Spatially Distributed Networked Sliding Mode Control

Jakob Ludwiger[1], Martin Steinberger[1], Martin Horn[1,2]

***Abstract*— A robust control algorithm for spatially distributed multi-input networked control systems with time-varying transmission delays is proposed. In order to increase the dependability of the spatially distributed scheme, a pivotal requirement to the algorithm is the exclusive usage of locally available information in each controller node. This ensures that no communication between the controller nodes is necessary. The algorithm consists of a buffering mechanism and discrete-time integral sliding mode control laws which are capable of rejecting matched perturbations with bounded change rate. The effectiveness of the proposed approach is demonstrated by means of a numerical simulation.**

I. Introduction

In modern control systems, the feedback loop is often closed using networked communication technologies. This means that sensors, controllers and actuators exchange their information using shared communication channels. Especially large scale systems with several actuators benefit from this architecture as there may be the need, e.g., due to restricted computational power of the controller nodes, that the control law has to be implemented in a spatially distributed fashion. Spatially distributed implementation refers to an architecture, where the control law is not executed on only one central controller node but is distributed over several controller nodes. Figure 1 shows a very general networked control system (NCS) where for each of the m input channels, a controller node is implemented. This architecture benefits also from increased flexibility, e.g. additional actuator and controller nodes can easily be added to the network. Additionally, large spatial distances can be overcome with a significantly reduced wiring effort or even wirelessly. However, this architecture comes along with additional challenges regarding the controller design. Depending on the network technology, the communication path could be affected by data loss and/or transmission delays.

There are several scientific publications targeting the design of control algorithms which are robust with respect to data losses (see e.g. [1], [2]). An additional big challenge is to design the control algorithm such that the networked induced delays are considered in the design process. Some approaches, e.g. [3] and [4] make use of Lyapunov-Krasovskii analysis to ensure stability of the closed loop system affected by time-varying delays. Others are based on prediction in order to compensate for the influence of delays and data losses, e.g. [5]. Furthermore, event triggered control is used to reduce the network load, which in consequence potentially lead to reduced delays and data losses (see e.g. [6] and [7]). Very good overviews on the existing methods are given in [8] and [9].

In [10] a sliding mode based control algorithm which deals with time-varying delays is proposed. It makes use of discrete-time first order sliding mode control in order to stabilize the NCS and alleviate matched perturbations. This approach was extended in [11] where two discrete-time sliding mode algorithms for sliding variables with higher relative degree where presented.

In the present paper, a control algorithm for multi input systems based on integral sliding mode control is presented. The algorithm is designed with a special focus on the possibility to implement it in a spatially distributed fashion. In order to achieve high dependability of the distributed system, no communication between the controller nodes is required. Additionally, the algorithm offers the possibility to consider the networked induced delay individually for each controller node which permits different priorities of the communication links or spatial location of the controller and actuator nodes. This paper significantly differs from [10], [11] since in these papers first order sliding mode approaches for single input systems applying eigenvalue assignment for the sliding surface design were proposed. Especially the requirement that the algorithm should be implementable in a

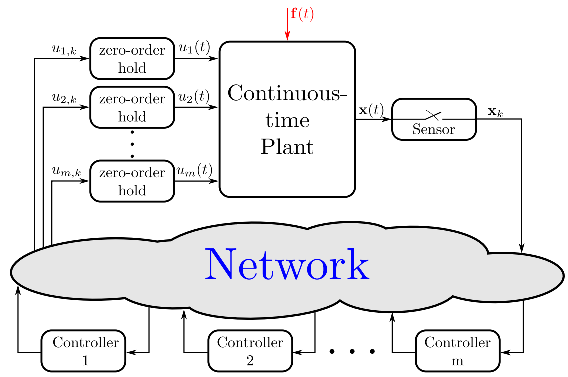

Fig. 1. Architecture

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spatially distributed fashion introduced additional challenges in the controller design.

II. Problem Statement

Consider the linear time-invariant plant

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}_c \boldsymbol{x}(t) + \boldsymbol{B}_c \left(\boldsymbol{u}(t) + \boldsymbol{f}(t)\right) \tag{1}$$

with state vector $\boldsymbol{x} \in \mathbb{R}^n$, inputs $\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^{\mathrm{T}}$ and matched perturbations $\boldsymbol{f} = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix}^{\mathrm{T}}$. The dynamic matrix $\boldsymbol{A}_c$ and input matrix $\boldsymbol{B}_c$ have appropriate dimensions. All elements of the state vector $\boldsymbol{x}$ are sampled with the constant sampling time T which results in the sampled states $\boldsymbol{x}_k = \boldsymbol{x}(kT)$. For each of the m input channels a controller node is implemented which receives data via a communication network. Due to network imperfections, the i$^{\text{th}}$ controller node receives the data delayed by the variable time delay $\tau^s_{i,k} \in \mathbb{R}^+$ and evaluates the control law based on the received data which introduces an additional variable time delay $\tau^c_{i,k} \in \mathbb{R}^+$ which accounts for limited computational resources. The i$^{\text{th}}$ controller node transmits the computed output to the i$^{\text{th}}$ actuator, who receives the data delayed by $\tau^a_{i,k} \in \mathbb{R}^+$. The following assumptions are made:

Assumption 1: System (1) is controllable, the input matrix $\boldsymbol{B}_c$ has full column rank and the constant sampling time T is non-pathological i.e., controllability is not lost due to sampling (see [12]).

