Auxiliary material to the lecture

Theorie der Elektrotechnik (engl.)

(437.253)

by

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Contents

1. Maxwell‘s equations

$$I.\quad curl\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$ generalized Ampere‘s law,

$$II.\quad curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$ Faraday‘s law,

$$III.\quad div\mathbf{B} = 0,$$ magnetic flux density is source free,

$$IV.\quad div\mathbf{D} = \rho,$$ Gauss‘ law.

Consequence: law of continuity

$$div\mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

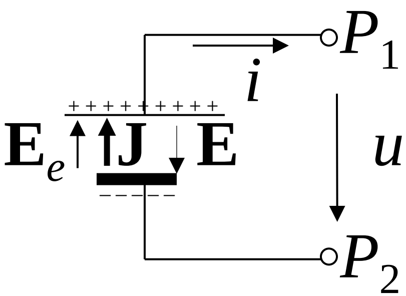
(implied by the divergence of the first equation + the fourth equation:

$$div\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0,\ div\mathbf{D} = \rho.)$$

Field quantities in voltage sources

Charges are separated in voltage sources due to non-electric (e.g. chemical) processes. The impressed field intensity $\mathbf{E}_e$ is a fictitious electric field intensity which would give rise to the same amount of charge separation:

$$u_q = \int_{P_2}^{P_1} \mathbf{E}_e \cdot d\mathbf{r} = -\int_{P_1}^{P_2} \mathbf{E}_e \cdot d\mathbf{r}, \quad u = \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{r}$$



Γ: Cross section

γ: conductivity

$$iR_q = \int_{P_2}^{P_1} \frac{\mathbf{J}\Gamma}{\gamma\Gamma} \cdot d\mathbf{r} = -\int_{P_1}^{P_2} \frac{\mathbf{J}}{\gamma} \cdot d\mathbf{r}$$

Non-ideal voltage source: $u = u_q - iR_q \Rightarrow \mathbf{E} = -\mathbf{E}_e + \dfrac{\mathbf{J}}{\gamma}$

Ideal voltage source: $u = u_q \Rightarrow \mathbf{E} = -\mathbf{E}_e, \gamma \to \infty$

$$\boxed{\mathbf{J} = \gamma(\mathbf{E} + \mathbf{E}_e)}$$

Material relationships:

$$\mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H},\ \mathbf{J} = \gamma(\mathbf{E} + \mathbf{E}_e).$$

Energy and power density:

$$w = w_e + w_m = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\ \left(= \int_0^D \mathbf{E}\cdot d\mathbf{D} + \int_0^B \mathbf{H}\cdot d\mathbf{B} \right),$$

$$p = \frac{|\mathbf{J}|^2}{\gamma}.$$

Energy and power loss in a volume Ω:

$$W = \int_\Omega w d\Omega, \quad P = \int_\Omega p d\Omega.$$

1.1 Classification of electromagnetics

1. Static fields $\left(\frac{\partial}{\partial t}=0\right)$

$$curl\mathbf{H}=\mathbf{J},\ curl\mathbf{E}=\mathbf{0},\ div\mathbf{B}=0,\ div\mathbf{D}=\rho,\ div\mathbf{J}=0.$$

electrostatic field: $curl\mathbf{E}=\mathbf{0},\ div\mathbf{D}=\rho,\ \mathbf{D}=\varepsilon\mathbf{E}.$

magnetostatic field: $curl\mathbf{H}=\mathbf{J},\ div\mathbf{B}=0,\ \mathbf{B}=\mu\mathbf{H}.$

static current field: $curl\mathbf{E}=\mathbf{0},\ div\mathbf{J}=0,\ \mathbf{J}=\gamma(\mathbf{E}+\mathbf{E}_e).$

2. Quasi-static field $\left(|\mathbf{J}| >> \left| \frac{\partial \mathbf{D}}{\partial t} \right| \right)$

$$curl\mathbf{H} = \mathbf{J},$$

$$curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$div\mathbf{B} = 0,$$

$$\mathbf{B} = \mu\mathbf{H},\ \mathbf{J} = \gamma(\mathbf{E} + \mathbf{E}_e).$$

3. Electromagnetic waves

$$curl\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$div\mathbf{B} = 0,$$

$$div\mathbf{D} = \rho,$$

$$\mathbf{D} = \varepsilon \mathbf{E},\ \mathbf{B} = \mu \mathbf{H},\ \mathbf{J} = \gamma (\mathbf{E} + \mathbf{E}_e).$$

1.2 Fundamentals of circuit theory

Circuit signals:

$$u_{12} = \int_1^2 \mathbf{E} \cdot d\mathbf{r}, \quad i_\Gamma = \int_\Gamma \mathbf{J} \cdot \mathbf{n} d\Gamma,$$

$$Q_\Omega = \int_\Omega \rho d\Omega = \int_\Omega div \mathbf{D} d\Omega = \oint_\Gamma \mathbf{D} \cdot \mathbf{n} d\Gamma, \quad \Phi_\Gamma = \int_\Gamma \mathbf{B} \cdot \mathbf{n} d\Gamma.$$

