Higher order sliding mode inspired nonlinear discrete-time observer ☆

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Abstract

This work proposes a design scheme for arbitrary order discrete-time sliding mode observers for input-affine nonlinear systems. The dynamics of the estimation errors are represented in a pseudo-linear form, where the coefficients of the characteristic polynomial comprise the nonlinearities of the algorithm. The design process is reduced to a state-dependent eigenvalue placement procedure. Moreover, two different discrete-time eigenvalue mappings are proposed. As basis for the eigenvalue mappings serves a modified version of the continuous-time uniform robust exact differentiator. Due on the chosen eigenvalue mapping the proposed algorithm does not suffer from discretization chattering. Global asymptotic stability of the estimation errors for observers of order 2 and 3 is proven and the method to prove stability for higher order observers is demonstrated. The performance of a 3-rd order observer is illustrated in simulation. Simulation studies indicate that proposed discrete-time observer might posses an upper bound of its convergence time independent of the initial conditions.

Keywords: Sliding mode control, fixed time convergence, discrete-time observer

1. Introduction

Estimators provide virtual measurements for fault detection, state and disturbance estimation, input reconstruction and on-line parameter identification. As the performance of the control algorithm depends on the virtual measurements, precision in the presence of parameter uncertainties and external disturbances is essential. Robust feedback loops can be achieved by applying Sliding Mode Control (SMC) [1], [2], [3], [4] and Sliding Mode Observers (SMO) [5], [6], [7], [8], [9], [10], [11], [12], [13]. As a result of their robustness SMO are widely investigated and adopted in industrial applications [14], [15], [16], [17]. In addition to the robustness property sliding mode algorithms also enforce finite or fixed time convergence [18], [19], [20], [21]. While the convergence times of finite time algorithms grow with increasing initial states, there exits an upper bound of the convergence times of fixed time algorithms independent of the initial state.

To implement continuous-time SMO on digital devices, some kind of time-discretization is necessary. While for most nonlinear algorithms the Euler forward method is the first choice to obtain a discrete-time approximation, applying this method to discontinuous algorithms yields discretization chattering [22], [23], or in the case of fixed time algorithms even unstable behaviour [24]. Using an implicit discretization approach can avoid this problem, i.e. the exact convergence to the origin in the unperturbed case is preserved, [25], [26], [27]. There are also some recent discrete-time SMC approaches which achieve disturbance rejection by using disturbance estimators [28], [29] and improve the performance of the discrete-time sliding mode controller with a chattering-free design technique [28].

While there exist numerous continuous-time SMO, there are far less discrete-time algorithms. They are commonly based on the Euler forward discretization method and provide an estimate of their precision based on the used sampling time, e.g. [5]. Furthermore, there are several SMO which are based on discrete-time first-order sliding mode techniques [30], [31]. These algorithms provide convergence to a boundary layer or, in the context of discrete-time sliding mode, often referred to as quasi sliding mode band. Once the sliding variable enters this boundary layer, it stays there for all further time steps and shows the typical discretization chattering. To overcome this phenomena some SMO use the idea of equivalent control [32], which eventually result in a linear observer, or replace the set-valued sign-function by a single-valued saturation function [33]. There are also papers dealing with the implementation of discrete-time SMO for industrial problemsm, see e.g. [34], [35]. What they all have in common is the above mentioned chattering in the unperturbed case and the fact

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that they converge to an invariant set in the presence of disturbances.

In this work a discrete-time sliding mode observer for nonlinear systems is proposed. It is based on a semi-implicit [36] eigenvalue mapping [37] of the continuous-time Uniform Robust Exact Differentiator (URED) proposed in [38]. As a result of the eigenvalue-mapping there is no need to replace the sign-function of the continuous-time algorithm by some approximation, as the desired variable structure property of the discrete-time system is considered by the implementation of state-dependent eigenvalues. Furthermore, the proposed algorithm does not suffer from discretization chattering in the unperturbed case and various simulation studies indicate that there might exists an upper bound of the observer's convergence time, independent of its initial errors. In contrast to the second-order discrete-time semi-implicit uniform robust exact differentiator proposed in [39] the algorithm is modified such that the extension to arbitrary order is straightforward. The observer algorithm for order 2 and 3 is studied in detail and global asymptotic stability of the origin for the estimation errors in the unperturbed case is proven. Moreover, it is shown that the stability proof works for different eigenvalue mappings. The novelties of the proposes observer can be summarized as:

- Inspired by higher order continuous-time sliding mode algorithms
- No discretization chattering
- straightforward design method for arbitrary order similar to the Formula of Ackerman
- easy to implement with low computational effort
- Applicable for a wide range of nonlinear systems with bounded uncertainties

2. Preliminaries

Consider the continuous-time URED as proposed in [38]

$$\frac{d\xi_1}{dt} = -k_1\tilde{\Phi}_{c1}(\zeta_1) + \xi_2, \tag{1a}$$
$$\frac{d\xi_2}{dt} = -k_2\tilde{\Phi}_{c2}(\zeta_1), \tag{1b}$$

with the non-linear functions

$$\tilde{\Phi}_{c1}(\zeta_1) = \lceil\zeta_1\rfloor^{\frac{1}{2}} + \mu\lceil\zeta_1\rfloor^{\frac{3}{2}}, \tag{2a}$$
$$\tilde{\Phi}_{c2}(\zeta_1) = \frac{1}{2}\lceil\zeta_1\rfloor^0 + 2\mu\zeta_1 + \frac{3}{2}\mu^2\lceil\zeta_1\rfloor^2, \tag{2b}$$

and $\lceil\zeta_1\rfloor^m = |\zeta_1|^m \operatorname{sign}(\zeta_1)$. The state variables ξ_1 and ξ_2 are estimates of the measured signal $f(t)$ and its derivative $\dot{f}(t)$. The Lebesgue-measurable signal $f(t)$ is assumed to be at least twice differentiable with the known Lipschitz-constant $L \geq |\ddot{f}(t)|$. The parameters k_1, k_2 and μ are positive tuning parameters. The estimation errors are defined as $\zeta_1 = \xi_1 - f(t)$ and $\zeta_2 = \xi_2 - \dot{f}(t)$ and their dynamics are

$$\frac{d\zeta_1}{dt} = -k_1\tilde{\Phi}_{c1}(\zeta_1) + \zeta_2, \tag{3a}$$
$$\frac{d\zeta_2}{dt} = -k_2\tilde{\Phi}_{c2}(\zeta_1) + d(t), \tag{3b}$$

with the unknown but bounded disturbance $d(t) = -\ddot{f}(t)$. The solutions of (3) are understood in the sense of Filippov [40]. In this paper we propose a modified version of the continuous-time algorithm, which serves as basis for the proposed discrete-time sliding mode observer. Due to the more simple structure of the modified algorithm, the proposed observer can be extended to arbitrary order.