Assumption 2: The communication network ensure loss-free communication (no packet dropouts occur) and the sum of all time delays for each input channel is bounded i.e.,

$$\tau_{i,k} = \tau^s_{i,k} + \tau^c_{i,k} + \tau^a_{i,k} \leq \delta_i T \quad \forall k, \quad i = 1, 2, \ldots, m \tag{2}$$

with $\delta_i \in \mathbb{N}$.

Assumption 3: The sampling time T is chosen small enough to ensure that the intersample behavior of the matched perturbation $\boldsymbol{f}(t)$ is negligible and can therefore be assumed as piece-wise constant i.e.,

$$\boldsymbol{f}(t) = \boldsymbol{f}(kT) = \boldsymbol{f}_k \quad kT \leq t \leq (k+1)T,\ k \in \mathbb{N}_0. \tag{3}$$

Additionally, the change rate of each component is bounded, i.e.

$$\sup \left| \frac{\boldsymbol{f}_{i,k+1} - \boldsymbol{f}_{i,k}}{T} \right| = L_i < \infty. \quad i = 1, 2, \ldots, m. \tag{4}$$

The control algorithm proposed in this paper aims for two goals. Firstly, system (1) should be robustly stabilized via a networked feedback. Secondly, there should be no communication necessary between the m controller nodes.

III. Overview of Proposed Approach

In fig. 2 the architecture of the proposed approach is depicted. The controller nodes are designed as discrete-time integral sliding mode controllers (D-ISMC). Using integral sliding mode control with an adequate choice of the sliding variables offers the possibility to cast the problem in a form,

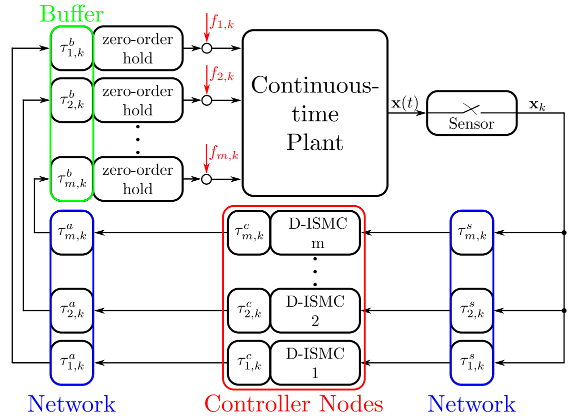

Fig. 2. Proposed Approach

where algorithms designed for systems with relative degree one, can be applied.

Additionally, a buffer is needed at each input channel, which keeps the round trip time (the delay from the sensor to the actuator) constant. Therefore, a time stamp is attached by the sensor to each measurement packet before transmission. The controller nodes receive the measurements and transmit their outputs with the time stamp from the sensor attached. The buffer is located directly at the zero-order hold and due to a time synchronization technique between buffers and sensor, the current delay $\tau_{i,k}$ can be computed by the buffer. The buffer stores the received packets and forwards them after the additional delay

$$\tau^b_{i,k} = \delta_i T - \tau_{i,k} \tag{5}$$

which ensures constant round trip times $\begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_m \end{bmatrix}^{\mathrm{T}} T = \boldsymbol{\delta} T$. In multimedia applications, this buffering mechanism is very common (see e.g. [13]) but has been introduced in the control community as well (see e.g. [14]). In the following sections, the model of the buffered networked control system as well as the discrete-time integral sliding mode controller design will be described in more detail.

IV. Model of the Plant with Buffered Network

Using the constant sampling time T, the constant round trip times $\boldsymbol{\delta} T$ ensured by the buffers in combination with plant (1), the discrete-time model of the NCS is given by