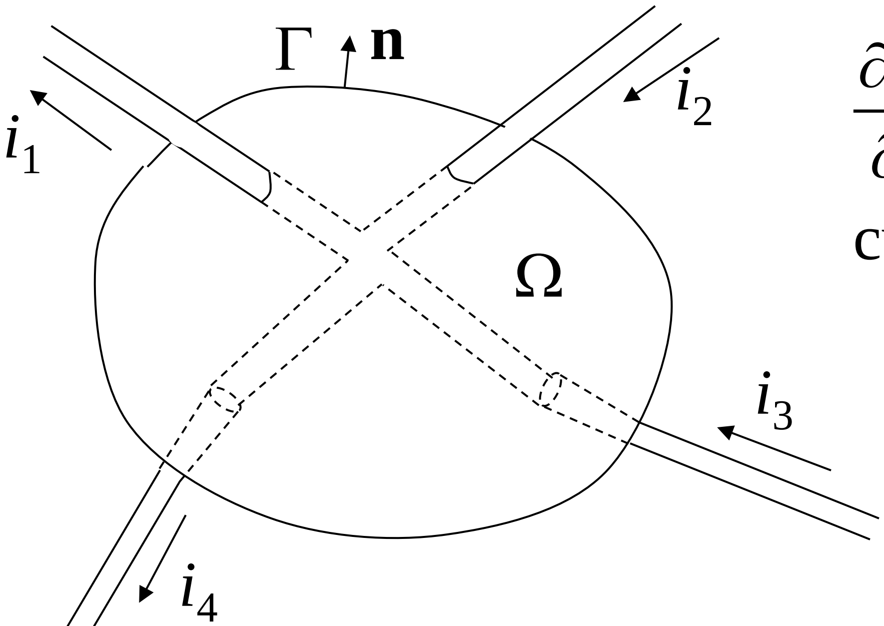
Circuit elements:

resistor: $u = Ri,$

ideal capacitor: $i = C\dfrac{du}{dt},$

ideal inductor: $u = L\dfrac{di}{dt}.$

Kirchhoff's current law:



$$\frac{\partial \mathbf{D}}{\partial t} = 0 \text{ on } \Gamma, \text{ since } \Gamma$$

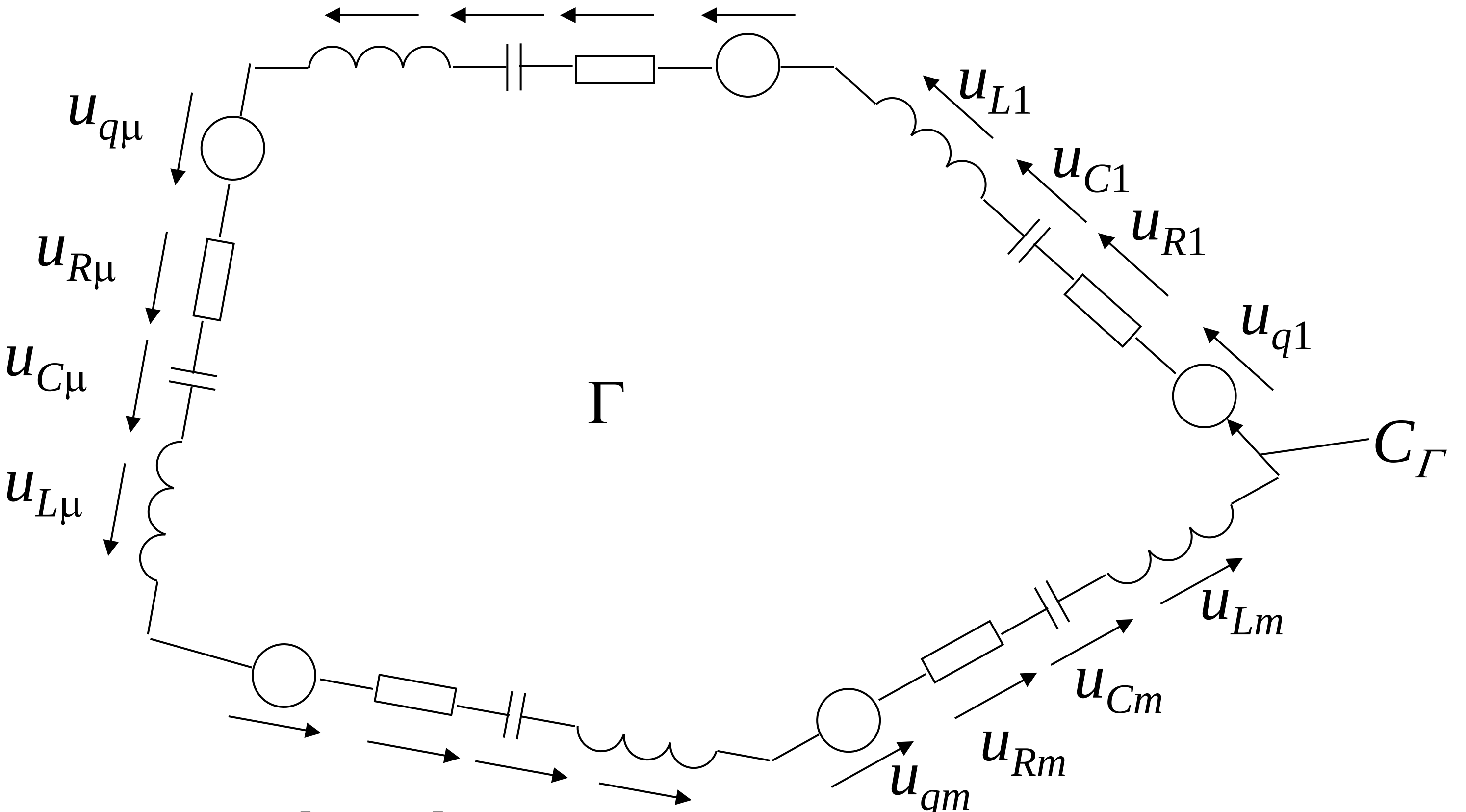
cuts no capacitor.