3. URED inspired Differentiator

The dynamics of the differentiator errors (3) can be represented in the pseudo-linear form

$$\dot{\boldsymbol{\zeta}} = \tilde{\mathbf{A}}(\zeta_1)\boldsymbol{\zeta} - \begin{bmatrix}0 & 1\end{bmatrix}^T \ddot{f}(t), \tag{4}$$

where $\boldsymbol{\zeta} = \begin{bmatrix}\zeta_1 & \zeta_2\end{bmatrix}^T$ and

$$\tilde{\mathbf{A}}(\zeta_1) = \begin{bmatrix} -k_1\frac{\tilde{\Phi}_{c1}(\zeta_1)}{\zeta_1} & 1 \\ -k_2\frac{\tilde{\Phi}_{c2}(\zeta_1)}{\zeta_1} & 0 \end{bmatrix}. \tag{5}$$

In [39] it has been shown that the state dependent matrix (5) has two state-dependent eigenvalues

$$\tilde{s}_i(\zeta_1) = -k_1\frac{1+\mu|\zeta_1|}{2\sqrt{|\zeta_1|}} \pm \sqrt{k_1^2\frac{(1+\mu|\zeta_1|)^2}{4|\zeta_1|} - k_2\frac{2+8\mu|\zeta_1|+6\mu^2\zeta_1^2}{4|\zeta_1|}} \quad i=1,2. \tag{6}$$

By slightly modifying the nonlinearity (2b), which in detail is just a variation of constant parameters, to

$$\Phi_{c2}(\zeta_1) = \lceil\zeta_1\rfloor^0 + 2\mu\zeta_1 + \mu^2\lceil\zeta_1\rfloor^2 \tag{7a}$$

while (2a)

$$\Phi_{c1}(\zeta_1) = \tilde{\Phi}_{c1}(\zeta_1) \tag{7b}$$

remains untouched and choosing the parameters as $k_1 = 2p_1$ and $k_2 = p_1^2$, where p_1 is a positive tuning gain, the modified differentiator is obtained. The modifed algorithm can also be represented in the pseudo-linear form

$$\dot{\boldsymbol{\zeta}} = \mathbf{A}(\zeta_1)\boldsymbol{\zeta} - \begin{bmatrix}0 & 1\end{bmatrix}^T \ddot{f}(t), \tag{8}$$

where

$$\mathbf{A}(\zeta_1) = \begin{bmatrix} -2p_1 \frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{2}}} & 1 \\ -p_1^2 \left(\frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{2}}}\right)^2 & 0 \end{bmatrix}. \tag{9}$$

The state-dependent eigenvalues (6) simplify due to the modified dynamic matrix (9) to

$$s_i(\zeta_1) = -p_1 \frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{2}}}, \quad i = 1, 2. \tag{10}$$

For $\mu = 0$ the closed loop dynamics of the Super Twisting Algorithm are recovered.

3.1. Extension to arbitrary order

Due to the simplicity of (10), system (1) can be extended to arbitrary order. Consider an at least n-times differentiable signal $f(t)$ with unknown derivatives $\dot{f}(t)$, $\ddot{f}(t), \ldots, f^{(n-1)}(t) = \frac{d^{n-1}f(t)}{dt^{n-1}}$. Its n-th derivative is bounded with the known constant $L \geq |f^{(n)}(t)|$. To estimate its first $n-1$ derivatives the n-th order continuous-time differentiator is proposed as

$$\frac{d\xi_1}{dt} = -k_1 \Phi_{c1}(\zeta_1) + \xi_2, \tag{11a}$$

$$\frac{d\xi_2}{dt} = -k_2 \Phi_{c2}(\zeta_1) + \xi_3, \tag{11b}$$

$$\vdots$$

$$\frac{d\xi_n}{dt} = -k_n \Phi_{cn}(\zeta_1), \tag{11c}$$

with

$$\Phi_{ci} = \left(\frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{n}}}\right)^i \zeta_1 \quad \forall \quad i = 1, \ldots, n, \tag{11d}$$

where μ is a positive tuning gain. The constants $k_1, \ldots, k_n$ are chosen such that

$$s^n + k_1 s^{n-1} + \cdots + k_{n-1}s + k_n = (s + p_1)^n \tag{11e}$$

holds, where p_1 is a positive tuning parameter. The i-th state variable ξ_i is the estimate of the signal's $(i-1)$-th derivative $f^{(i-1)}(t)$. The dynamics of the errors can be written in the pseudo-linear form

$$\dot{\boldsymbol{\zeta}} = \mathbf{A}(\zeta_1)\boldsymbol{\zeta} - \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T f^{(n)}(t), \tag{12}$$

where

$$\boldsymbol{\zeta} = \begin{bmatrix} \zeta_1 & \ldots & \zeta_n \end{bmatrix}^T = \begin{bmatrix} \xi_1 - f(t) & \ldots & \xi_n - f^{(n-1)}(t) \end{bmatrix}^T.$$

The state dependent dynamic matrix

$$\mathbf{A}(\zeta_1) = \begin{bmatrix} -k_1 \left(\frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{n}}}\right) & 1 & 0 & \ldots & 0 \\ -k_2 \left(\frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{n}}}\right)^2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \ldots & 0 & 1 \\ -k_n \left(\frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{n}}}\right)^n & 0 & \ldots & \ldots & 0 \end{bmatrix} \tag{13}$$

has n identical state dependent eigenvalues

$$s_n = -p_1 \left(\frac{1+\mu|\zeta_1|}{|\zeta_1|^{\frac{1}{n}}}\right). \tag{14}$$

Remark: Note that the stability of the proposed continuous-time algorithm is not investigated in this paper, as it only serves as basis for the discrete-time algorithm.

4. Discrete-time Sliding Mode Observer

The implementation of the above mentioned differentiator can be seen as observer for an integrator chain. The observer proposed in this paper is a generalization of the discretized differentiator proposed in Chapter 3.1.

4.1. Discrete-time eigenvalue mapping

In the following subsections ihe proposed observers are designed such that the eigenvalues (14) of the pseudo-linear continuous-time system are mapped to discrete-time eigenvalues of the corresponding discrete-time system. The eigenvalue mappings used in this work ensure that there exists no discretization chattering in the unperturbed case. In Section 5 it will be shown that only the lower and upper bounds of the state-dependent discrete-time eigenvalues are crucial for the proof of global asymptotic stability of the origin of the observer errors. Hence, various discrete-time eigenvalues are possible. Taking the proposed continuous-time eigenvalues for order n (14) into account, the semi-implicit eigenvalue mapping proposed in [39] yields the discrete-time eigenvalues

$$z_n = z_s(e_{n,k}) = \frac{|e_{n,k}|^{\frac{1}{n}}}{hp_1\mu|e_{n,k}| + |e_{n,k}|^{\frac{1}{n}} + hp_1}, \tag{15}$$

while the matching eigenvalue mapping proposed in [37] yields

$$z_n = z_m(e_{n,k}) = \exp\left(-hp_1 \frac{1+\mu|e_{n,k}|}{|e_{n,k}|^{\frac{1}{n}}}\right) \tag{16}$$

where h is the constant sampling time and $e_{n,k}$ is the difference between the measured and the estimated signal and is defined in the following subsection. In Fig. 1 the two different eigenvalue mappings as function of $e_{n,k}$ for $n = 2$