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}(\hat{\boldsymbol{u}}_k + \boldsymbol{f}_k) \tag{6}$$

with $\boldsymbol{A} = e^{\boldsymbol{A}_c T}$, $\boldsymbol{B} = \int_0^T e^{\boldsymbol{A}_c s} \boldsymbol{B}_c ds = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_m \end{bmatrix}$ and $\hat{\boldsymbol{u}}_k = \begin{bmatrix} u_{1,k-\delta_1} & u_{2,k-\delta_2} & \cdots & u_{m,k-\delta_m} \end{bmatrix}^{\mathrm{T}}$. Defining the lifted state vector

$$\boldsymbol{\xi}_k = \Big[\boldsymbol{x}_k^{\mathrm{T}} \quad u_{1,k-1} \quad \cdots \quad u_{1,k-\delta_1} \quad \cdots \quad \cdots \quad u_{m,k-1} \quad \cdots \quad u_{m,k-\delta_m} \Big]^{\mathrm{T}} \tag{7}$$

results in a lifted model

$$\boldsymbol{\xi}_{k+1} = \hat{\boldsymbol{A}}\boldsymbol{\xi}_k + \hat{\boldsymbol{B}}\boldsymbol{u}_k + \hat{\boldsymbol{B}}_f \boldsymbol{f}_k \tag{8}$$

with

$$\hat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \mathbf{0}_{n\times(\delta_1-1)} & \boldsymbol{b}_1 & \cdots & \mathbf{0}_{n\times(\delta_m-1)} & \boldsymbol{b}_m \\ \mathbf{0}_{1\times n} & \mathbf{0} & 0 & \cdots & \mathbf{0} & 0 \\ \mathbf{0}_{(\delta_1-1)\times n} & \boldsymbol{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{1\times n} & \mathbf{0} & 0 & \cdots & \mathbf{0} & 0 \\ \mathbf{0}_{(\delta_m-1)\times n} & \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I} & \mathbf{0} \end{bmatrix} \tag{9a}$$

$$\hat{\boldsymbol{B}} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \\ 1 & \cdots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & 1 \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \qquad \hat{\boldsymbol{B}}_f = \begin{bmatrix} \boldsymbol{B} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{9b}$$

The matrices of the lifted model have dimensions $\hat{\boldsymbol{A}} \in \mathbb{R}^{(n+\theta)\times(n+\theta)}$ and $\hat{\boldsymbol{B}}, \hat{\boldsymbol{B}}_f \in \mathbb{R}^{(n+\theta)\times m}$ with $\theta = \sum_{i=1}^{m} \delta_i$. Please note that the perturbation $\boldsymbol{f}_k$ does not fulfill the matching condition for the lifted model (8) (see [15]) but it is fulfilled for the original system (6). Hence, a sliding mode control law can be designed in order to robustly stabilize the origin of state vector $\boldsymbol{x}_k$ but not state vector $\boldsymbol{\xi}_k$ which complies with the problem statement.

V. Controller Design

Integral sliding mode control is a very effective method to robustify a control loop, which was designed for the unperturbed case i.e. $\boldsymbol{f}_k = \mathbf{0}$, $\forall k$. This is achieved by defining the control law as

$$\boldsymbol{u}_k = \boldsymbol{u}_k^N + \boldsymbol{u}_k^S \tag{10}$$

where $\boldsymbol{u}_k^N$ represents a nominal control law and $\boldsymbol{u}_k^S$ the sliding mode part of the control law. In order to comply with the problem statement, the control law will be designed not only to achieve asymptotical stability of the nominal lifted plant, i.e. (8) with $\boldsymbol{f}_k = \mathbf{0}$, $\forall k$, but also no data exchange between the controller nodes should be necessary. As a consequence, the i$^{\text{th}}$ controller node cannot use the whole state vector $\boldsymbol{\xi}_k$ because this vector would include the controller outputs of all other controller nodes and their history.

A. Nominal Control Law

The nominal lifted model

$$\hat{\boldsymbol{\xi}}_{k+1} = \hat{\boldsymbol{A}}\hat{\boldsymbol{\xi}}_k + \hat{\boldsymbol{B}}\boldsymbol{u}_k^N \tag{11}$$

with nominal lifted state vector

$$\hat{\boldsymbol{\xi}}_k = \Big[\boldsymbol{x}_k^{\mathrm{T}} \quad u_{1,k-1}^N \quad \cdots \quad u_{1,k-\delta_1}^N \quad \cdots \quad \cdots \quad u_{m,k-1}^N \quad \cdots \quad u_{m,k-\delta_m}^N\Big]^{\mathrm{T}} \tag{12}$$

results from (8) for $\boldsymbol{f}_k = \boldsymbol{u}_k^S = \mathbf{0}$, $\forall k$. In the following theorem, the nominal control law is designed based on theorem 5 in [16] in such a way that global asymptotical stability of the nominal closed loop system is achieved. Additionally, each controller node uses only the measurements $\boldsymbol{x}_k$ and locally available information.

