Adaptive Gains Super-Twisting-Algorithm: Design and Discretization

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***Abstract*— In this paper, an eigenvalue-based discretization scheme is applied to a novel adaptive super-twisting-algorithm. Following the proposed procedure the discretization chattering effect is avoided entirely. An attractive property of the adaptation law is the insensitivity of the closed-loop system to overly large gains which in existing laws potentially leads to instability. Using Lyapunov's direct method the stability of the feedback is loop shown. Numerical examples underline the beneficial properties of the proposed methodology.**

I. INTRODUCTION

Design techniques based on sliding mode control are often the first choice when dealing with systems subject to external non-vanishing disturbances and model uncertainties [1], [2], [3], [4], [5]. However, the tuning of the control laws is not straightforward and often requires a priori knowledge about the disturbances acting on the plant, e.g., an upper bound on the amplitude. This drawback can be overcome by introducing adaptation schemes for the controller gains. see, e.g., [6], [7], [8]. Commonly, sliding mode controllers are expressed in continuous-time whereas the practical implementation requires a discrete-time formulation. It is well-known that the discretization of sliding mode based controllers is a critical step in the design process as an improper discretization leads to undesired phenomena in the resulting feedback loop such as discretization chattering, i.e, high frequency oscillations [9], [10], [11]. These oscillations are in particular problematic when using some gain adaptation as this constellation may trigger unbounded growth of controller gains which in consequence leads to loop instability.

In [12] the implicit discretization is applied to a Super-Twisting observer with time varying gain which allows to adjust a desired convergence time. The implicit scheme allows to avoid the discretization chattering irrespective of unbounded gains. However, it is assumed that the upper bound of the disturbances is known a priori. Other adaptive discrete-time sliding mode approaches can be found in [13], [14]. To the best knowledge of the authors, the discretization of adaptive gain super-twisting-algorithms (AGSTAs) has not yet been addressed in literature.

In this work a continuous-time AGSTA together with a proper discretization scheme is proposed. The presented adaptation law is based on an algorithm as introduced in [15]. In contrast to the original algorithm it provides an increased adaptation rate. The discretization scheme is based on the techniques published in [16]. The algorithm presented in this paper provides the above mentioned attractive features such as gain adaptation, discretization chattering avoidance and, does not require any a priori knowledge about the disturbance slew rate. Furthermore, the control law is given by explicit recursions guaranteeing a straightforward implementation.

The paper is structured as follows: In Section II a motivating example, illustrating loop instability when applying explicit Euler discretization to the AGSTA [7] is given. In Section III the used discretization scheme is revisited, Section IV outlines the proposed adaption scheme and the main result is presented. An illustrative example is discussed in Section V and Section VI concludes the paper.

II. MOTIVATION

As motivating example consider a scalar plant $\dot{x} = u + \varphi(t)$, where u and φ represent the control signal and an external (possibly non-vanishing) disturbance, respectively. The disturbance is assumed to satisfy $|\dot{\varphi}| \leq L$ with the positive constant L. The goal is to drive the system state x to zero despite the disturbance φ. If L is known the problem can be solved by applying the super-twisting-algorithm

$$\begin{aligned} u &= -\beta_1 |x|^{\frac{1}{2}} \operatorname{sign}(x) + v \\ \dot{v} &= -\beta_2 \operatorname{sign}(x) \end{aligned} \tag{1}$$

as controller with the positive real constant parameters β_1, β_2 and the controller state variable v, see [17]. In the case of an unknown bound L the problem can typically be solved by applying a suitable controller parameter adaptation to the Super-Twisting-Algorithm (STA). The adaptation scheme

$$\begin{aligned} \dot{\beta}_1 &= \begin{cases} \omega_1 \operatorname{sign}(|x| - \mu) & \beta_1 > \beta_m \\ \omega_2 & \beta_1 \leq \beta_m \end{cases} \\ \beta_2 &= \omega_3 \beta_1 \end{aligned} \tag{2}$$

originally proposed in [7], with tuning parameters ω_1, ω_2, ω_3, μ and β_m, allows to drive the state variable x into the vicinity of $x = 0$. In contrast to other adaptation algorithms see, e.g., [15] this approach also allows to reduce the gains of the controller by introducing the parameter μ. This is usually desirable as overly large controller gains will inevitably lead to chattering. In general, the chattering, however, is