Integrating the law of continuity over Ω:

$$\int_\Omega \left(div\mathbf{J} + \frac{\partial \rho}{\partial t} \right) d\Omega = \int_\Omega div \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) d\Omega = \oint_\Gamma \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n} d\Gamma = \oint_\Gamma \mathbf{J} \cdot \mathbf{n} d\Gamma = 0.$$

$$\boxed{\sum_\nu i_\nu = 0}$$

Kirchhoff's voltage law:



Integrating Faraday's law over Γ:

$$\int_\Gamma \left(curl\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} d\Gamma = \oint_{C_\Gamma} \mathbf{E} \cdot d\mathbf{r} + \frac{d}{dt} \int_\Gamma \mathbf{B} \cdot \mathbf{n} d\Gamma = 0.$$

$$\sum_{\mu=1}^{m} \left(u_{q\mu} + u_{R\mu} + u_{C\mu} + u_{L\mu} \right) = 0$$

1.3 Energy conversion in electromagnetic field

$$-\mathbf{E}\cdot curl\mathbf{H} = -\mathbf{E}\cdot\mathbf{J} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$ I. M.Eq.· (**-E**)

$$+\quad \mathbf{H}\cdot curl\mathbf{E} = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}$$ II. M.Eq.· **H**

$$\mathbf{H}\cdot curl\mathbf{E} - \mathbf{E}\cdot curl\mathbf{H} = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\mathbf{J} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

$$\mathbf{H}\cdot curl\mathbf{E} - \mathbf{E}\cdot curl\mathbf{H} = \mathbf{H}\cdot\left(\nabla\times\mathbf{E}\right) - \mathbf{E}\cdot\left(\nabla\times\mathbf{H}\right) =$$

$$= \nabla\cdot\left(\mathbf{E}\times\mathbf{H}_c\right) + \nabla\cdot\left(\mathbf{E}_c\times\mathbf{H}\right) = \nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right) = div\left(\mathbf{E}\times\mathbf{H}\right)$$

Integration over a volume Ω:

$$-\int_{\Omega}\left(\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)d\Omega=\int_{\Omega}\mathbf{E}\cdot\mathbf{J}\,d\Omega+\int_{\Omega}div\left(\mathbf{E}\times\mathbf{H}\right)d\Omega$$

$$\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial}{\partial t}\int_{0}^{B}\mathbf{H}\cdot d\mathbf{B}=\frac{\partial w_m}{\partial t}\left(=\frac{\partial}{\partial t}\frac{1}{2}\mathbf{H}\cdot\mathbf{B}=\frac{\partial}{\partial t}\frac{1}{2}\mu\left|\mathbf{H}\right|^2\right)$$

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\frac{\partial}{\partial t}\int_{0}^{D}\mathbf{E}\cdot d\mathbf{D}=\frac{\partial w_e}{\partial t}\left(=\frac{\partial}{\partial t}\frac{1}{2}\mathbf{E}\cdot\mathbf{D}=\frac{\partial}{\partial t}\frac{1}{2}\varepsilon\left|\mathbf{E}\right|^2\right)$$

for linear media

$$\mathbf{J}=\gamma\left(\mathbf{E}+\mathbf{E}_e\right)\Rightarrow\mathbf{E}=\frac{\mathbf{J}}{\gamma}-\mathbf{E}_e\Rightarrow\mathbf{E}\cdot\mathbf{J}=\frac{\left|\mathbf{J}\right|^2}{\gamma}-\mathbf{E}_e\cdot\mathbf{J}=p-\mathbf{E}_e\cdot\mathbf{J}$$

$$\int_{\Omega}div\left(\mathbf{E}\times\mathbf{H}\right)d\Omega=\oint_{\Gamma}\left(\mathbf{E}\times\mathbf{H}\right)\cdot\mathbf{n}\,d\Gamma$$ (Gauss’ theorem)

Poynting's theorem:

$$-\frac{d}{dt}\int_{\Omega}\left(w_m + w_e\right)d\Omega = \int_{\Omega}\frac{\left|\mathbf{J}\right|^2}{\gamma}d\Omega - \int_{\Omega}\mathbf{E}_e\cdot\mathbf{J}d\Omega + \oint_{\Gamma}\left(\mathbf{E}\times\mathbf{H}\right)\cdot\mathbf{n}d\Gamma$$

right hand side:

reasons for the decrease of energy in a volume

$\mathbf{S} = \mathbf{E}\times\mathbf{H}$ Poynting's vector: power through unit surface

$\oint_{\Gamma}\mathbf{S}\cdot\mathbf{n}d\Gamma$ is the power leaving the volume Ω through radiation.