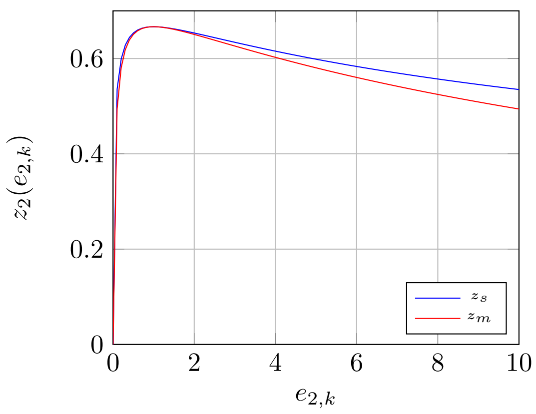

Figure 1: Discrete-time eigenvalue mappings (15) and (16) as function of the observer output error $e_{2,k}$

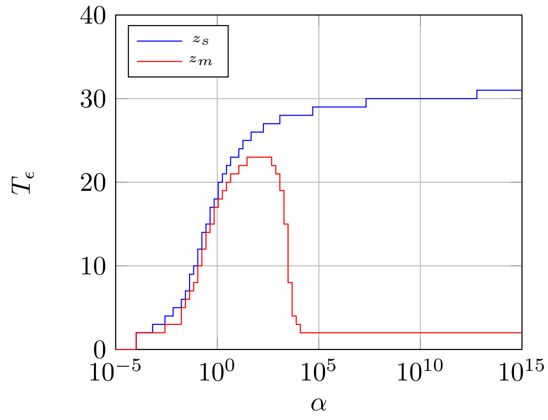

Figure 2: Practical convergence time as function of the initial observer errors for the eigenvalue mappings (15) and (16)

are illustrated. The parameters are chosen as $h = 1$, $\mu = 1$ and p_1 is chosen separately for every mapping such that $\max(z_s(e_{n,k})) = \max(z_m(e_{n,k})) = \frac{2}{3}$. Implementing two nonlinear observers according to (23) with the above mentioned eigenvalue configurations and comparing the practical convergence time T_ϵ, i.e. $T_\epsilon = k_\epsilon h$ such that $||\mathbf{e}_k||_2 \leq \epsilon$ $\forall k \geq k_\epsilon$, of both observers for varying initial observer errors $\mathbf{e}_0^T = \begin{bmatrix}\alpha & \alpha\end{bmatrix}$ yields Fig. 2.

The evolution of T_ϵ indicates that both observers have an upper bound for their practical convergence time. While the parameter h is more or less fixed, as it is the used sampling time, the positive parameters p_1 and μ can be used to tune the observer. For the following considerations the cases $e_{n,k} = 0$ and $|e_{n,k}| \to \infty$ are excluded from parameter tuning discussion as both eigenvalue mappings yield $z_n(0) = 0$ and $\lim_{|e_{n,k}|\to\infty} z_n = 0$ independent of the parameters. The parameter μ can be used to tune the convergence rate of the system dependent of the observer error $e_{n,k}$:

- decrease μ: faster convergence for small errors $|e_{nk}|$
- increase μ: faster convergence for large errors $|e_{n,k}|$

The parameter p_1 can be used to tune the overall convergence rate, i.e., larger p_1 yields faster convergence.

4.2. Observer design for linear systems

Assume a linear n-th order discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{b}u_k, \tag{17a}$$
$$y_k = \mathbf{c}^T\mathbf{x}_k \tag{17b}$$

with known matrices and vectors $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$, which can be obtained, for example, via zero-order-hold discretization with sampling time h of the n-th order continuous-time system

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}\bar{u}, \tag{18}$$
$$\bar{y} = \bar{\mathbf{c}}^T\bar{\mathbf{x}}. \tag{19}$$

If the pair $(\mathbf{A}, \mathbf{c}^T)$ is observable, then there exists a regular state transformation $\mathbf{w} = \mathbf{T}\mathbf{x}$ such that

$$\mathbf{w}_{k+1} = \tilde{\mathbf{A}}\mathbf{w}_k + \tilde{\mathbf{b}}u_k, \tag{20a}$$
$$y_k = \tilde{\mathbf{c}}^T\mathbf{w}_k, \tag{20b}$$

is in observable canonical form with $\tilde{\mathbf{c}}^T = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}$ and $\mathbf{T}$ is a regular $n \times n$ matrix. The observer

$$\hat{\mathbf{w}}_{k+1} = \tilde{\mathbf{A}}\hat{\mathbf{w}}_k + \tilde{\mathbf{b}}u_k + \mathbf{l}e_{n,k} \tag{21}$$

with $e_{n,k} = y_k - \tilde{\mathbf{c}}^T\hat{\mathbf{w}}_k$ can be implemented such that the dynamics of the observer error $\mathbf{e}_k = \begin{bmatrix}e_{1,k} & \dots e_{n,k}\end{bmatrix}^T = \mathbf{w}_k - \hat{\mathbf{w}}_k$ are

$$\mathbf{e}_{k+1} = \mathbf{A}_o\mathbf{e}_k, \tag{22}$$

with

$$\mathbf{A}_o = \mathbf{A}_o(z_n) = \begin{bmatrix} 0 & \dots & \dots & 0 & -\Phi_1(z_n) \\ 1 & 0 & \dots & 0 & -\Phi_2(z_n) \\ 0 & \ddots & \ddots & 0 & -\Phi_3(z_n) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & -\Phi_n(z_n) \end{bmatrix}. \tag{23}$$

The correction term $\mathbf{l} = \mathbf{l}(z_n)$ can be determined via Ackermann's formula such that the nonlinear functions $\Phi_i(z_n)$ are the coefficients of the polynomial

$$\begin{aligned}&(z - z_n)^n = \\ &= z^n + \Phi_n(z_n)z^{n-1} + \dots + \Phi_2(z_n)z + \Phi_1(z_n)\end{aligned} \tag{24}$$

and the state dependent matrix $\mathbf{A}_o(z_n)$ has one single state dependent eigenvalue z_n of multiplicity n. The eigenvalue $z_n = z_n$ is chosen according to one of the proposed eigenvalue mappings (15) - (16). The choice of the used eigenvalue mappings ensures, that there exists no discretization chattering, which is a common problem when The Euler forward method is used to discretize discontinuous algorithms.

4.3. Generalization to observable input-affine non-linear systems

The above mentioned sliding mode observer concept can be extended to a far more general class of systems. Assume a nominal non-linear input-affine single input-single output system of order n:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \tag{25}$$

$$y = h(\mathbf{x}), \tag{26}$$

If the nominal system (25) with output (26) is completely uniformly locally weakly observable, there exists an local diffeomorphism, such that the transformed nominal system reads as [41]

$$\dot{\xi}_1 = \xi_2 + \gamma_1(\xi_1, u) = \tilde{f}_1(\xi_1, \xi_2, u) \tag{27a}$$

$$\dot{\xi}_2 = \xi_3 + \gamma_2(\xi_1, \xi_2, u) = \tilde{f}_2(\xi_1, \dots, \xi_3, u) \tag{27b}$$

$$\vdots$$

$$\dot{\xi}_{n-1} = \xi_n + \gamma_{n-1}(\xi_1, \dots, \xi_{n-1}, u) = \tilde{f}_{n-1}(\boldsymbol{\xi}, u) \tag{27c}$$

$$\dot{\xi}_n = \gamma_n(\boldsymbol{\xi}, u) = \tilde{f}_n(\boldsymbol{\xi}, u) \tag{27d}$$