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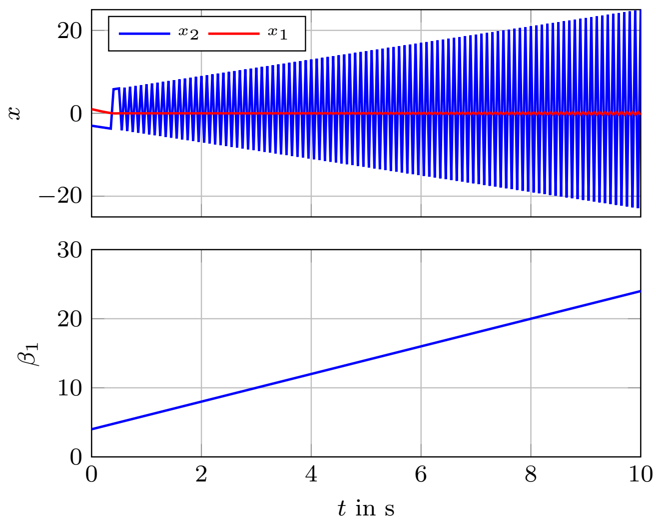

Fig. 1. Simulation of the closed loop system: The upper plot shows the states x_1 and x_2 of the system. The lower plot shows the temporal evolution of the adaptation variable γ

evoked by unmodeled dynamics in the control loop such as, e.g., actuator and sensor dynamics and time delays as well as the discrete-time realization of the controller in a digital environment. The chattering due to the discrete-time realization of the controller, which goes along with sampling of the control output, often is termed discretization chattering, see [16]. For the super-twisting-algorithm it has been shown that without any parameter adaptation and in the presence of sampling with sampling time T_s, the trajectories of the closed loop, that is, $x_1 := x$ and $x_2 := v - \varphi$ do not converge to the origin but stay in a vicinity of it, i.e., the homogeneous ball

$$\mathscr{R} = \{x_1, x_2 \in \mathbb{R} : |x_1| \leq \mu_1 T_\mathrm{s}^2, |x_2| \leq \mu_2 T_\mathrm{s}\} \tag{3}$$

with constants $\mu_1, \mu_2 \in \mathbb{R}^+$ [18]. Note that, with properly tuned controller gains β_1 and β_2 the constants μ_1, μ_2 are typically small so that the closed-loop performance is satisfactory.

When enhancing the algorithm with some adaptation mechanism, the parameter tuning, however requires special attention. This is illustrated by the following simulation example. Fig. 1 shows the temporal evolution of the system states of the closed loop governed by the plant and the AGSTA (1) with gain adaptation (2). The values for this simulations were chosen as: $T_\mathrm{s} = 0.05$ s, the initial values $x_{1,0} = 1$, $x_{2,0} = 1$, $\beta_{1,0} = 4$ and the parameters $\omega_1 = 2$, $\mu = 0.001$, $\omega_3 = \beta_m$, $\omega_2 = 1$ and $\beta_m = 0.01$. The continuous-time AGSTA has been discretized using the forward Euler discretization.

It is clearly visible that in the beginning the algorithm works as intended. The system states approach the origin and also the adaptation law adapts the gains accordingly. However, after approximately $t = 0.5$ s the system-states start to oscillate, i.e., chattering is present. Since the plant state oscillates and do not decrease any more also the adaptation law cannot decrease and will continue to grow, which, in turn, will further increase the amplitude of the oscillation. Consequently, the system states will grow unbounded.

It is noteworthy that this effect can be avoided by increasing the parameter μ. However proper tuning requires knowledge on the chattering amplitude which depends on the controller gains and thus also on the parameters of the adaptation algorithm and its initialization. This renders the tuning cumbersome and often unpractical. This issue calls for advanced discretization schemes that allow to avoid the discretization chattering and, in particular, render the closed loop insensitive to overly large gains which means that the chattering amplitude should not grow when growing the gains. One such discretization scheme for the STA is the implicit discretization proposed in [19]. Others have been proposed in [20], [16], [18]. In the following the technique described in [16] is exploited to discretize an adaptive STA.

