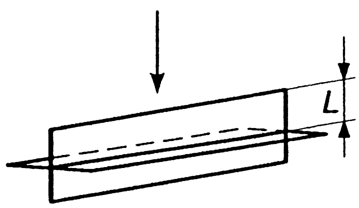
Sphere	$Sh_{min} = 2$
Infinitely long cylinder	$Sh_{min} = 0.3$
Plate	$Sh_{min} = 0$

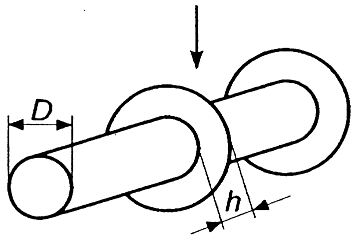
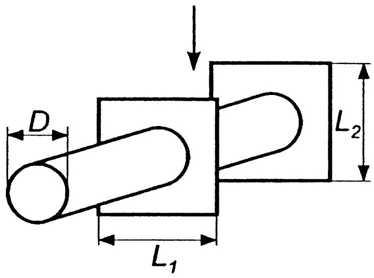
Tabelle 1: Characteristic contact length L' for various bodies.

SKIZZE	BESCHREIBUNG	ANSTRÖMLÄNGE
	**Ebene Platte** *längs angeströmt*	$L' = L$
	**Kreiszylinder** *quer angeströmt*	$L' = \frac{\pi}{2} D$
	**Kugel**	$L = D$
	**Kreisscheibe** *in Richtung eines Durchmessers angeströmt*	$L' = \frac{\pi}{4} D$

SKIZZE	BESCHREIBUNG	ANSTRÖMLÄNGE
	**Rechteckförmiges Prisma** *quer angeströmt* *a) Strömung ⊥ auf eine Fläche*	$L' = L_1 + L_2$
	*b) Strömung ⊥ auf eine Kante*	$L' = L_1 + L_2$
	**Würfel** *a) Strömung ⊥ auf eine Fläche*	$L' = 1{,}50L$
	*b) Strömung ⊥ auf eine Kante*	$L' = 1{,}24L$ *in einer ungeordneten Schüttung* $L' = 1{,}3L$
	*c) Strömung ⊥ auf ein Eck*	$L' = 1{,}16L$

SKIZZE	BESCHREIBUNG	ANSTRÖMLÄNGE
	**Ellipsenförmiger Zylinder** *quer angeströmt*	$L' = \frac{\pi}{2}\left[1{,}5\,(a+b) - \sqrt{ab}\right]$ $= \frac{\pi}{2}(a+b)$
 	**Ellipsenförmig Scheibe** *a) Strömung ⊥ zur kleinen Halbachse* *b) Strömung ⊥ zur großen Halbachse*	 $L' = \frac{\pi}{2}a$ $L' = \frac{\pi}{2}b$
 	**Rotationsellipsoid** *a) Strömung ⊥ zur kleinen Halbachse* *b) Strömung ⊥ zur großen Halbachse*	 $L' = \frac{(a+b)^2}{2b}$ $L' = \frac{(a+b)^2}{2a}$

SKIZZE	BESCHREIBUNG	ANSTRÖMLÄNGE
	**Dreieckförmiges Prisma** *quer angeströmt* *a) Strömung $\perp$ auf eine Kante*	$L' = \frac{3}{2} L$
	*b) Strömung $\perp$ auf eine Fläche*	$L' = \frac{3}{2} L$
	**Winkelförmiges Prisma** *quer angeströmt* *a) Strömung $\perp$ auf Winkelkante* *1) $\alpha > 60°$* *2) $\alpha < 60°$*	$L' = 2L$ $L' = L$
	*b) Strömung $\perp$ in den Winkel* *1) $\alpha > 60°$* *2) $\alpha < 60°$*	$L' = 2L$ $L' = L$
	**Kreuzförmiges Prisma** *quer angeströmt* *Strömung $\perp$ auf Kante*	$L' = 4L$

SKIZZE	BESCHREIBUNG	ANSTRÖMLÄNGE
	**Berippte Rohre** *quer angeströmt* *a) kreisförmiges Rippe*	$L' = \frac{\pi}{2}\sqrt{D^2 + h^2}$
	*b) rechteckförmige Rippe*	$L' = \frac{\pi}{2}\sqrt{D^2 + h^2}$ mit $h = 0{,}565 \cdot L_1\sqrt{\frac{L_1}{L_2}} - \frac{D}{2}$