$$y = \xi_1, \tag{27e}$$

where $\gamma_i(\xi_1, \dots, \xi_i, u) = \beta_i(\xi_1, \dots, \xi_i)u$ for $i = 1, \dots, n-1$ and $\gamma_n(\boldsymbol{\xi}, u) = \alpha(\boldsymbol{\xi}) + \beta_n(\boldsymbol{\xi})u$. It is assumed that for bounded states ξ_i and bounded input u the right sides of all differential equations are bounded, i.e. $|\gamma_i| \leq L_{\gamma_i}$ $\forall \quad i = 1, \dots, n$. Additionally, perturbations are introduced in each channel

$$\dot{\xi}_i = \tilde{f}_i(\xi_1, \dots, \xi_{i+1}, u) + \delta_i(\boldsymbol{\xi}, u), \quad i = 1, \dots, n, \tag{28}$$

which represent parameter uncertainties and external disturbances, are unknown and bounded $|\delta_i| \leq L_{\delta_i}$ as well. A discrete-time approximation of the above transformed system with sampling time h is

$$\boldsymbol{\xi}_{k+1} = \mathbf{A}\boldsymbol{\xi}_k + \mathbf{B}\begin{bmatrix} \gamma_1(\xi_{1,k}, u_k) + \delta_1(\boldsymbol{\xi}_k, u_k) \\ \vdots \\ \gamma_n(\boldsymbol{\xi}_k, u_k) + \delta_n(\boldsymbol{\xi}_k, u_k) \end{bmatrix} \tag{29}$$

$$y_k = \mathbf{c}^T\boldsymbol{\xi}_k \tag{30}$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & h & \dots & \frac{h^{n-1}}{(n-1)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & h \\ 0 & \dots & 0 & 1 \end{bmatrix}, \tag{31}$$

$$\mathbf{B} = \begin{bmatrix} h & \frac{h^2}{2} & \dots & \frac{h^n}{n!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{h^2}{2} \\ 0 & \dots & 0 & h \end{bmatrix}, \quad \mathbf{c}^T = \begin{bmatrix} 1 & 0 \dots & 0 \end{bmatrix} \tag{32}$$

The proposed generalized observer is defined as

$$\begin{aligned} \hat{\boldsymbol{\xi}}_{k+1} &= \mathbf{A}\hat{\xi}_k \\ &+ \mathbf{B}\begin{bmatrix} \gamma_1(\xi_{1,k}, u_k) \\ \mathrm{sat}_2(\hat{\gamma}_2(\xi_{1,k}, \hat{\xi}_{2,k}, u_k)) \\ \vdots \\ \mathrm{sat}_n(\hat{\gamma}_n(\xi_{1,k}, \hat{\xi}_{2,k}, \dots, \hat{\xi}_{n,k}, u_k)) \end{bmatrix} + \tilde{\mathbf{l}}\sigma_{1,k}, \end{aligned} \tag{33}$$

$$\mathrm{sat}_i(\hat{\gamma}_i) = \begin{cases} L_{\gamma_i}, & \hat{\gamma}_i > L_{\gamma_i} \\ \hat{\gamma}_i, & |\hat{\gamma}_i| \leq L_{\gamma_i} \\ -L_{\gamma_i}, & \hat{\gamma}_i < L_{\gamma_i} \end{cases} \tag{34}$$

where the observer errors are defined as $\boldsymbol{\sigma}_k = \begin{bmatrix} \sigma_{1,k} & \dots & \sigma_{n,k} \end{bmatrix}^T = \boldsymbol{\xi}_k - \hat{\boldsymbol{\xi}}_k$ and the correction term $\tilde{\mathbf{l}} = \tilde{\mathbf{l}}(\sigma_{1,k})$ can be computed using Ackermann's formula

$$\tilde{\mathbf{l}} = (\mathbf{A} - z\mathbf{I})^n \begin{bmatrix} \mathbf{c}^T \\ \mathbf{c}^T\mathbf{A} \\ \vdots \\ \mathbf{c}^T\mathbf{A}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_1 + 1 \\ \frac{1}{h}\tilde{\Phi}_2 \\ \vdots \\ \frac{1}{h^{n-1}}\tilde{\Phi}_n \end{bmatrix}. \tag{35}$$

Therein, $\mathbf{I}$ is the identity matrix of dimension $n \times n$ and $\tilde{\Phi}_i = \tilde{\Phi}_i(z_n)$ for $i = 1, \dots, n$ where $z_n = z_n(\sigma_{1,k})$ are the discrete-time eigenvalues according to one of the two eigenvalue mappings (15) and (16).
The dynamics of the observer errors reads as

$$\boldsymbol{\sigma}_{k+1} = \tilde{\mathbf{A}}_o\boldsymbol{\sigma}_k + \tilde{\mathbf{B}}\mathbf{d}_k \tag{36}$$

with

$$\tilde{\mathbf{A}}_o = \tilde{\mathbf{A}}_o(z_n) = \begin{bmatrix} -\tilde{\Phi}_1 & h & \dots & \dots & \frac{h^{n-1}}{(n-1)!} \\ -\frac{1}{h}\tilde{\Phi}_2 & 1 & h & \dots & \frac{h^{n-2}}{(n-2)!} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & h \\ \frac{1}{h^{n-1}}\tilde{\Phi}_n & 0 & \dots & 0 & 1 \end{bmatrix}, \tag{37}$$

$$\tilde{\mathbf{B}} = \mathbf{B} \tag{38}$$

$$\mathbf{d}_k = \begin{bmatrix} d_{1,k} & \dots & d_{n,k} \end{bmatrix}^T \tag{39}$$

$$d_{1,k} = \delta_1(\boldsymbol{\xi}_k, u_k) \tag{40}$$

$$\begin{aligned} d_{i,k} &= \delta_i(\boldsymbol{\xi}_k, u_k) + \gamma_i(\xi_{1,k}, \dots, \xi_{i,k}, u_k) \\ &- \mathrm{sat}_{L_i}(\hat{\gamma}_i(\xi_{1,k}, \hat{\xi}_{2,k}, \dots, \hat{\xi}_{i,k}, u_k)), \quad i = 2, \dots, n \end{aligned} \tag{41}$$

where each element of the disturbance vector is bounded with $|d_{1,k}| \leq L_{\delta_1}$ and $|d_{i,k}| \leq 2L_{\gamma_i} + L_{\delta_i}$. If $\gamma_i(\xi_{1,k}, \dots, \xi_{i,k}, u_k) = \gamma_i(\xi_{1,k}, u_k)$ depends only on the measured state state $\xi_{1,k}$ and the known input u_k and $\delta_i(\boldsymbol{\xi}_k, u_k) = 0$ the corresponding element in the disturbance vector is zero, i.e. $d_{i,k} = 0$ for $i = 1, \dots, n$.