III. DISCRETIZATION SCHEME

Consider the plant dynamics written as

$$\begin{aligned} \dot{x} &= u + \varphi, \\ \dot{\varphi} &= \Delta. \end{aligned} \tag{4}$$

Under zero-order-hold discretization the plant dynamics (4) are governed by the recursions

$$\begin{bmatrix} x_{k+1} \\ \varphi_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & T_\mathrm{s} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ \varphi_k \end{bmatrix} + \begin{bmatrix} T_\mathrm{s} \\ 0 \end{bmatrix} u_k + \begin{bmatrix} \frac{T_\mathrm{s}^2}{2} \\ T_\mathrm{s} \end{bmatrix} [-L, L]. \tag{5}$$

For the realization of the STA in a digital environment, the continuous-time STA (1) needs to be discretized. To that end the continuous time closed-loop plant, i.e.,

$$\begin{aligned} \dot{x_1} &= -\beta_1 |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1) + x_2, \\ \dot{x_2} &= -\beta_2 \operatorname{sign}(x_1) + \Delta \end{aligned} \tag{6}$$

is rewritten in so-called pseudo-linear representation. In general, the pseudo-linear form of a non-linear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{7}$$

with state vector $\mathbf{x} \in \mathbb{R}^n$ and vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\dot{\mathbf{x}} = \mathbf{M}(\mathbf{x})\mathbf{x}. \tag{8}$$

Note that, in general the state dependent matrix $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is nonlinear and not unique. The closed loop system (6) can be expressed as

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\beta_1 |x_1|^{-\frac{1}{2}} & 1 \\ -\beta_2 |x_1|^{-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta \tag{9}$$

in pseudolinear form for all $x_1 \in \mathbb{R}, x_1 \neq 0$. The basic idea of the discretization scheme is to obtain the discrete-time STA by mapping the state-dependent eigenvalues

$$s_{1,2} = |x_1|^{-\frac{1}{2}} p_{1,2} \tag{10}$$

of the the dynamic matrix

$$\mathbf{M}(\mathbf{x}) = \begin{bmatrix} -\beta_1 |x_1|^{-\frac{1}{2}} & 1 \\ -\beta_2 |x_1|^{-1} & 0 \end{bmatrix} \tag{11}$$

where $p_{1,2}$ are the roots of the polynomial

$$\omega(s) = s^2 + \beta_1 s + \beta_2 \tag{12}$$

to the discrete-time domain. The parameter p_i are related to the parameters β_i in the following manner:

$$\beta_2 = p_1 p_2, \quad \beta_1 = -(p_1 + p_2), \quad p_1, p_2 \in \mathbb{R}^-. \tag{13}$$

The discrete-time controller u_k is computed by assigning this discrete-time eigenvalues q_i to the dynamic matrix of the discrete-time closed-loop. This is achieved via

$$\begin{aligned} u_k &= -\beta_1 \phi_1 + v_k \\ v_{k+1} &= v_k - T_\mathrm{s} \beta_2 \phi_2 \end{aligned} \tag{14}$$

with

$$\phi_1 = -\frac{1}{T_\mathrm{s}\beta_1}(q_1 + q_2 - 2)x_{1,k} \tag{15a}$$

$$\phi_2 = \frac{1}{T_\mathrm{s}^2 \beta_2}(q_1 - 1)(q_2 - 1)x_{1,k} \tag{15b}$$

and defining $x_{2,k} = \varphi_k + v_k$ and $\mathbf{x_k} = [x_{1,k}\ x_{2,k}]^T$ results in

$$\mathbf{x}_{k+1} = \mathbf{M}_\mathrm{d}\mathbf{x}_k \tag{16}$$

with

$$\mathbf{M}_{\mathrm{d},m} = \begin{bmatrix} q_m(s_1) + q_m(s_2) - 1 & T_\mathrm{s} \\ \frac{1}{T_\mathrm{s}}\left[q_m(s_1) + q_m(s_2) - 1 - q_m(s_1)q_m(s_2)\right] & 1 \end{bmatrix} \quad m \in \mathscr{A} \tag{17}$$

$\mathscr{A} = \{E, I, M, P\}$. Controller (14) was proposed in [18]. The matrix $\mathbf{M}_\mathrm{d}$ has eigenvalues at $z_i = q_i$. Therefore, q_1 and q_2 are the discretized eigenvalues of the continuous-time system (6).

As proposed in [16], the mapping of the continuous-time eigenvalues can be done in different ways, e.g.