1.4 Existence of a unique solution to Maxwell's equations

The solution of Maxwell's equations in a volume Ω with the boundary Γ is unique for $t > t_0$, provided that the

1. initial conditions

$$\mathbf{H}(\mathbf{r}, t = t_0) = \mathbf{H}_{0anf}(\mathbf{r}), \mathbf{E}(\mathbf{r}, t = t_0) = \mathbf{E}_{0anf}(\mathbf{r}), \quad \forall \mathbf{r} \in \Omega,$$ and the

2. boundary conditions for the tangential components

$$\mathbf{H}_t(\mathbf{r}, t) = \mathbf{H}_{0tan}(\mathbf{r}, t) \ \underline{\text{or}} \ \mathbf{E}_t(\mathbf{r}, t) = \mathbf{E}_{0tan}(\mathbf{r}, t), \ \forall \mathbf{r} \in \Gamma, t \geq t_0$$

are fulfilled. Both the functions $\mathbf{H}_{0anf}(\mathbf{r}), \mathbf{E}_{0anf}(\mathbf{r})$, $\mathbf{H}_{0tan}(\mathbf{r}, t)$ and $\mathbf{E}_{0tan}(\mathbf{r}, t)$ and the impressed field intensity $\mathbf{E}_e$ must be known. The material properties are assumed to be linear.

Proof:

Assumption:

two solutions $\mathbf{E}', \mathbf{H}'$ and $\mathbf{E}'', \mathbf{H}''$ exist.

Their differences: $\mathbf{E}_0 = \mathbf{E}' - \mathbf{E}'', \mathbf{H}_0 = \mathbf{H}' - \mathbf{H}''$
satisfy Maxwell’s equations (these are linear). The initial and boundary conditions for the difference fields are homogenous and $\mathbf{E}_{e0}$=0. Therefore, Poynting’s theorem :

$$-\frac{d}{dt}\int_{\Omega}\left(\frac{1}{2}\mu\left|\mathbf{H}_0\right|^2 + \frac{1}{2}\varepsilon\left|\mathbf{E}_0\right|^2\right)d\Omega = \int_{\Omega}\frac{\left|\mathbf{J}_0\right|^2}{\gamma}d\Omega + \oint_{\Gamma}\left(\mathbf{E}_0 \times \mathbf{H}_0\right)\cdot\mathbf{n}\,d\Gamma$$

Since the boundary conditions imply that either $\mathbf{E}_0$ or $\mathbf{H}_0$ point in the normal direction **n**, the vector $\mathbf{S}_0 = \mathbf{E}_0 \times \mathbf{H}_0$ has no normal component. Therefore, the surface integral is zero.

$$-\frac{d}{dt}\int_{\Omega}\left(\frac{1}{2}\mu\left|\mathbf{H}_0\right|^2+\frac{1}{2}\varepsilon\left|\mathbf{E}_0\right|^2\right)d\Omega=\int_{\Omega}\frac{\left|\mathbf{J}_0\right|^2}{\gamma}\,d\Omega$$

The right hand side is obviously non-negative, therefore the quantity whose negative time derivative is written in the left hand side can never increase. The initial conditions, however, fix it as zero at the time instant t_0 and, since it is obviously non-negative, it can never decrease either. Therefore it is always zero:

$$\int_{\Omega}\left(\frac{1}{2}\mu\left|\mathbf{H}_0\right|^2+\frac{1}{2}\varepsilon\left|\mathbf{E}_0\right|^2\right)d\Omega=0.$$

Therefore, $\mathbf{E}_0=\mathbf{0},\ \mathbf{H}_0=\mathbf{0},$ i.e. $\mathbf{E}'=\mathbf{E}''$ und $\mathbf{H}'=\mathbf{H}''$.

q. e. d.