Theorem 1: Let assumptions 1 and 2 hold. Consider the constant round trip times $\boldsymbol{\delta}T$ ensured by the buffers, the nominal lifted model (11), and controller

$$\boldsymbol{u}_k^N = \boldsymbol{K}\hat{\boldsymbol{\xi}}_k = \begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{K}_2\end{bmatrix}\hat{\boldsymbol{\xi}}_k \tag{13}$$

with

$$\boldsymbol{K}_x \in \mathbb{R}^{m\times n}, \quad \boldsymbol{K}_2 = \begin{bmatrix} \boldsymbol{k}_1^{\mathrm{T}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{k}_2^{\mathrm{T}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{k}_m^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{k}_i^{\mathrm{T}} \in \mathbb{R}^{1\times\delta_i}. \tag{14}$$

If there exist a symmetric positive definite matrix $\boldsymbol{Y} \in \mathbb{R}^{n+\theta\times n+\theta}$, matrices

$$\boldsymbol{Z} = \begin{bmatrix}\boldsymbol{Z}_x & \boldsymbol{Z}_2\end{bmatrix} \qquad \text{with} \quad \boldsymbol{Z}_x \in \mathbb{R}^{m\times n}, \tag{15a}$$

$$\boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{z}_1^{\mathrm{T}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{z}_2^{\mathrm{T}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{z}_m^{\mathrm{T}} \end{bmatrix}, \qquad \boldsymbol{z}_i^{\mathrm{T}} \in \mathbb{R}^{1\times\delta_i} \tag{15b}$$

and $\boldsymbol{X} = \begin{bmatrix}\boldsymbol{X}_1 & \mathbf{0} \\ \boldsymbol{X}_2 & \boldsymbol{X}_3\end{bmatrix}$ with $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$, $\boldsymbol{X}_2 \in \mathbb{R}^{\theta\times n}$,

$$\boldsymbol{X}_3 = \begin{bmatrix} \bar{\boldsymbol{X}}_{3,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \bar{\boldsymbol{X}}_{3,2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \bar{\boldsymbol{X}}_{3,m} \end{bmatrix} \quad \text{with} \quad \bar{\boldsymbol{X}}_{3,i} \in \mathbb{R}^{\delta_i\times\delta_i} \tag{16}$$

and a scalar $0 \le \gamma < 1$ that satisfy

$$\begin{bmatrix} \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} - \boldsymbol{Y} & \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{A}}^{\mathrm{T}} - \boldsymbol{Z}^{\mathrm{T}}\hat{\boldsymbol{B}}^{\mathrm{T}} \\ \hat{\boldsymbol{A}}\boldsymbol{X} - \hat{\boldsymbol{B}}\boldsymbol{Z} & (1-\gamma)\boldsymbol{Y} \end{bmatrix} > 0, \tag{17}$$

then the nominal lifted closed loop system is globally asymptotically stable with

$$\boldsymbol{K}_2 = \boldsymbol{Z}_2\boldsymbol{X}_3^{-1} \qquad \text{and} \qquad \boldsymbol{K}_x = (\boldsymbol{Z}_x - \boldsymbol{K}_2\boldsymbol{X}_2)\boldsymbol{X}_1^{-1}. \tag{18}$$

Additionally, only measurements $\boldsymbol{x}_k$ and local information, i.e. the history of the own control signals $u_{i,k}$, is used in each controller node.

Proof: Considering the multiplication

$$\boldsymbol{K}\boldsymbol{X} = \begin{bmatrix}\boldsymbol{K}_x\boldsymbol{X}_1 + \boldsymbol{K}_2\boldsymbol{X}_2 & \boldsymbol{K}_2\boldsymbol{X}_3\end{bmatrix} \tag{19}$$

and applying (18) gives

$$\boldsymbol{K}\boldsymbol{X} = \boldsymbol{Z}. \tag{20}$$

Using (20) in (17) gives

$$\begin{bmatrix} \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} - \boldsymbol{Y} & \boldsymbol{X}^{\mathrm{T}}(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\boldsymbol{K})^{\mathrm{T}} \\ (\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\boldsymbol{K})\boldsymbol{X} & (1-\gamma)\boldsymbol{Y} \end{bmatrix} > 0. \tag{21}$$

Applying theorem 3 in [17] to (21) with $N = 1$, $G_1 = \boldsymbol{X}$, $S_1 = \boldsymbol{Y}$ and $A_1 = \hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\boldsymbol{K}$ proofs the global asymptotical stability of the nominal closed loop system.
Additionally, due to the specific structure of $\boldsymbol{K}_2$, the i$^{\text{th}}$ controller node uses only locally available information. ■

Remark 1: Some applications trigger the need to design the nominal control law in such a way that it depends exclusively on the measurements $\boldsymbol{x}_k$ and not on the history of the actuating signals. The solution to this problem represents a special case of theorem 1. Setting $\boldsymbol{Z}_2 = \boldsymbol{0}$ and using a dense matrix $\boldsymbol{X}_3 \in \mathbb{R}^{\theta \times \theta}$ results in a nominal control law (13) with $\boldsymbol{K}_2 = \boldsymbol{0}$.

B. Sliding Mode Control Law

In the following theorem, the sliding mode based part of the control law is designed such that the input channels are decoupled and no communication between the controller nodes is necessary.