1) Explicit Euler

$$q_E(s) = f_\mathrm{E}(s_i) = 1 + s_i(x_1)T_\mathrm{s} \tag{18a}$$

2) Implicit

$$q_I = f_\mathrm{I}(s_i) = \frac{1}{1 - s_i(x_1)T_\mathrm{s}} \tag{18b}$$

3) Matching

$$q_M = f_\mathrm{M}(s_i) = e^{s_i(x_1)T_\mathrm{s}} \tag{18c}$$

4) Pade Approximation

$$q_P = f_\mathrm{P}(s_i) = \frac{1 + s_i(x_1)\frac{T_\mathrm{s}}{2}}{1 - s_i(x_1)\frac{T_\mathrm{s}}{2}}. \tag{18d}$$

Note that the discretized eigenvalues q_i are functions of the continuous-time eigenvalues. Substituting the forward Euler mapping (18a) into (14) yields the forward Euler discretization of the continuous STA, see, [16]. In contrast to the explicit Euler mapping the other approaches provide for the elimination of the discretization chattering and the insensitivity to overly large gains. Thus, the latter three approaches are in particular suitable for the discretization of an adaptive gains STA. In the following it is focused on the Matching approach and the impact of this procedure on the discretization of an adaptive algorithm is investigated. The application of these procedure is shown in the next section. It is noteworthy that by using the Matching-Approach and assuming real continuous-time eigenvalues, the discrete-time eigenvalues are $q_i \in [0, 1]$, i.e. the eigenvalues are always contained in the unit disk in the complex plain. However, stability of the closed-loop system can not be concluded from the eigenvalues, as they depend on the state variable x_1.

IV. ADAPTIVE DISCRETE-TIME SUPER-TWISTING-ALGORITHM

The proposed discrete-time algorithm is motivated by the continuous-time adaptive STA

$$\begin{aligned} u &= -\beta_1 \gamma |x|^{\frac{1}{2}} \mathrm{sign}(x) + v \\ \dot{v} &= -\beta_2 \gamma^2 \mathrm{sign}(x) \end{aligned} \tag{19}$$

where the adaptation variable γ is governed by

$$\dot{\gamma} = \frac{\gamma}{2}\alpha \begin{cases} |x|^{-\frac{1}{2}}\gamma & |x_1| \geq \gamma^2 \\ |x|^{\frac{1}{2}}\frac{1}{\gamma} & |x_1| < \gamma^2 \end{cases} \tag{20}$$

where $\gamma(0)$ needs to be a positive real number and α represents a positive constant. This adaptive gain STA in combination with plant (4) yields the closed-loop system

$$\begin{aligned} \dot{x}_1 &= -\beta_1 \gamma |x_1|^{\frac{1}{2}} + x_2, \\ \dot{x}_2 &= -\beta_2 \gamma^2 \mathrm{sign}(x_1) + \Delta, \end{aligned} \tag{21}$$

where x_1 and x_2 represent the state variables. Note that the adaptation law (20) is a modified version of the algorithm proposed in [15] and provides an increased adaptation rate in the case of large deviations from $|x_1| = 0$. As in the previous section, also this closed-loop system can be represented in pseudo-linear form, i.e.,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\beta_1 \gamma |x_1|^{-\frac{1}{2}} & 1 \\ -\beta_2 \gamma^2 |x_1|^{-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta. \tag{22}$$

Proposition 1: Consider the closed loop system (21) consisting of the plant (4) with controller (19) and adaptation (20). The parameter satisfy (13) and $\alpha < \beta_1$. In the unperturbed case, i.e., $\varphi(t) = 0, \Delta = 0, \forall t$, the origin $x_1 = x_2 = 0$ is globally asymptotically stable.

Proof: The same Lyapunov-function as used in [15], i.e.,

$$V(x_1, x_2) = \gamma^2 |x_1| + \frac{1}{2p_1 p_2} x_2^2. \tag{23}$$

is used to show the asymptotic convergence to zero of the state variables in system (23). The time derivative of (23) along the trajectories of system (21) is given

$$\dot{V}(x_1) = -\gamma^3 \beta_1 |x_1|^{\frac{1}{2}} + 2\gamma\dot{\gamma}|x_1|. \tag{24}$$

Using the adaptation law (20) this can be written as

$$\dot{V} = -\gamma |x_1|^{\frac{1}{2}} \begin{cases} \gamma^2(\beta_1 - \alpha) & |x_1| \geq \gamma^2 \\ \beta_1 \gamma^2 - |x_1|\alpha & |x_1| < \gamma^2 \end{cases} \tag{25}$$

which is negative semi-definite if $\alpha < \beta_1$. The asymptotic stability of $x_1 = x_2 = 0$ can be concluded by the application of the Extended Invariance Principle as outlined in [21]. ■