2. Static fields

Maxwell's equations $\left(\frac{\partial}{\partial t}=0\right)$:

$$curl\mathbf{H}=\mathbf{J},$$
$$curl\mathbf{E}=\mathbf{0},$$
$$div\mathbf{B}=0,$$
$$div\mathbf{D}=\rho,$$
$$div\mathbf{J}=0.$$

electrostatic field:

$$curl\mathbf{E}=\mathbf{0},$$
$$div\mathbf{D}=\rho,$$
$$\mathbf{D}=\varepsilon\mathbf{E}.$$

magnetostatic field:

$$curl\mathbf{H}=\mathbf{J},$$
$$div\mathbf{B}=0,$$
$$\mathbf{B}=\mu\mathbf{H}.$$

static current field:

$$curl\mathbf{E}=\mathbf{0},$$
$$div\mathbf{J}=0,$$
$$\mathbf{J}=\gamma(\mathbf{E}+\mathbf{E}_e).$$

2.1 Boundary value problems for the scalar potential

Electrostatic field and static current field:

$curl\mathbf{E} = \mathbf{0} \Rightarrow \mathbf{E} = -gradV$, V: electric scalar potential

Magnetostatic field, if **J**=**0**:

$curl\mathbf{H} = 0 \Rightarrow \mathbf{H} = -grad\psi$, ψ: magnetic scalar potential

Differential equations:

$$div\mathbf{D} = \rho \Rightarrow -div(\varepsilon gradV) = \rho,$$

$$div\mathbf{B} = 0 \Rightarrow -div(\mu grad\psi) = 0,$$

$$div\mathbf{J} = 0 \Rightarrow -div(\gamma gradV) = 0,$$

generalized Laplace-Poisson and Laplace equation.

The solution of the Laplace-Poisson equation in unbounded free space ($\varepsilon = \varepsilon_0$):

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \frac{\rho(\mathbf{r}')d\Omega'}{|\mathbf{r} - \mathbf{r}'|},\ V(\mathbf{r} \to \infty) = 0.$$

In regions free of charges, the electrostatic field is also described by the generalized Laplace equation:

$$-div(\varepsilon grad V) = 0.$$

Boundary conditions:

Dirichlet boundary condition: $V = V_0$ (known) on Γ_D,

$\psi = \psi_0$ (known) on Γ_D.

This means the prescription of $\mathbf{E_t}$ or $\mathbf{H_t}$.

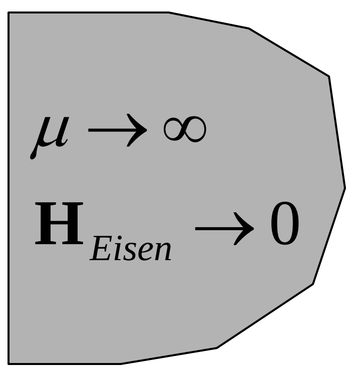
Γ_D is typically constituted by electrodes in case of electrostatic and static current field. Indeed:

$$\mathbf{E_t} = \mathbf{0} \Rightarrow V = \text{konstant.}$$

Magnetostatic field: interface to highly permeable regions, to magnetic walls ($\mu \rightarrow \infty$).

$\mu = \mu_0$

$\mathbf{H_t} = \mathbf{0}$



since $\mathbf{H_t} = \mathbf{H}_{\mathbf{t}Eisen}$

Neumann boundary condition:

$$\varepsilon \frac{\partial V}{\partial n} = \sigma \text{ (known) on } \Gamma_N,$$

$$\gamma \frac{\partial V}{\partial n} = 0 \text{ on } \Gamma_N,$$

$$\mu \frac{\partial \psi}{\partial n} = b \text{ (known) on } \Gamma_N.$$

This means the prescription of D_n, J_n or B_n.

In case of static current field, Γ_N is the interface to the non-conducting region.

If σ=0 in the electrostatic case and b=0 in the magnetostatic case, Γ_N is a surface parallel to the field lines. Otherwise, D_n, B_n is known on Γ_N.

Boundary value problems:

electrostatic field:

$$-div(\varepsilon\, grad V) = \rho \text{ in } \Omega,$$

$$V = V_0 \text{ on } \Gamma_D,\ \varepsilon \frac{\partial V}{\partial n} = \sigma \text{ on } \Gamma_N.$$

magnetostatic field:

$$-div(\mu\, grad \psi) = 0 \text{ in } \Omega,$$

$$\psi = \psi_0 \text{ on } \Gamma_D,\ \mu \frac{\partial \psi}{\partial n} = b \text{ on } \Gamma_N.$$

static current field:

$$-div(\gamma\, grad V) = 0 \text{ in } \Omega,$$

$$V = V_0 \text{ on } \Gamma_D,\ \gamma \frac{\partial V}{\partial n} = 0 \text{ on } \Gamma_N.$$

Uniqueness of the solution of the boundary value problem:

Due to the analogy, it is sufficient to treat the electrostatic case.

Proof:

Assumption: two solutions V' und V'' exist.

Their difference: $V = V' - V''$ satisfies the boundary value problem $-div(\varepsilon\, grad V) = 0$ in Ω,

$$V = 0 \text{ on } \Gamma_D, \; \varepsilon \frac{\partial V}{\partial n} = 0 \text{ on } \Gamma_N.$$

Identity:

$$div(V \varepsilon\, grad V) = \varepsilon \left| grad V \right|^2 + V div(\varepsilon\, grad V).$$

Integration over a domain Ω with boundary Γ:

$$\int_{\Omega} \varepsilon \left| gradV \right|^2 d\Omega = -\int_{\Omega} V div(\varepsilon gradV) d\Omega + \oint_{\Gamma} V \varepsilon gradV \cdot \mathbf{n} d\Gamma.$$

According to the boundary value problems for *V*, the right hand side is zero $(\varepsilon gradV \cdot \mathbf{n} = \varepsilon \frac{\partial V}{\partial n})$.