Theorem 2: Let assumptions 1–3 hold. Consider the constant round trip times $\boldsymbol{\delta}T$, ensured by the buffers (5), and lifted model (8) with state vector $\boldsymbol{\xi}_k$.
The discrete-time integral sliding variables are defined as

$$\boldsymbol{\sigma}_k = \begin{bmatrix} \sigma_{1,k} & \cdots & \sigma_{m,k} \end{bmatrix}^{\mathrm{T}} = \boldsymbol{M}\boldsymbol{\xi}_k + \boldsymbol{w}_k \tag{22}$$

with

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1^{\mathrm{T}} & 1 & \boldsymbol{0}_{1\times(\delta_1-1)} & \cdots & 0 & \boldsymbol{0}_{1\times(\delta_m-1)} \\ \vdots & \boldsymbol{0}_{(m-1)\times 1} & \ddots & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{m}_m^{\mathrm{T}} & 0 & \boldsymbol{0} & \cdots & 1 & \boldsymbol{0} \end{bmatrix} \tag{23a}$$

$$\boldsymbol{M}_x = \begin{bmatrix} \boldsymbol{m}_1 & \cdots & \boldsymbol{m}_m \end{bmatrix}^{\mathrm{T}} = \boldsymbol{B}^+ \tag{23b}$$

where $\boldsymbol{B}^+ = (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})^{-1}\boldsymbol{B}$ denotes the left inverse of $\boldsymbol{B}$.
Let the nominal control law in (10) be given by (13) and

$$\boldsymbol{w}_{k+1} = \begin{bmatrix} w_{1,k} & \cdots & w_{m,k} \end{bmatrix}^{\mathrm{T}} = -\boldsymbol{M}(\hat{\boldsymbol{A}}\boldsymbol{\xi}_k + \hat{\boldsymbol{B}}\boldsymbol{u}_k^N). \tag{24}$$

The sliding mode part $\boldsymbol{u}_k^S$ in (10) is given by

$$\begin{aligned} u_{i,k}^S &= \sigma_{i,k} - T\alpha_i\sqrt{|\sigma_{i,k}|}\,\mathrm{sign}\left(\sigma_{i,k}\right) + T\nu_{i,k} \\ \nu_{i,k+1} &= \nu_{i,k} - T\beta_i\,\mathrm{sign}\left(\sigma_{i,k}\right) \end{aligned} \tag{25}$$

where

$$\alpha_i = 1.5\sqrt{\Lambda_i}, \qquad \beta_i = 1.1\Lambda_i, \qquad \Lambda_i \geq \frac{L_i}{T}, \tag{26}$$

with change rates L_i (4).
Then

1) the states $\boldsymbol{x}_k$ of system (6) are ultimately bounded.
2) only measurements $\boldsymbol{x}_k$ and local information, i.e. the history of the own control signals $u_{i,k}$, is used in each controller node.

Remark 2: The super twisting is capable of compensating for perturbations with bounded change rate but unbounded amplitude which fits perfectly the problem statement of this paper, therefore this algorithm was chosen.

Proof: The proof consists of two parts. In the first part it will be shown, that a sliding mode controller based on theorem 2 results in the ultimate boundedness of the states $\boldsymbol{x}_k$.
In the second part it will be shown that the resulting control law uses only the locally available information. Considering the forward increment of (22) and using (8), (10) results in

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{M}\hat{\boldsymbol{A}}\boldsymbol{\xi}_k + \boldsymbol{M}\hat{\boldsymbol{B}}\left(\boldsymbol{u}_k^N + \boldsymbol{u}_k^S\right) + \boldsymbol{M}\hat{\boldsymbol{B}}_f\boldsymbol{f}_k + \boldsymbol{w}_{k+1}. \tag{27}$$

Using (24), (27) simplifies to

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{M}\hat{\boldsymbol{B}}\boldsymbol{u}_k^S + \boldsymbol{M}\hat{\boldsymbol{B}}_f\boldsymbol{f}_k \tag{28}$$

According to assumption 1, $\boldsymbol{B}_c$ has full column rank and the sampling time not equal to a pathological sampling time. Thus, the left inverse $\boldsymbol{M}_x = \boldsymbol{B}^+$ exists since $\boldsymbol{B}$ has full column rank as well.
As a result,

$$\boldsymbol{M}\hat{\boldsymbol{B}} = \boldsymbol{M}\hat{\boldsymbol{B}}_f = \boldsymbol{I}_m \tag{29}$$

is satisfied using (23a) and (23b). This simplifies (28) even further to

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{u}_k^S + \boldsymbol{f}_k. \tag{30}$$

From (30) one can see, that the i$^{\text{th}}$ sliding variable $\sigma_{i,k}$ is affected only by the corresponding control signal $u_{i,k}^S$ and perturbation $f_{i,k}$. Due to this property, m discrete-time sliding mode controllers can be independently designed.