The eigenvalues of the state-dependent dynamic matrix in system (22) are

$$s^*_{1,2} = \gamma s_{1,2} = \gamma |x_1|^{-\frac{1}{2}} p_{1,2}, \tag{26}$$

which relates the eigenvalues $s_{1,2}$ from the closed-loop system without adaptation with the eigenvalues $s^*_{1,2}$ from the closed-loop system with adaptation. Adaptation law (20) is also a pseudo-linear system representation with eigenvalue

$$s_a = \frac{\alpha}{2} \begin{cases} \gamma |x_1|^{-\frac{1}{2}} & |x_1| \geq \gamma^2 \\ \frac{|x_1|^{\frac{1}{2}}}{\gamma} & |x_1| < \gamma^2 \end{cases}. \tag{27}$$

The adaptive discrete-time STA proposed in this paper is obtained by the discrete-time STA (14) with an eigenvalue mapping $q(s^*_{1,2}(x_{1,k}))$ in combination with discrete-time adaptation law

$$\gamma_{k+1} = \gamma_k \cdot q(s_a(x_{1,k})), \tag{28}$$

which is also established by the same discretization methodology. Therein $x_{1,k} = x_1(kT_s)$ and $k = 0, 1, 2, \ldots$.

The focus in this paper stays on the Matching-approach, which means that all discretizations take place using the mapping procedure (18c).

With this mapping, the closed loop is globally asymptotically stable.

Proposition 2: Consider the closed loop composed of the plant (5) and the controller (14), (15a), (15b) with adaptation (26), (27), (28) and mapping (18c). Let the parameters satisfy $\beta_1 = -2p$, $\beta_2 = p^2$ with $p \in \mathbb{R}$ and $p < 0$ and

$$\alpha < \frac{1}{T_s}\left[\ln\left(-\frac{\left(-1+e^{pT_s}\right)^2\left(-1-2e^{pT_s}+e^{2pT_s}\right)}{2p^2T_s^2}\right) - 2pT_s\right].$$

Then, in the unperturbed case, i.e., $\varphi(t) = 0, \forall t$, the origin $x_{1,k} = x_{2,k} = 0$ is globally asymptotically stable.

Proof: Consider the candidate Lyapunov-function

$$V_k = \gamma_k^2 |x_{1,k} - T_s x_{2,k}| + \frac{1}{2p^2} x_{2,k}^2. \tag{29}$$

Using (5), (14) and (15a), (15b) the first difference yields

$$\Delta V_k = \gamma_{k+1}^2 |\bar{q}^2 x_{1,k}| + \frac{1}{2p^2}\left(-\frac{1}{T_s}(\bar{q}-1)^2 x_{1,k} + x_{2,k}\right)^2 - \gamma_k^2 |x_{1,k} - T_s x_{2,k}| - \frac{1}{2p^2} x_{2,k}^2 \tag{30}$$

where, due to the particular choice of the parameters, $\bar{q} = q_M(s_1(x_{1,k})) = q_M(s_2(x_{1,k}))$ holds. Reformulating (30) as

$$\Delta V_k = \left(\gamma_{k+1}^2 q^2 \operatorname{sign}(x_{1,k}) + \frac{1}{2p^2T_s^2}(q-1)^4 x_{1,k}\right) x_{1,k} - \left(\frac{1}{p^2T_s^2}(q-1)^2 x_{1,k}\right) T_s x_{2,k} - \gamma_k^2 |x_{1,k} - T_s x_{2,k}| \tag{31}$$

and defining

$$f_1(x_{1,k}) := \gamma_{k+1}^2 q^2 \operatorname{sign}(x_{1,k}) + \frac{1}{2p^2T_s^2}(q-1)^4 x_{1,k} \tag{32}$$

$$f_2(x_{1,k}) := \frac{1}{p^2T_s^2}(q-1)^2 x_{1,k} \tag{33}$$

leads to

$$\Delta V_k = f_1(x_{1,k}) x_{1,k} - f_2(x_{1,k}) T_s x_{2,k} - \gamma_k^2 |x_{1,k} - T_s x_{2,k}|. \tag{34}$$