Hence $gradV = \mathbf{0} \Rightarrow V = \text{constant}$ in Ω.

Since, however $V = 0$ on Γ_D, $V = 0$ in $\Omega \Rightarrow V' = V''$.

q. e. d.

2.2 Analytic methods for solving the Laplace equation

Laplace equation: $$divgradV = \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

$$(divgrad\,\psi = 0).$$

electrostatic field: $\rho = 0,\ \varepsilon =$ constant $(=\varepsilon_0)$,

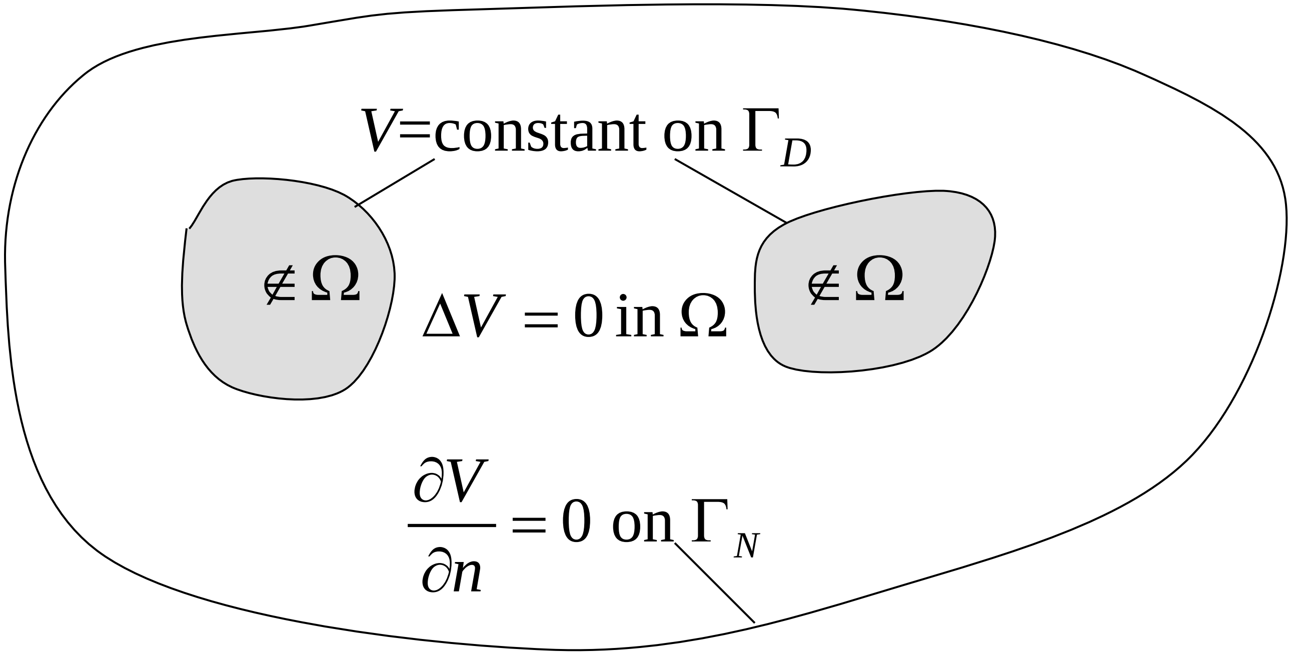
magnetostatic field: $\mathbf{J} = \mathbf{0},\ \mu =$ constant $(=\mu_0)$,

static current field: $\gamma =$ constant.

$\Delta V = 0 \Rightarrow V$ is a harmonic function.

There exist infinitely many harmonic functions!

Dirichlet or Neumann boundary conditions :



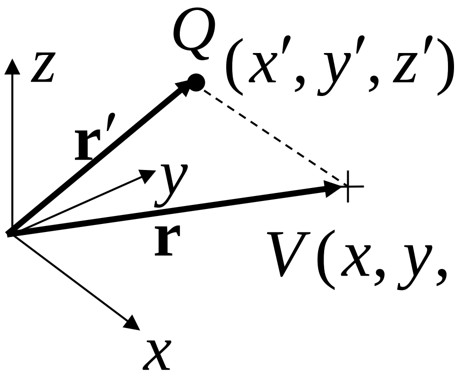
Analytic methods:

Generation of harmonic functions and selection of the one satisfying the boundary conditions.

This yields the true solution, since that is unique.

2.2.1 Method of fictitious charges (method of images)

The potential function of a point charge is harmonic in all points in space except in the point where the charge is located.