Mass transfer in internal flow in channels and pipes

$$\dot{N}_A = A\beta\overline{\Delta c_A}$$

$$\overline{\Delta c_A} = \frac{\Delta c_{A,0} - \Delta c_{A,L}}{ln\frac{\Delta c_{A,0}}{\Delta c_{A,L}}}$$

In hydraulically developed laminar flow in pipes with circular cross section

$$Sh = \frac{\beta d}{D} = 3.66 + \frac{0.188\left(Re\,Sc\frac{d}{L}\right)^{0.8}}{1 + 0.117\left(Re\,Sc\frac{d}{L}\right)^{0.467}}$$

Ratio of *concentration* entrance length to pipe diameter

$$\frac{L_{0,c}}{d} = 1.365\,Re\,Sc$$

In hydraulic and concentration entrance in pipe with circular cross section (laminar)

Ratio of hydraulic entrance length to pipe diameter

$$\frac{L_{0,u}}{d} = 0.0575\,Re$$

Ratio of concentration to dynamic entrance lengths

$$\frac{L_{0,c}}{L_{0,u}} = 23.7 \cdot Sc$$

$$Sh = 3.66 + \frac{0.0677\left(Re\,Sc\frac{d}{L}\right)^{1.33}}{1 + 0.1 \cdot Sc^{0.17}\left(Re\,Sc\frac{d}{L}\right)^{0.83}} \qquad \text{for } 0.1 \leq Sc \leq 100 \text{ and } Re\,Sc \cdot \frac{d}{L} \geq 0$$

Mass transfer in turbulent flow in pipes with circular cross section

Hydraulically smooth pipe wall

$$Sh = \frac{\lambda_0}{8} \frac{Re \cdot Sc}{1.07 + 12.7\,(Sc^{2/3} - 1)\sqrt{\frac{\lambda_0}{8}}} \left(1 - \frac{180}{Re^{0.75}}\right) \left[1 + \left(\frac{d}{L}\right)^{2/3}\right]$$

The above relation is valid for the regimes

$$Re \geq 2300 \qquad Sc \geq 0.5 \qquad 0 \leq \frac{d}{L} < \infty$$

The friction factor λ_0 for smooth pipe wall may be calculated for arbitrary Reynolds number of turbulent pipe flow using the relation by Filonenko:

$$\lambda_0 = \frac{1}{\left(1.82 \cdot log_{10} Re - 1.64\right)^2}$$

Rough pipe wall

$$Sh_r = \frac{\lambda_r}{8} \frac{Re \cdot Sc}{1 + \left(Sc\frac{\lambda_r}{\lambda_0} - 1\right) \cdot 1.5 \cdot Re^{-1/8} \cdot Sc^{-1/6}}$$

$$\frac{1}{\sqrt{\lambda_r}} = -2 \cdot log_{10}\left(\frac{2.51}{Re\sqrt{\lambda_r}} + \frac{1}{3.71}\frac{K}{d}\right)$$

Phase equilibria

Range of low concentrations ($x_i \to 0$):

Here we may apply Henry's law:

$$p_i = H_{i,x} \cdot x_i \quad \text{where} \quad H_{i,x} = f(T) \text{ is only a function of temperature}$$

$$\text{or} \quad p_i = H_{i,c} \cdot c_i \quad \text{where} \quad H_{i,c} = \frac{H_{i,x}}{c_f}$$

$$\text{or} \quad y_i = H_i^* \cdot x_i \quad \text{where} \quad H_i^* = \frac{H_{i,x}}{p}$$

where the Henry constant has the following dimensions:

$$H_{i,x} \ldots\ldots [Pa] \ \ (\text{ or } [bar])$$

$$H_{i,c} \ldots\ldots \left[\tfrac{m^3 \cdot Pa}{mol}\right] \ \ (\text{or } \left[\tfrac{m^3 \cdot bar}{mol}\right])$$

$H_i^* \ldots\ldots [-]$

Range of high concentrations ($x_i \rightarrow 1$):

Here we may apply RAOULT's law:

$$p_i = x_i \cdot p_i^\circ$$

where p_i° $[Pa]$ is the vapour pressure of the pure component i.