5. Proof of global asymptotic stability

In the unperturbed case, i.e. $\mathbf{d}_k = \mathbf{0}$ for the generalized observer, there exists a regular state transformation $\mathbf{e}_k =$

$\mathbf{S}\boldsymbol{\sigma}_k$ with $e_{n,k} = \sigma_{1,k}$ such that the error dynamics of the generalized observer (36) are equivalent with the error dynamics of the observer for linear systems (22). Hence, the following stability proof for the observer (21) holds also for the unperturbed generalized observer.
The dynamic matrix $\mathbf{A}_o(z_n)$ can be split into a constant matrix $\mathbf{A}_L$ and a matrix $\mathbf{A}_N(z_n)$ which contains the nonlinearities, i.e.,

$$\mathbf{A}_o(z_n) = \mathbf{A}_L + \mathbf{A}_N(z_n) \tag{42}$$

where

$$\mathbf{A}_L = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}, \tag{43}$$

$$\mathbf{A}_N(z_n) = \begin{bmatrix} 0 & \dots & 0 & -\Phi_1(z_n) \\ 0 & \dots & 0 & -\Phi_2(z_n) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -\Phi_n(z_n) \end{bmatrix}. \tag{44}$$

Note that according to the eigenvalue mappings (15) and (16) $\lim_{e_n \to \infty} z_n(e_n) = 0$ and according to (24) $\Phi_i(0) = 0$, hence

$$\lim_{e_n \to \infty} A_0(z_n) = A_L. \tag{45}$$

Theorem 1. *There exist positive parameters $p_1^\star$, $\mu^\star$ and $h^\star$ such that the origin of system (22) for $n \geq 2$ is global asymptotically stable if its parameters are selected as $p_1 \geq p_1^\star$, $\mu \geq \mu^\star$ and $h \geq h^\star$.*

Proof. There exist positive definite matrices $\mathbf{P} \succ 0$ and $\mathbf{Q} \succ 0$ of appropriate dimension with

$$\mathbf{A}_L^T \mathbf{P} \mathbf{A}_L - \mathbf{P} = -\mathbf{Q}, \tag{46}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{n-1} & \mathbf{q}_{n-1} \\ \mathbf{q}_{n-1}^T & q_n \end{bmatrix}, \tag{47}$$

$\mathbf{Q}_{n-1} \succ 0$, $\mathbf{Q}_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\mathbf{q}_{n-1} \in \mathbb{R}^{n-1}$, $q_n \in \mathbb{R}$ and $\mathbf{A}_L$ according to (45). Let $V_k = \mathbf{e}_k^T \mathbf{P} \mathbf{e}_k$ be a Lyapunov function with

$$\begin{aligned} \Delta V_k &= \mathbf{e}_k^T \left(\mathbf{A}_o^T(z_n) \mathbf{P} \mathbf{A}_o^T(z_n) - \mathbf{P} \right) \mathbf{e}_k \\ &= -\mathbf{e}_k^T \mathbf{M}(z_n) \mathbf{e}_k, \end{aligned} \tag{48}$$

$$\mathbf{M}(z_n) = \begin{bmatrix} \mathbf{Q}_{n-1} & \mathbf{m}_{n-1}(z_n) \\ \mathbf{m}_{n-1}^T(z_n) & m_n(z_n) \end{bmatrix}, \tag{49}$$

$$\mathbf{m}_{n-1}^T(z_n) = \begin{bmatrix} m_1(z_n) & \dots & m_{n-1}(z_n) \end{bmatrix}, \tag{50}$$

where $m_1(z_n), \dots, m_n(z_n)$ are polynomials of z_n and $\mathbf{m}_{n-1}^T(0) = \mathbf{q}_{n-1}^T$, $m_n(0) = q_n$. As $\mathbf{Q}_{n-1}$ is positive definite, a sufficient condition for global asymptotic stability of the origin is that $\det(\mathbf{M}(e_{n,k})) > 0$ holds. As $\det(\mathbf{M}(z_n))$ is a polynomial of z_n and $\mathbf{M}(0) = \mathbf{Q}$, $\mathbf{M}(z_n)$ is positive definite for $0 \leq z_n < z^\star < 1$. Both eigenvalue mappings have their maximum $z_{n,max} = \max(z_n)$ at $|e_{n,k}| = \frac{1}{\mu(n-1)}$ with

$$z_{s,max} = \frac{n-1}{n-1+n\left(\mu\left(n-1\right)\right)^{\frac{1}{n}} hp_1} \tag{51}$$

$$z_{m,max} = \exp\left(-hp_1 n \left(\mu\left(n-1\right)\right)^{\frac{1}{n}}\right). \tag{52}$$

Hence, there exist parameter settings $h^\star$, $\mu^\star$ and $p_1^\star$ such that $\Delta V_k < 0$ for all $h \geq h^\star$, $\mu \geq \mu^\star$, $p_1 \geq p_1^\star$ and $\mathbf{e}_k \neq \mathbf{0}$. □

5.1. Existence of an invariant set

In the case that the disturbance vector of the generalized observer is not equal to zero, i.e. $\mathbf{d}_k \neq \mathbf{0}$, it can be shown that there exists an attractive invariant set $\mathcal{X}$ around the origin. Using the same regular state transformation $\mathbf{e}_k = \mathbf{S}\boldsymbol{\sigma}_k$ as for the proof of global asymptotic stability, one can show that the error dynamics of the generalized observer can be written as

$$\mathbf{e}_{k+1} = \mathbf{A}_o(z_n)\mathbf{e}_k + \bar{\mathbf{d}}_k, \tag{53}$$

with $\bar{\mathbf{d}}_k = \mathbf{S}\mathbf{d}_k$. The perturbation $\bar{\mathbf{d}}_k$ is bounded as $\mathbf{d}_k$ is bounded by assumption.

Theorem 2. *There exist positive parameters $p_1^\star$, $\mu^\star$ and $h^\star$ such that for bounded perturbation $\bar{\mathbf{d}}_k$ there exists an invariant set $\mathcal{X}$ around the origin of system (53) such that $V_k \leq c^\star$ if $\mathbf{e}_k \in \mathcal{X}$, $\Delta V_k < 0$ if $\mathbf{e}_k \notin \mathcal{X}$ and $V_{k+1} \leq c^\star$ if $\mathbf{e}_k \in \mathcal{X}$, if its parameters are selected as $p_1 \geq p_1^\star$, $\mu \geq \mu^\star$ and $h \geq h^\star$.*

Proof. Using the same Lyapunov function as for the proof of Theorem 1 yields

$$\Delta V_k = -\mathbf{e}_k^T \mathbf{M}(z_n)\mathbf{e}_k + 2\mathbf{e}_k^T \mathbf{A}_o(z_n)^T \mathbf{P}\bar{\mathbf{d}}_k + \bar{\mathbf{d}}_k^T \mathbf{P}\bar{\mathbf{d}}_k. \tag{54}$$

It is proven in Theorem 1 that $\mathbf{M}(z_n)$ is positive definite. As $\bar{\mathbf{d}}_k$ is bounded and (45) holds there exists a set $\mathcal{X}_1$ such that if $\mathbf{e}_k \notin \mathcal{X}_1$ the quadratic term of (54) dominates and $\Delta V_k < 0$ holds. For the second part of the proof the Definition 2.1 and Theorem 2.1 of the paper [42] is used:
Definition: System (53) is D,$\mathcal{X}_2$-BIBS (Bounded Input-Bounded State) stable if for each initial condition $\mathbf{e}_0$ in $\mathcal{X}_2$ and for every input $\bar{\mathbf{d}}$ with $||\bar{\mathbf{d}}_k|| < D$ for all $k \geq 0$, the state $\mathbf{e}_k$ remains bounded for all $k \geq 0$.