Note that $\Delta V_k < 0$ holds if

$$f_1(x_{1,k}) x_{1,k} - f_2(x_{1,k}) T_s x_{2,k} < \gamma_k^2 |x_{1,k} - T_s x_{2,k}| \tag{35}$$

holds. Functions $f_1(x_{1,k})$ and $f_2(x_{1,k})$ are odd and $\operatorname{sign}(f_1(x_{1,k})) = \operatorname{sign}(f_2(x_{1,k})) = \operatorname{sign}(x_{1,k})$, which results from $q \in [0, 1]$. The inequality is fulfilled if $|f_1(x_{1,k})| < |f_2(x_{1,k})| < \gamma_k^2$. Note that for this inequality to hold true in the limit the adaptation law must satisfy $\gamma_{k+1}^2 = \gamma_k^2$ as

$$\lim_{x_{1,k}\to\infty} f_1(x_{1,k}) = \gamma_{k+1}^2, \quad \lim_{x_{1,k}\to\infty} f_2(x_{1,k}) = \gamma_k^2. \tag{36}$$

It can be verified from (28) that in the limit $\lim_{x_{1,k}\to\infty} f_i(x_{1,k})$ the relation $\gamma_{k+1}^2 = \gamma_k^2$ indeed holds. In the following, due to symmetry, only $x_{1,k} > 0$ is considered to show $|f_1(x_{1,k})| < |f_2(x_{1,k})|$, which then, using (32) and (33) takes the form

$$\gamma_{k+1}^2 2p^2T_s^2 \frac{x_{1,k}}{|x_{1,k}|} + q^2(q-2)^2 x_{1,k} < x_{1,k}. \tag{37}$$

The cases $x_{1,k} \geq \gamma^2$ and $x_{1,k} < \gamma^2$ of the adaptation law (27), (28) are now being investigated by substituting γ_{k+1} in (37).

Case $|x_{1,k}| \geq \gamma^2$:
Inequality (37) yields

$$\gamma_k^2 e^{2\frac{\alpha}{2}\gamma_k |x_{1,k}|^{-\frac{1}{2}} T_s} \frac{q^2 2p^2T_s^2}{|x_{1,k}|} + q^2(q-2)^2 < 1. \tag{38}$$

Introducing the variable $\tau = \frac{\gamma_k p T_s}{\sqrt{|x_{1,k}|}}$ leads to

$$g_1(\tau) := 1 - e^{2\tau}(2\tau^2 e^{\frac{\alpha}{p}\tau} + (e^{\tau}-2)^2) > 0. \tag{39}$$

Note that τ incorporates both, γ and $x_{1,k}$ and, therefore, the interval $|x_{1,k}| \geq \gamma^2$ which covers case 1 is equivalent to $\tau \in [pT_s, 0)$. Recall that $p < 0$.

Case $|x_{1,k}| < \gamma^2$:
Using the second part of the adaptation law and following the same procedure as above one obtains

$$g_2(\tau) := 1 - \left(e^{\frac{\alpha p T_s^2}{-\tau}} 2\tau^2 + e^{4\tau} - 4e^{3\tau} + 4e^{2\tau}\right) > 0 \tag{40}$$

for $\tau \in (-\infty, pT_s]$.

Thus, if (39) and (40) hold true, (29) is a Lyapunov function. To verify (39) and (40) the roots of the two exponential polynomials need to be determined which is done numerically in this paper.

In order to enhance the investigation of the stability the influence of the parameter α is taken into account. At the switching point $\tau = pT_s$, $g_1(\tau = pT_s) = g_2(\tau = pT_s) > 0$ must hold. In this regard the maximum value α_m for α can be calculated by setting $g_1(\tau = pT_s) = g_2(\tau = pT_s) = 0$ which yields the expression

$$\alpha_m = -\frac{(2pT_s - \ln(-\frac{(-1+e^{pT_s})^2 \cdot (-1-2e^{pT_s}+e^{2pT_s})}{2p^2T_s^2}))}{T_s}. \tag{41}$$

Hence, if $\alpha < \alpha_m$, $g_1(\tau = pT_s) = g_2(\tau = pT_s) > 0$.

Functions $g_1(\tau)$ and $g_2(\tau)$ are plotted in Fig. 2 with $\alpha = \alpha_m$, $T_s = 0.05$s and $p = -5$. From this plot it can be seen that the two functions are always greater than zero except on two occasions. One is at $\tau = 0$, which equals $x_{1,k} \to \infty$, the other one depends on α at the switching point. ■

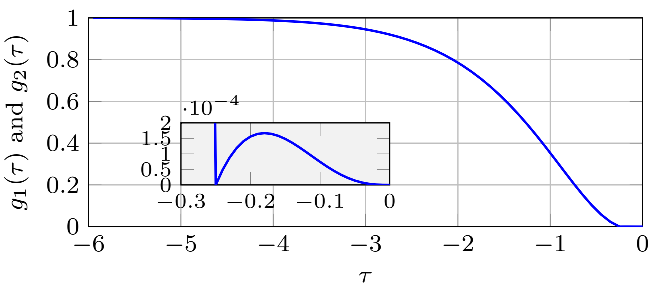

Fig. 2. Function values of $g_1(\tau)$ and $g_2(\tau)$ as evolution ofer τ.