Consider a continuous-time perturbed integrator

$$\begin{aligned} \frac{\mathrm{d}\sigma_i}{\mathrm{d}t} &= \tilde{u}_i + \varphi_i \\ \frac{\mathrm{d}\varphi_i}{\mathrm{d}t} &= \Delta_i(t) \end{aligned} \tag{31}$$

with bounded perturbation $\sup(|\Delta_i(t)|) = L_{\varphi_i} < \infty$. Applying the super twisting algorithm

$$\begin{aligned} \tilde{u}_i &= -\alpha_i\sqrt{\sigma_i}\,\mathrm{sign}\left(\sigma_i\right) + \nu_i \\ \frac{\mathrm{d}\nu_i}{\mathrm{d}t} &= -\beta_i\,\mathrm{sign}\left(\sigma_i\right) \end{aligned} \tag{32}$$

proposed in [18] results in a finite time stable closed loop system for suitably chosen parameters α_i and β_i, e.g. the well-established setting

$$\alpha_i = 1.5\sqrt{\Phi_i}, \qquad \beta_i = 1.1\Phi_i, \qquad \Phi_i \geq L_{\varphi_i}, \tag{33}$$

whose stability was recently proven in [19]. Applying Euler forward discretization of (31) and (32) results in the discrete time closed loop system

$$\begin{aligned} \sigma_{i,k+1} &= \sigma_{i,k} - T\alpha_i\sqrt{|\sigma_{i,k}|}\,\mathrm{sign}\left(\sigma_{i,k}\right) + T\bar{\sigma}_{i,k} \\ \bar{\sigma}_{i,k+1} &= \bar{\sigma}_{i,k} + T\Delta_{i,k} - T\beta_i\,\mathrm{sign}\left(\sigma_{i,k}\right) \end{aligned} \tag{34}$$

with $\bar{\sigma}_{i,k} = \nu_{i,k} + \varphi_{i,k}$. Applying (25) to the i$^{\text{th}}$ component of (30) and considering the bounded change rate (4) of perturbation (3) and $\varphi_{i,k+1} = \varphi_{i,k} + T\Delta_{i,k}$ results in (34) with $f_{i,k} = T\varphi_{i,k}$. Therefore, the change rate L_{φ_i} is derived from (4) by

$$L_{\varphi_i} = \sup\left|\frac{\varphi_{i,k+1} - \varphi_{i,k}}{T}\right| = \sup\left|\frac{f_{i,k+1} - f_{i,k}}{T^2}\right| = \frac{L_i}{T} \tag{35}$$

which proofs the equivalence of (26) and (33). As a consequence, the ultimate boundedness of the sliding variables $\sigma_{i,k}$ and states $\boldsymbol{x}_k$ is ensured (see [20]).

To show that the i$^{\text{th}}$ controller node uses only $\boldsymbol{x}_k$ and the history of $u_{i,k}$, variables $\boldsymbol{\sigma}_k$ and $\boldsymbol{w}_{k+1}$ have to be analyzed, because in theorem 2 they depend on the whole lifted state vector. Evaluating the ith component of (22) considering (7) and (23a) results in

$$\sigma_{i,k} = \boldsymbol{m}_i^{\mathrm{T}} \boldsymbol{x}_k + u_{i,k-1} + w_{i,k}. \tag{36}$$

Computing

$$\boldsymbol{M}\hat{\boldsymbol{A}}\boldsymbol{\xi}_k = \boldsymbol{M}_x \boldsymbol{A}\boldsymbol{x}_k + \underbrace{\boldsymbol{M}_x \begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_m\end{bmatrix}}_{\boldsymbol{I}_m} \begin{bmatrix} u_{1,k-\delta_1} \\ \vdots \\ u_{m,k-\delta_m} \end{bmatrix} \tag{37}$$

applying (7), (9a), (13) and (23a), gives the ith component of (24) as

$$w_{i,k+1} = -\boldsymbol{m}_i^{\mathrm{T}} \boldsymbol{A} - u_{i,k}^{N} - u_{i,k-\delta_i}. \tag{38}$$

From (36) and (38) it is clearly visible that apart from the measurements $\boldsymbol{x}_k$ only local information is used in each controller node.

■

VI. Numerical Simulation

The effectiveness of the proposed approach is verified by means of a numerical simulation. For this simulation, the unstable continuous-time plant (1) with

$$\boldsymbol{A}_c = \begin{bmatrix} -3 & -3 & -2 & 1 \\ 2 & -3 & -2 & 2 \\ 1 & 2 & -3 & 1 \\ -3 & -3 & 3 & 0 \end{bmatrix} \quad \boldsymbol{B}_c = \begin{bmatrix} -1 & -3 & 0 \\ 1 & 3 & 1 \\ -3 & 3 & -1 \\ -2 & -3 & 2 \end{bmatrix} \tag{39}$$