Theorem 3. *Assume that the origin is an asymptotically stable equilibrium point of system (53) with $\bar{\mathbf{d}}_k = \mathbf{0}$ and let V_k be an associated Lyapunov function which is assumed to be continuously differentiable. Then there exists an positive constant D and a bounded open neighbourhood $\mathcal{X}_2$ of the origin in $\mathbb{R}^n$ so that system (53) is D,$\mathcal{X}_2$-BIBS stable.*

The proof can be found in [42].
By defining $\mathcal{X}_2 = \{\mathbf{e} \in \mathbb{R}^n | V_k(\mathbf{e}) \leq c\}$ and utilizing the continuity of V_k and V_{k+1} yields that every set $\mathcal{X}_2 \supseteq \mathcal{X}_1$

is in attractive invariant set of system (53). The smallest invariant set is $\mathcal{X}$ with

$$\boxed{\begin{aligned}&\mathcal{X} = \{\mathbf{e} \in \mathbb{R}^n | V_k(\mathbf{e}) \leq c^\star\} = \min \mathcal{X}_2\\ &\text{s.t.}\\ &\mathcal{X} \supseteq \mathcal{X}_1^\star = \min \mathcal{X}_1.\end{aligned}}$$

$\square$

5.2. Second order observer

For an observer of order $n = 2$ the nonlinearities are according to (24)

$$\Phi_1 = \Phi_1(z_2) = z_2^2, \tag{56a}$$

$$\Phi_2 = \Phi_2(z_2) = -2z_2. \tag{56b}$$

Using, e.g. the eigenvalue mapping (15), the observer error dynamics are

$$e_{1,k+1} = -\frac{1}{\left(hp_1|e_{2,k}| + |e_{2,k}|^{\frac{1}{2}} + hp_1\right)^2} \lceil e_{2,k} \rfloor^2,$$

$$e_{2,k+1} = e_{1,k} + \frac{2}{hp_1|e_{2,k}| + |e_{2,k}|^{\frac{1}{2}} + hp_1} \lceil e_{2,k} \rfloor^{\frac{3}{2}}. \tag{57}$$

Theorem 4. *There exist positive parameters $p_1^\star$, $\mu^\star$ and $h^\star$ such that the origin of system (22) for order $n = 2$ is global asymptotically stable if its parameters are selected as $p_1 \geq p_1^\star$, $\mu \geq \mu^\star$ and $h \geq h^\star$.*

Proof. Let $V_k = \mathbf{e}_k^T \mathbf{P} \mathbf{e}_k$ be a Lyapunov function candidate where

$$\mathbf{P} = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \tag{58}$$

is the positive definite solution of

$$\mathbf{A}_L^T \mathbf{P} \mathbf{A}_L - \mathbf{P} = -\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}. \tag{59}$$

Remark: Note that although in the proof of theorem 1 $\mathbf{Q}$ is assumed to be positive definite, $\mathbf{Q}$ is positive semi-definite. Numerous evaluation have shown that the chosen $\mathbf{Q}$ yields the largest set of parameters p_1, h and μ such that the origin of system (22) for order $n = 2$ is global asymptotically stable. Nevertheless, one can also choose a positve definite matrix for the stability proof.
Computing $\Delta V_k = V_{k+1} - V_k$ yields the quadratic form

$$\Delta V_k(z_2) = -\mathbf{e}_k^T \mathbf{M}(z_2) \mathbf{e}_k, \tag{60a}$$

with the state dependent matrix

$$\mathbf{M}(z_2) = \begin{bmatrix} 2 & m_1 \\ m_1 & m_2 \end{bmatrix}, \tag{60b}$$

where

$$m_1 = m_1(z_2) = \sqrt{2} + \sqrt{2}\Phi_1 + \Phi_2, \tag{60c}$$

$$m_2 = m_2(z_2) = 1 - 3\Phi_1^2 - 2\sqrt{2}\Phi_1\Phi_2 - \Phi_2^2. \tag{60d}$$

A sufficient condition for global asymptotic stability of the origin is that $\det(\mathbf{M}(z_2)) > 0$ holds, which under consideration of (56) is equivalent to

$$0 < z_n < \frac{1}{\sqrt{2}}. \tag{61}$$

Note that $\mathbf{M}(z_2)$ is only positive semi-definite for $z_n = 0$, which occurs only for (15) and (16) iff $e_{2,k} = 0$. However, $\Delta V_k < 0$ for $e_{2,k} = 0$, $e_{1,k} \neq 0$. Hence, the origin of system (23) with $n = 2$ is global asymptotically stable for

$$0 \leq z_2 < \frac{1}{\sqrt{2}}, \tag{62}$$

which is equivalent to

$$h^\star p_1^\star > \frac{\sqrt{2} - 1}{2\sqrt{\mu}}, \qquad \text{for } z_3 = z_s(e_{2,k}) \tag{63}$$

$$h^\star p_1^\star > \frac{\ln 2}{4\sqrt{\mu}}, \qquad \text{for } z_3 = z_m(e_{2,k}), \tag{64}$$

depending on the choice of the discrete-time eigenvalue mapping. $\square$

5.3. Third order observer

For an observer of order $n = 3$ the nonlinearities according to (24) are

$$\Phi_1 = \Phi_1(z_3) = -z_3^3, \tag{65a}$$

$$\Phi_2 = \Phi_2(z_3) = 3z_3^2, \tag{65b}$$

$$\Phi_3 = \Phi_3(z_3) = -3z_3. \tag{65c}$$

Theorem 5. *There exist positive parameters $p_1^\star$, $\mu^\star$ and $h^\star$ such that the origin of system (22) for order $n = 3$ is global asymptotically stable if its parameters are selected as $p_1 \geq p_1^\star$, $\mu \geq \mu^\star$ and $h \geq h^\star$.*

Proof. Let $V_k = \mathbf{e}_k^T \mathbf{P} \mathbf{e}_k$ be a Lyapunov function candidate where

$$\mathbf{P} = \begin{bmatrix} \frac{197}{16} & 6 & 1 \\ 6 & \frac{133}{16} & 4 \\ 1 & 4 & \frac{53}{16} \end{bmatrix} \tag{66}$$

is the positive definite solution of

$$\mathbf{A}_L^T \mathbf{P} \mathbf{A}_L - \mathbf{P} = -\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & \frac{53}{16} \end{bmatrix}. \tag{67}$$

Remark: Note that for the choice of $\mathbf{Q}$ the same considerations as for (59) hold.
Computing $\Delta V_k = V_{k+1} - V_k$ yields the quadratic form

$$\Delta V_k(z_3) = -\mathbf{e}_k^T \mathbf{M}(z_3) \mathbf{e}_k, \tag{68a}$$

with the state dependent matrix

$$\mathbf{M}(z_3) = \begin{bmatrix} 4 & 2 & m_1 \\ 2 & 5 & m_2 \\ m_1 & m_2 & m_3 \end{bmatrix}, \tag{68b}$$

where

$$m_1 = m_1(z_3) = 2 + 12\Phi_1 + \frac{133}{8}\Phi_2 + 8\Phi_3 \tag{68c}$$

$$m_2 = m_2(z_3) = 8 + 2\Phi_1 + 8\Phi_2 + \frac{53}{8}\Phi_3 \tag{68d}$$

$$\begin{aligned} m_3 = m_3(z_3) = \frac{1}{16}(&53 - 197\Phi_1^2 - 133\Phi_2^2 - 53\Phi_3^2 \\ &- 128\Phi_2\Phi_3 - 32\Phi_1(6\Phi_2 + \Phi_3)) \end{aligned} \tag{68e}$$