$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z,\ \mathbf{r}' = x'\mathbf{e}_x + y'\mathbf{e}_y + z'\mathbf{e}_z$$

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-\frac{1}{2}}$$

Proof:

$$\frac{\partial V}{\partial x} = -\frac{Q}{4\pi\varepsilon_0}(x-x')\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-\frac{3}{2}} =$$

$$= -\frac{Q}{4\pi\varepsilon_0}(x-x')|\mathbf{r}-\mathbf{r}'|^{-3}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{Q}{4\pi\varepsilon_0}\left[|\mathbf{r}-\mathbf{r}'|^{-3} - 3(x-x')^2|\mathbf{r}-\mathbf{r}'|^{-5}\right]$$

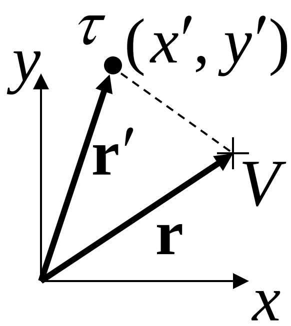
$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{|\mathbf{r}-\mathbf{r}'|^3} - 3\frac{(x-x')^2+(y-y')^2+(z-z')^2}{|\mathbf{r}-\mathbf{r}'|^5}\right] =$$

$$= -\frac{Q}{4\pi\varepsilon_0}3\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{|\mathbf{r}-\mathbf{r}'|^2}{|\mathbf{r}-\mathbf{r}'|^5}\right] = 0, \text{ if } \mathbf{r} \neq \mathbf{r}'.$$

q. e. d.

In two dimensions (planar problems, $\frac{\partial}{\partial z} = 0$),

the potential function of an infinitely long line charge is harmonic.



$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y, \ \mathbf{r}' = x'\mathbf{e}_x + y'\mathbf{e}_y$$

$$V(x,y) = \frac{\tau}{2\pi\varepsilon_0} \ln \frac{r_0}{\left[(x-x')^2 + (y-y')^2\right]^{\frac{1}{2}}}$$

$$V(x,y) = \frac{\tau}{2\pi\varepsilon_0} \left\{ \ln r_0 - \frac{1}{2} \ln\left[(x-x')^2 + (y-y')^2\right] \right\}$$

$$\frac{\partial V}{\partial x} = -\frac{\tau}{2\pi\varepsilon_0}\frac{x - x'}{(x - x')^2 + (y - y')^2}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\tau}{2\pi\varepsilon_0}\frac{(x - x')^2 + (y - y')^2 - 2(x - x')^2}{\left[(x - x')^2 + (y - y')^2\right]^2} =$$

$$= \frac{\tau}{2\pi\varepsilon_0}\frac{(x - x')^2 - (y - y')^2}{\left[(x - x')^2 + (y - y')^2\right]^2}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\tau}{2\pi\varepsilon_0}\frac{(x - x')^2 - (y - y')^2 + (y - y')^2 - (x - x')^2}{\left[(x - x')^2 + (y - y')^2\right]^2} = 0$$

q. e. d.

Due to its linearity, the Laplace equation is satisfied by the potential function of an *arbitrary charge distribution* (in regions free of charges).

Satisfaction of the boundary conditions: a fictitious charge distribution within the electrodes should have equipotential surfaces which coincide with the electrodes.

Examples:

- *One point charge:* concentric spherical surfaces (spherical capacitor)
- *One infinitely long line charge:* concentric cylindrical surfaces (cylindrical capacitor)
- *Line dipole:* non-concentric cylindrical surfaces
- *Mirror image over a plane:* infinite conducting plane
- *Multiple mirror images over planes:* planar electrodes forming an angle ($\alpha = 180 / n$)

2.2.2 Separation of variables

Laplacian in a general orthogonal coordinate system:

$$\Delta u = div(gradu)$$

$$\Delta u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial u}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right]$$

$$\Delta u(x,y,z) = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$\Delta u(r,\phi,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}$$

$$\Delta u(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

Method of separation:

Assumption: $V(x_1, x_2, x_3) = X_1(x_1) X_2(x_2) X_3(x_3)$

It is attempted to reduce the Laplace equation (a partial differential equation) for *V* to three ordinary differential equations for the functions X_1, X_2 and X_3. The general solutions of these equations yield special solutions of the Laplace equation by means of the above assumption. Due to the linearity of the Laplace equation, it is satisfied by any linear combination of these solutions. The solution of a particular boundary value problem is the linear combination which satisfies the boundary conditions of the problem. This is only possible if the boundary conditions are specified along surfaces x_i = constant.

Cartesian coordinates, 2D case:

Assumption: $V(x, y) = X(x)Y(y)$

Laplace equation: $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

$$Y(y)\frac{d^2 X(x)}{dx^2} + X(x)\frac{d^2 Y(y)}{dy^2} = 0$$

Division by $X(x)Y(y)$ yields:

$$\underbrace{\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2}}_{f(x)} + \underbrace{\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2}}_{g(y)} = 0$$

This is only possible if

$f(x) = \text{constant}, \; g(y) = \text{constant}.$

Hence, the ordinary differential equations are:

$$\frac{d^2X(x)}{dx^2} = fX(x), \quad \frac{d^2Y(y)}{dy^2} = gY(y),$$

where $g = -f$.