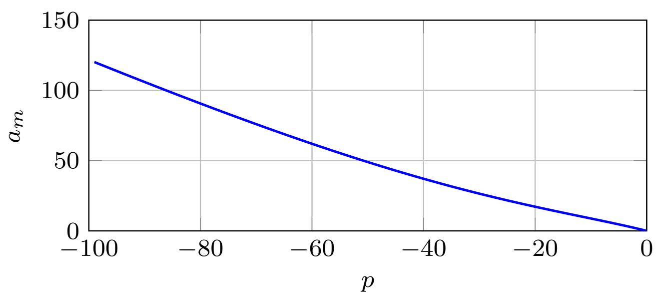

Fig. 3. Relation of the maximum adaptation speed α_m to controller gain p

Fig. 3 shows the relation from the adaptation speed α_m over the controller gain p, thus representing a visualization of (41). The relation is not linear, but depends on the controller gain and therefore changes with the original chosen values for p.

The dynamical system (17) together with (18c) and (26) approaches the dead beat system (42) if $\gamma \to \infty$.

$$\mathbf{x}_{k+1} = \begin{bmatrix} -1 & T_s \\ -\frac{1}{T_s} & 1 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix} [-L, L] \tag{42}$$

The maximum factor ξ by which the dead beat system can increase the peak of its input $[-L, L]$ is called peak gain [22]. It is defined as

$$\xi = ||h||_1 \tag{43}$$

with h being the impulse response of the dead bead system (42) and specifying $x_{1,k}$ as the output.

V. SIMULATION EXAMPLE

Results which are obtained by simulating the unperturbed closed loop system consisting of the plant (5) and the proposed discrete-time control strategy using controller (14), (15a), (15b) with adaptation (26), (27), (28) and mapping (18c) are plotted in Fig. 4. The sampling time is $T_s = 0.05$s, initial values are $\begin{bmatrix} x_{1,0} & x_{2,0} \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\gamma_0 = 1.$ and the gain parameters are chosen to $p = -2$ and $\alpha = 1$.

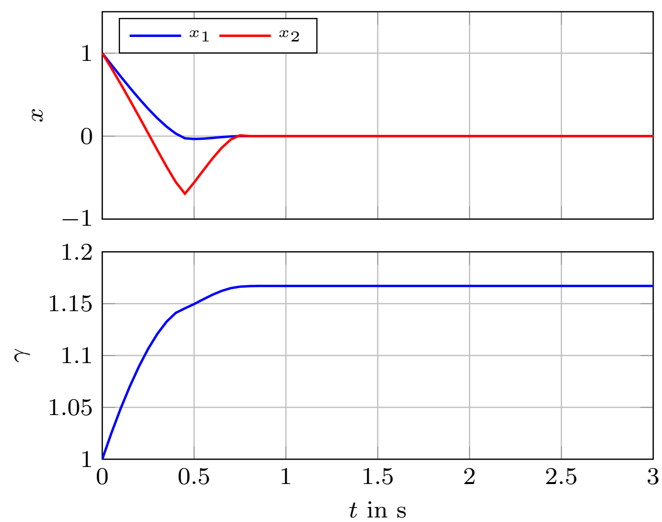

Fig. 4. Simulation of the closed loop system: The upper plot shows the states x_1 and x_2 of the system. The lower plot shows the temporal evolution of the adaptation variable γ, discretized with the Matching-approach

The temporal evolutions of the states and the adaptation variable are visible. In comparison to the motivating example given in section II, see Fig. 1, the system is stable and also under the adaptation there is no discretization chattering, which is avoided effectively. This also lets the adaptation variable to converge to a constant value and a steady state is reached.