A sufficient condition for global asymptotic stability of the origin is that $\det(\mathbf{M}(z_3)) > 0$ holds, which, under consideration of (65), is equivalent to

$$0 < z_3 < 0.498582. \tag{69}$$

Note that $\mathbf{M}(z_3)$ is only positive semi-definite for $z_n = 0$, which occurs only for (15) and (16) iff $e_{3,k} = 0$. However, $\Delta V_k < 0$ for $e_{3,k} = 0$ and $\begin{bmatrix} e_{1,k} & e_{2,k} \end{bmatrix}^T \neq \mathbf{0}$. Hence, the origin of system (23) with $n = 3$ is global asymptotically stable for

$$0 \leq z_n < 0.498582, \tag{70}$$

which is equivalent to

$$h^\star p_1^\star > \frac{2 - 0.9972}{3\sqrt[3]{2\mu}}, \qquad \text{for } z_3 = z_s(e_{3,k}) \tag{71}$$

$$h^\star p_1^\star > -\frac{2}{3\sqrt[3]{2\mu}} \ln 0.498582, \qquad \text{for } z_3 = z_m(e_{3,k}) \tag{72}$$

depending on the choice of the discrete-time eigenvalue mapping. □

6. Examples

For simulation study the following nonlinear input-affine continuous-time system

$$\dot{\xi}_1 = \xi_2, \tag{73a}$$

$$\dot{\xi}_2 = \xi_3 + \sin(\xi_1)u \tag{73b}$$

$$\dot{\xi}_3 = -\sin(\xi_2)u \tag{73c}$$

$$y_k = \xi_1 \tag{73d}$$

with $u = 1 = \text{const.}$ and initial value $\boldsymbol{\xi}_0 = \begin{bmatrix} \frac{\pi}{8} & 0 & 0 \end{bmatrix}^T$ is considered. For the observer study with sampling time $h = 0.01$ three different observers are compared:

- The first observer Σ_1 is designed as discrete-time high gain observer, i.e. all eigenvalues are chosen as $z = 0$. The correction term is computed via the formula of Ackerman and all nonlinearities of the system are neglected.

$$\hat{\boldsymbol{\xi}}_{k+1} = \mathbf{A}\hat{\boldsymbol{\xi}}_k + \mathbf{l}\sigma_{1,k}. \tag{74}$$

- The second observer Σ_2 is implemented according to the structure proposed in this paper:

$$\hat{\boldsymbol{\xi}}_{k+1} = \mathbf{A}\hat{\boldsymbol{\xi}}_k + \mathbf{B}\begin{bmatrix} 0 \\ \mathrm{sat}_1(\hat{\gamma}_2(\xi_{1,k}, u_k)) \\ \mathrm{sat}_1(\hat{\gamma}_3(\hat{\xi}_{2,k}, u_k)) \end{bmatrix} + \mathbf{l}\sigma_{1,k} \tag{75}$$

with

$$\mathbf{B} = \begin{bmatrix} h & \frac{h^2}{2} & \frac{h^3}{6} \\ 0 & h & \frac{h^2}{2} \\ 0 & 0 & h \end{bmatrix}, \tag{76}$$

$$\hat{\gamma}_2(\xi_{1,k}, u_k) = \sin(\xi_{1,k})u_k, \tag{77}$$

$$\hat{\gamma}_3(\hat{\xi}_{2,k}, u_k) = -\sin(\hat{\xi}_{2,k})u_k. \tag{78}$$

Eigenvalue mapping (15) is used and

$$\mathbf{A} = \begin{bmatrix} 1 & h & \frac{h^2}{2} \\ 0 & 1 & h \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} \Phi_1 + 1 \\ \frac{1}{h}\Phi_2 \\ \frac{1}{h^2}\Phi_3 \end{bmatrix} \tag{79}$$

$$\Phi_1 = 3z_3 - 2 \tag{80}$$

$$\Phi_2 = -\frac{1}{2}z_3^3 - \frac{3}{2}z_3^2 + \frac{9}{2}z_3 - \frac{5}{2}, \tag{81}$$

$$\Phi_3 = z_3^3 - 3z_3^2 + 3z_3 - 1 \tag{82}$$

$$z_3 = z_3(\sigma_{1,k}) = z_3(\xi_{1,k} - \hat{\xi}_{1,k}) \tag{83}$$

hold. The parameters $\mu = 1$ and $p_1 = 30$ are chosen such that condition (71) is fulfilled.

- For the third observer Σ_3 the system (73) was linearized around the equilibrium point $\boldsymbol{\xi}_R = \mathbf{0}$, $u_R = 1$ and ZOH-discretized with sampling time $h = 0.01$, this yields

$$\Delta\boldsymbol{\xi}_{k+1} = \bar{\mathbf{A}}\Delta\boldsymbol{\xi}_k + \bar{\mathbf{b}}\Delta u_k, \tag{84}$$

$$\bar{\mathbf{A}} = \begin{bmatrix} 1.00005 & 0.01 & 0.00005 \\ 0.01 & 1 & 0.01 \\ -0.00005 & -0.01 & 0.99995 \end{bmatrix}, \quad \bar{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{85}$$

$$\Delta\xi_k = \xi_k - \xi_R, \quad \Delta u_k = u_k - u_R \tag{86}$$

and a classical linear Luenberger observer was designed such that the dynamic matrix of the observer error has a eigenvalue $z = 0.1$ with multiplicity $n = 3$.

All three observers are initialized with $\hat{\boldsymbol{\xi}}_k = \mathbf{0}$ and $\Delta\hat{\boldsymbol{\xi}}_k = \mathbf{0}$ respectively. In Fig. 3 the evolution of the states $\xi_1(t)$ to $\xi_3(t)$ of the continuous-time model and the corresponding observer errors σ_1 to σ_3 of all three observers Σ_1, Σ_2 and Σ_3 are displayed. One can see that the proposed observer Σ_2 achieves much better accuracy that the other two observers. In fact, for σ_3 the observer Σ_2 achieves to drive the state to a band of approximately $|\sigma_3| < 2.10^{-3}$, while using Σ_1 and Σ_3 the band in which σ_3 converges increases by the factor 500.

To illustrate the performance of higher order observers the following 4-th order nonlinear input-affine continuous-time system is considered:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 - \sin(\xi_1)u \\
\dot{\xi}_3 &= \xi_4 - \sin(\xi_2)u \\
\dot{\xi}_4 &= \delta_4(\xi_3)u,
\end{aligned}
$$

where $\delta_4(\xi_3) = -\sin(x_3)$ is assumed as unknown, but bounded parameter uncertainty. The nonlinearites in channel 2 and 3 are considered in the observer design. The observer is implemented according to subsection 4.3 with the observer parameters $p_1 = 40$, $h = 0.01$ and $\mu = 10$. The continuous-time system is initialized with $\boldsymbol{\xi}_0 = \begin{bmatrix} 0 & 0 & 0 & \frac{\pi}{8} \end{bmatrix}^T$, while the observer states are initialized with $\hat{\boldsymbol{\xi}}_0 = \mathbf{0}$. Despite of the uncertainty, the observer error states stay in a neighbourhood around the origin, as can be seen in Fig. 4. The first subfigure shows the time evolution of the states of the continuous-time system, while subfigures 2-5 show the time evolution of the observer errors. One can see the achieved precision for every system state which is from approximately 10^{-7} for the first state to approximately 0.4 for the last observer state.