was chosen. The constant sampling time is set to $T = 0.1s$ and the constant round trip times ensured by the buffers are known to be $\boldsymbol{\delta}T = \begin{bmatrix}4 & 7 & 6\end{bmatrix} T$. The sensor, controllers and buffers were implemented using the "TrueTime" toolbox [21]. Constructing the lifted model (8) and solving the LMI given in theorem 1 for $\gamma = 0.98$ results in the nominal control law (13) with

$$\boldsymbol{K}_x = \begin{bmatrix} 0.25 & -0.0224 & -0.487 & -0.509 \\ -0.00241 & 0.0011 & 0.00261 & 0.00405 \\ -0.00449 & 0.00172 & 0.00553 & 0.00655 \end{bmatrix} \tag{40}$$

$$\boldsymbol{k}_1^{\mathrm{T}} = \begin{bmatrix}0.461 & 0.365 & 0.213\end{bmatrix}$$

$$\boldsymbol{k}_2^{\mathrm{T}} = \begin{bmatrix}80.5 & 57.9 & 32.0 & 16.5 & 9.08 & 4.68 & 2.19\end{bmatrix} \cdot 10^{-3}$$

$$\boldsymbol{k}_3^{\mathrm{T}} = \begin{bmatrix}47.3 & 33.0 & 20.9 & 12.6 & 6.77 & 2.88\end{bmatrix} \cdot 10^{-3}$$

The perturbation was chosen as

$$\boldsymbol{f}_k = \boldsymbol{\rho}_1 \sin(\boldsymbol{\omega}_1 kT) + \boldsymbol{\rho}_2 \sin(\boldsymbol{\omega}_2 kT) + \boldsymbol{\rho}_3 \tag{41}$$

with

$$\boldsymbol{\rho}_1 = \boldsymbol{\rho}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad \boldsymbol{\omega}_1 = \begin{bmatrix}0.1\\0.2\\0.3\end{bmatrix}, \quad \boldsymbol{\rho}_2 = \begin{bmatrix}3\\2\\1\end{bmatrix}, \quad \boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 \pi. \tag{42}$$

Computing the exact change rates (4) of (41) results in

$$\boldsymbol{L} = \begin{bmatrix}L_1 & L_2 & L_3\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1.04 & 1.66 & 1.84\end{bmatrix}^{\mathrm{T}}. \tag{43}$$

TABLE I
Parameter setting for sliding mode controllers

i	Λ_i	α_i	β_i
1	10.6	4.88	11.66
2	16.8	6.15	18.48
3	18.6	6.47	20.46

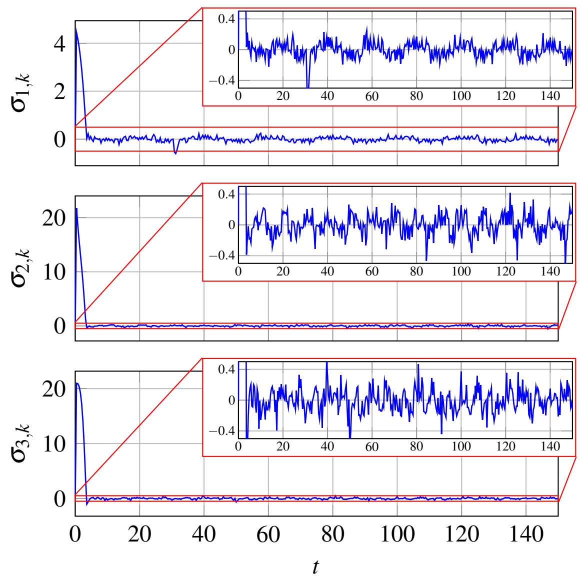

Fig. 3. Sliding variables $\boldsymbol{\sigma}_k$

Based on theorem 2, the sliding variable $\boldsymbol{\sigma}_k$ is given by (22) with

$$\boldsymbol{M}_x = \begin{bmatrix}\boldsymbol{m}_1^{\mathrm{T}} \\ \boldsymbol{m}_2^{\mathrm{T}} \\ \boldsymbol{m}_3^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix} -0.384 & 0.661 & -2.24 & -1.31 \\ -1.38 & 1.14 & 1.07 & -0.302 \\ -1.28 & 3.55 & -1.26 & 3.05 \end{bmatrix} \tag{44}$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1^{\mathrm{T}} & 1 & \boldsymbol{0}_{1\times 2} & 0 & \boldsymbol{0}_{1\times 6} & 0 & \boldsymbol{0}_{1\times 5} \\ \boldsymbol{m}_2^{\mathrm{T}} & 0 & \boldsymbol{0} & 1 & \boldsymbol{0} & 0 & \boldsymbol{0} \\ \boldsymbol{m}_3^{\mathrm{T}} & 0 & \boldsymbol{0} & 0 & \boldsymbol{0} & 1 & \boldsymbol{0} \end{bmatrix}$$