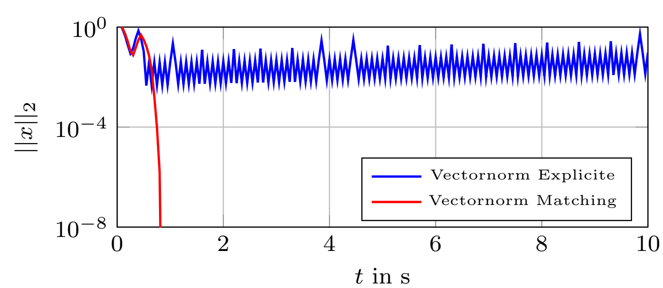

Fig. 5. Comparison of Matching- and Euler-forward approach: The vector norm of x_1 and x_2 shows a discretization chattering which increases in amplitude with the Euler-forward discretization whereas the Matching-approach allows for a smooth transition to zero of the vector norm.

The vector-norm of the state-variables x_1 and x_2 is visible as the red line in Fig. 5. Additionally the simulation was carried out by using the Euler forward discretization method (18a) instead of the Matching approach (18c), where the result can bee seen as the blue line. Due to the discretization chattering and the continuous adaptation of γ the Euler forward discretized system shows an increasing vector-norm in the temporal evolution whereas the system discretized by the Matching method shows no fluctuations of the states and the vector-norm converges to zero.

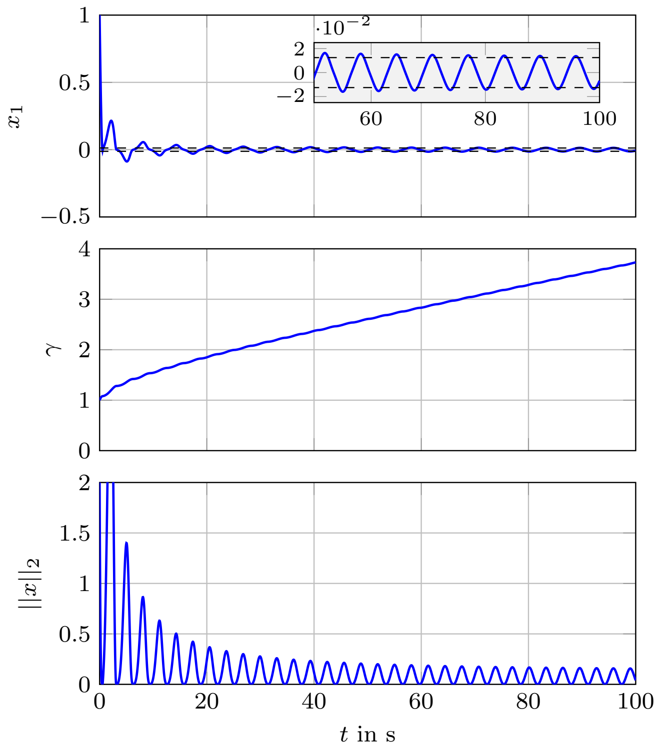

Fig. 6. Simulation of the closed loop system: The upper plot shows the states x_1 and x_2 of the system. The lower plot shows the temporal evolution of the adaptation variable γ, discretized with Euler-forward

In Fig. 6 the proposed algorithm, again consisting of the plant (5) and the discrete-time control strategy using controller (14), (15a), (15b) with adaptation (26), (27), (28) and mapping (18c), was used with an external disturbance $\Delta = 5\sin(t)$ present. The initial controller gain $p = -2$ is not sufficient to compensate for the disturbance. In order to show the adaptation of the variable γ a small $\alpha = 0.5$ was chosen, all other initial values remain unchanged to the previous example. The results in Fig. 6 show that the algorithm is adapting to the present disturbance by increasing the adaptation variable γ in order to increase the controller gain. The state space variable x_1 is converging into a band with the size of $\xi \cdot \max(\Delta_k) = 0.0025 \cdot 5 = 0.0125$, which is indeed the result of the disturbance present with the gain of the dead beat system (43) and displayed as the dashed lines in Fig. 6. Although the adaptation variable γ continues to increase, the vector-norm does not which underlines the insensitivity of the proposed methodology to overly large gains.

VI. Conclusion

An eigenvalue-based discretization scheme was applied to a novel adaptive super-twisting-algorithm. This approach evades the discretization chattering effectively and therefore an unbounded growth in chattering amplitude, see Fig. 1. The resulting algorithms are insensitive to overly large gains and are given in explicit recursions which is advantageous for the application in real world control hardware.

Although the final analytical proof of the positiveness of the inequalities (39) and (40) is still outstanding, global asymptotic stability is evaluated with the help of Lyapunov's direct method and by interpreting Fig. 2.

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