7. Conclusion

A new design scheme for arbitrary order sliding mode observer for nonlinear input-affine systems has been proposed, where the design procedure of the sliding mode algorithm is reduced to state-dependent eigenvalue placement. Two discrete-time eigenvalue mappings have been proposed, which all suppresses discretization chattering. Simulation studies indicate that the proposed observer posses ses an upper bound of its practical convergence time of its estimation errors.
It is proven that for order 2 and 3 the estimation errors converge to the origin in the unperturbed case. Furthermore, the method to prove global asymptotic stability for higher order observers ($n \geq 4$) has been demonstrated. The proposed observer for order $n = 3$ was evaluated in simulation studies and was compared to a classical Luenberger observer and a discrete-time High gain observer. Additionally, the performance of the observer was demonstrated on a 4-th order system.

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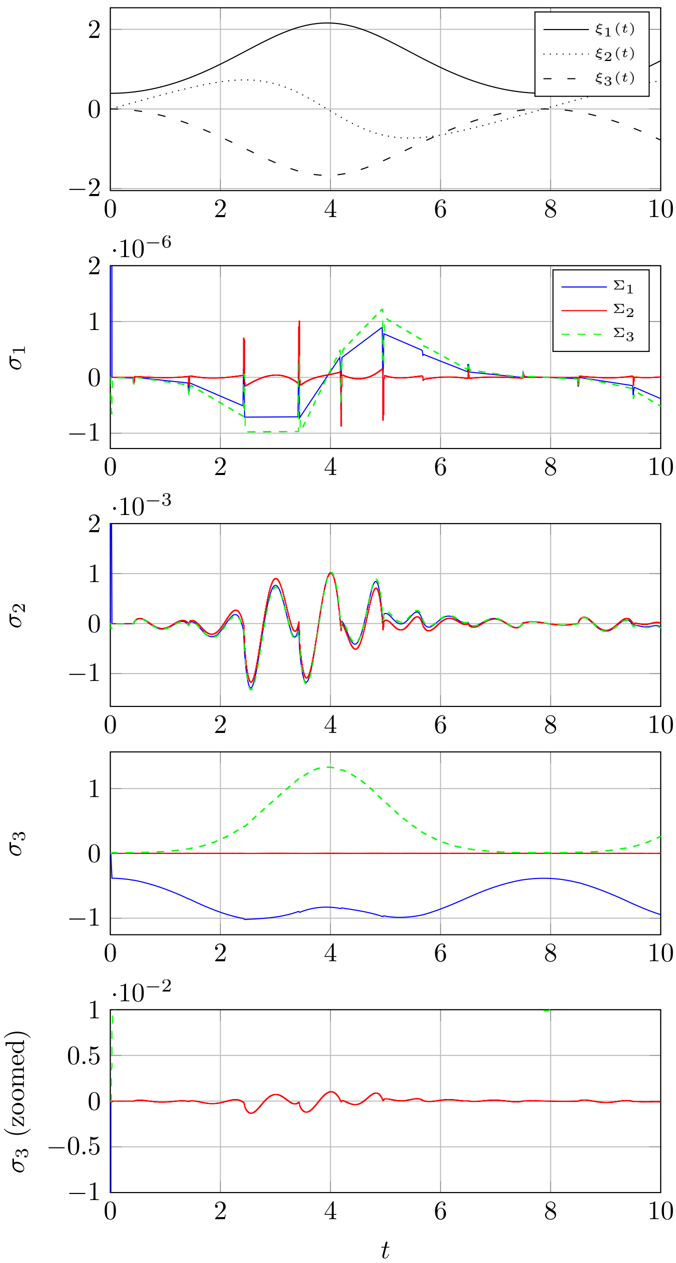

Figure 3: Comparison of the proposed observer algorithms to a classical Luenberger observer

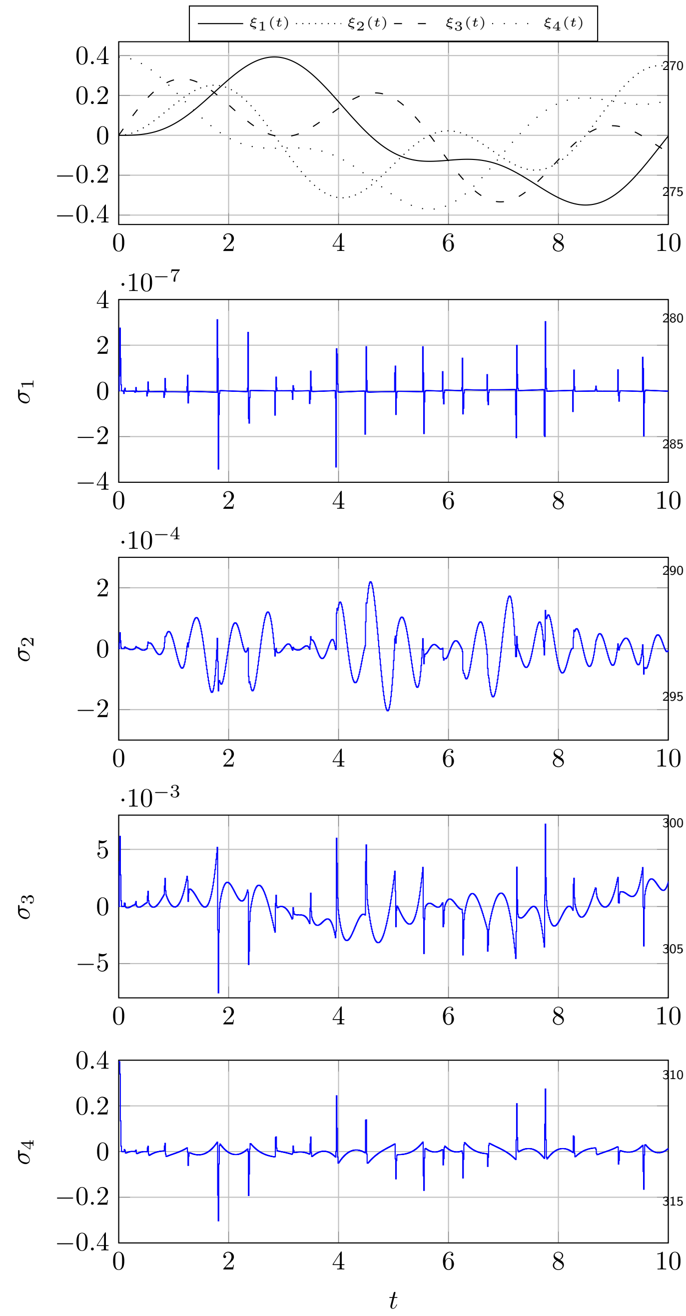

Figure 4: Simulation study to illustrate the performance of the proposed observer for higher order

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