and

$$\begin{bmatrix}\boldsymbol{w}_1^{\mathrm{T}} \\ \boldsymbol{w}_2^{\mathrm{T}} \\ \boldsymbol{w}_3^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix} 0.0121 & -0.522 & 2.09 & 1.43 \\ 0.684 & -1.3 & -0.697 & 0.0633 \\ 1.4 & -2.0 & 0.33 & -3.38 \end{bmatrix} \tag{45}$$

$$\boldsymbol{w}_{k+1} = \begin{bmatrix} \boldsymbol{w}_1^{\mathrm{T}} & \boldsymbol{0} & -1 & \boldsymbol{0} & 0 & \boldsymbol{0} & 0 \\ \boldsymbol{w}_2^{\mathrm{T}} & \boldsymbol{0} & 0 & \boldsymbol{0} & -1 & \boldsymbol{0} & 0 \\ \boldsymbol{w}_3^{\mathrm{T}} & \boldsymbol{0} & 0 & \boldsymbol{0} & 0 & \boldsymbol{0} & -1 \end{bmatrix} \boldsymbol{\xi}_k - \boldsymbol{u}_k^{N}.$$

One possible choice for parameters (26) considering (43) is given in table I.

Simulation results for the sliding variables $\boldsymbol{\sigma}_k$ using the proposed approach is depicted in fig. 3. This figure shows the ultimate boundedness of the sliding variable. The blue lines in fig. 4 show the evolution of the plant states $\boldsymbol{x}_k$. The control signals $\boldsymbol{u}_k$ are depicted in fig. 5.

To demonstrate the massively increased accuracy achieved by using sliding mode control techniques, a simulation was performed without using the sliding mode part of the control law, i.e. $\boldsymbol{u}_k = \boldsymbol{u}_k^N$.

The resulting plant states $\boldsymbol{x}_k$ are represented by the red curves in fig. 4. A comparison of the curves in fig. 4 reveals

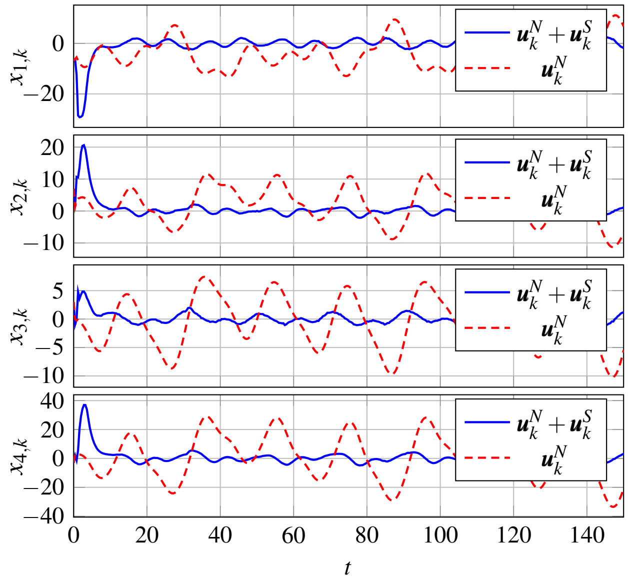

Fig. 4. States $\boldsymbol{x}_k$

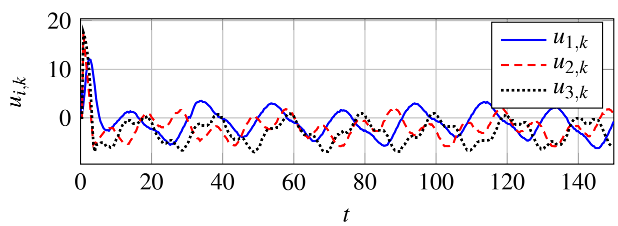

Fig. 5. Control signals $\boldsymbol{u}_k$

the significant increase in accuracy. As the sliding variables act as an accuracy measure, the increased accuracy can also be verified by analyzing the sliding variables. For $\boldsymbol{u}_k^s = 0$ the sliding variables equal the perturbations (41). Therefore, comparing the amplitude of the sliding variables in fig. 3 with the amplitudes of the perturbations in (41) corroborates the increased accuracy as well.

VII. Conclusions and Outlook

A spatially distributed integral sliding mode control based algorithm for multiple input NCS with time-varying transmission delays was presented in this paper. The nominal control law as well as the sliding mode part of the control law are designed such that, apart from the measurement values, only local information is used in each controller node. As a consequence, no communication between the controller nodes is necessary which increases the dependability. The specific choice of the sliding variables offer the possibility to use a discretized version of the super-twisting algorithm to improve the accuracy when perturbations act on the NCS. A numerical simulation exemplified the effectiveness of the proposed approach. In future research, the proposed approach will be extended in order to explicitly consider additional network effects like data loss. In order to reduce discretization induced chattering, the proposed approach will be extended by more sophisticated discretization methods of the super twisting algorithm (e.g. [22], [23]).

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