The mass transport coefficient

Mass transport coefficient related to the gas phase

$$\boxed{k_{i,g} = \frac{1}{\frac{1}{\beta_{i,g}} + \frac{H_{i,c}}{R\,T_I \cdot \beta_{i,f}}}}$$

Mass transport coefficient related to the liquid phase

$$\boxed{k_{i,f} = \frac{1}{\frac{R\,T_I}{H_{i,c} \cdot \beta_{i,g}} + \frac{1}{\beta_{i,f}}}}$$

Mass transfer resistances

We define the following mass transfer resistances:

A) For the gas phase:

related to the gas phase: $c_{i,g,\infty} - c_{i,g,I}^* = \frac{1}{\beta_{i,g}} \cdot \dot{n}_i$ $\qquad \boxed{r_{i,g,g} = \frac{1}{\beta_{i,g}}}$

related to the liquid phase: $\underbrace{\frac{H_{i,c}}{R\,T_I} c_{i,f,I}^*}_{c_{i,g,I}^*} - \underbrace{\frac{H_{i,c}}{R\,T_I} c_{i,f,\infty}}_{c_{i,g,\infty}^*} = \frac{H_{i,c}}{R\,T_I} \cdot \frac{1}{\beta_{i,f}} \cdot \dot{n}_i$ $\qquad \boxed{r_{i,g,f} = \frac{H_{i,c}}{R\,T_I} \frac{1}{\beta_{i,f}}}$

B) For the liquid phase:

related to the gas phase: $\underbrace{\frac{R\,T_I}{H_{i,c}} c_{i,g,\infty}}_{c_{i,f,\infty}^*} - \underbrace{\frac{R\,T_I}{H_{i,c}} c_{i,g,I}^*}_{c_{i,f,I}^*} = \frac{R\,T_I}{H_{i,c}} \cdot \frac{1}{\beta_{i,g}} \cdot \dot{n}_i$ $\qquad \boxed{r_{i,f,g} = \frac{R\,T_I}{H_{i,c}} \frac{1}{\beta_{i,g}}}$

related to the liquid phase: $c_{i,f,I}^* - c_{i,f,\infty} = \frac{1}{\beta_{i,f}} \cdot \dot{n}_i$ $\qquad \boxed{r_{i,f,f} = \frac{1}{\beta_{i,f}}}$

HTU/NTU approach for constant phase flow rates

$$HTU_1 = \frac{\dot{N}_1}{U\,c_1 k_{i,g}} \qquad NTU_1 = \pm \int_{y_{i,\alpha}}^{y_{i,\omega}} \frac{dy_i}{y_i^* - y_i}$$

or

$$HTU_2 = \frac{\dot{N}_2}{U\, c_2 k_{i,f}} \qquad\qquad NTU_2 = \pm \int\limits_{x_{i,\alpha}}^{x_{i,\omega}} \frac{dx_i}{x_i^* - x_i}$$

Dependency of the mass transfer coefficient on the molar fraction (binary system)

STEFAN correction

$$\boxed{\frac{\beta}{\beta_0} \approx \frac{1}{1 - y_{i,\infty}} \approx \frac{1}{1 - y_{i,I}^*}}$$

For gases the correction may be written in terms of partial pressure as per:

$$\boxed{\frac{\beta}{\beta_0} \approx \frac{p}{p - p_{i,\infty}} \approx \frac{p}{p - p_{i,I}^*}}$$

List of Translations

The present list puts together translations of German words in figures of the present materials where the figures were not easily transformed into a fully English version. The meanings of words still contained in those figures are listed here in alphabetic order, starting from the German word.

Anlauf, Anlaufbereich	entrance region
Anströmlänge	wetted length
aus	out
berippte Rohre	finned pipes
Beschreibung	description
dreieckförmiges Prisma	triangular cylinder
ebene Platte	flat plate
ein	in
ellipsenförmige Scheibe	elliptic disk
ellipsenförmiger Zylinder	elliptic cylinder
entwickelt	developed
Grenzschicht	boundary layer
in einer untergeordneten Schüttung	in random packing
in Richtung eines Durchmessers angeströmt	in flow along one diameter
kreisförmige Rippe	circular fin
Kreisscheibe	circular disk
Kreiszylinder	circular cylinder
kreuzförmiges Prisma	crossed plates
Kugel	sphere
längs angeströmt	in parallel flow
mit	with
quer angeströmt	in transverse flow
Randbedingung	boundary condition
rechteckförmige Rippe	rectangular fin
rechteckförmiges Prisma	rectangular cylinder
reibungsfrei	frictionless
Rotationsellipsoid	spheroid
Skizze	sketch

Strömung $\perp$ auf ein Eck	flow normal on one tip
Strömung $\perp$ auf eine Fläche	flow normal on one face
Strömung $\perp$ auf eine (Winkel-) Kante	flow normal on one edge
Strömung $\perp$ in den Winkel	flow normal into the angle
Strömung $\perp$ zur großen Halbachse	flow normal to the long semi-axis
Strömung $\perp$ zur kleinen Halbachse	flow normal to the short semi-axis
thermisch entwickelt	thermally developed
winkelförmiges Prisma	angled plates
Würfel	cube