$$f = -p^2,\ g = p^2: \quad X(x) = C_1 \cos px + C_2 \sin px$$

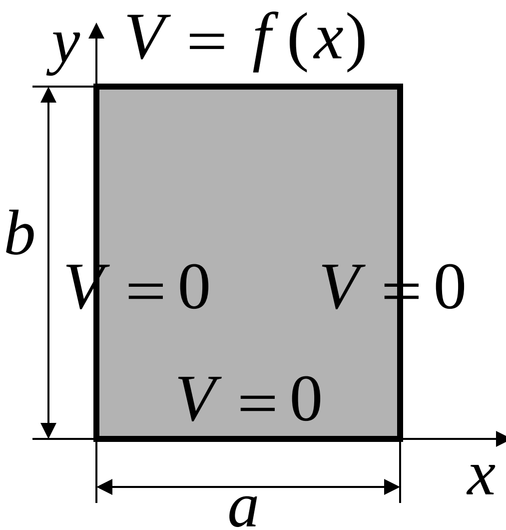
$$Y(y) = C_3 e^{py} + C_4 e^{-py} = C_3' \cosh py + C_4' \sinh py$$

One solution: $$\boxed{V = \sum_n A_n^{\pm} e^{\pm p_n y} \cos p_n x + B_n^{\pm} e^{\pm p_n y} \sin p_n x}$$

A further solution is obtained by interchanging *x* and *y*:

$$\boxed{V = \sum_n A_n^{\pm} e^{\pm p_n x} \cos p_n y + B_n^{\pm} e^{\pm p_n x} \sin p_n y}$$

Example:



boundary conditions:

$$(1)\quad y=b,\quad V=f(x);$$

$$(2)\quad y=0,\quad V=0; \quad \Rightarrow \sinh p_n y,$$

$$\left.\begin{aligned}&(3)\quad x=0,\quad V=0;\\&(4)\quad x=a,\quad V=0.\end{aligned}\right\}\Rightarrow \sin p_n x,\ p_n = n\pi/a,\quad n=1,2,\dots.$$

$$V(x,y)=\sum_{n=1}^{\infty}B_n\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \therefore \text{(2), (3), (4) satisfied.}$$

Fourier series of $f(x)$: $$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{a}\right).$$

$$V(x,b)=\sum_{n=1}^{\infty}B_n\sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\Rightarrow B_n\sinh\left(\frac{n\pi b}{a}\right)=b_n$$

$$V(x,y)=\sum_{n=1}^{\infty}b_n\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right).$$

Cylindrical coordinates, 2D case:

Assumption: $V(r,\phi) = R(r)\Phi(\phi)$

Laplace equation: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

$$\Phi(\phi)\frac{1}{r}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + R(r)\frac{1}{r^2}\frac{d^2\Phi(\phi)}{d\phi^2} = 0.$$

Multiplication by r^2 and division by $R(r)\Phi(\phi)$ yield:

$$\underbrace{\frac{1}{R(r)}r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)}_{f(r)} + \underbrace{\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}}_{g(\phi)} = 0.$$

This is only possible if

$f(r) = \text{constant},\ g(\phi) = \text{constant}.$

Hence, the ordinary differential equations are:

$$r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = fR(r), \quad \frac{d^2\Phi(\phi)}{d\phi^2} = g\Phi(\phi),$$

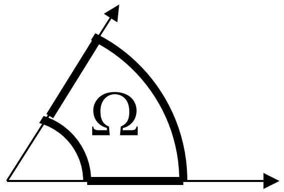
where $g = -f$.

Since $\Phi(\phi)$ must be periodic with the period 2π, the only possibility is $g = -n^2$ (n: integer)

$$\Phi(\phi) = C_3 \cos n\phi + C_4 \sin n\phi \; (n > 0).$$

The case $n = 0$ yields $\Phi(\phi) = C_3 + C_4\phi$.

This is only periodic if $C_4 = 0$. The case $C_4 \neq 0$, makes sense only if the problem region is $0 \leq \phi \leq \phi_{\max}$ where $\phi_{\max} < 2\pi$.



$$r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)=n^2R(r).$$

a) $n=0$: $r\dfrac{dR(r)}{dr}=C_1,\ R(r)=C_1\ln r+C_2.$

b) $n>0$: Assumption: $R(r)=r^{\alpha}$,

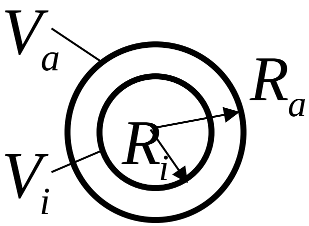
$$\frac{dR(r)}{dr}=\alpha r^{\alpha-1},\quad r\frac{dR(r)}{dr}=\alpha r^{\alpha},\quad \frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)=\alpha^2r^{\alpha-1},$$

$$\alpha^2r^{\alpha}=n^2r^{\alpha}\Rightarrow\ \alpha=\pm n:\quad R(r)=C_1r^n+C_2r^{-n}$$

Solution:

$$\boxed{V=\left(C_1\ln r+C_2\right)\left(C_3+C_4\phi\right)+\sum_n\left(A_n^{\pm}r^{\pm n}\cos n\phi+B_n^{\pm}r^{\pm n}\sin n\phi\right)}$$

Example: cylindrical capacitor



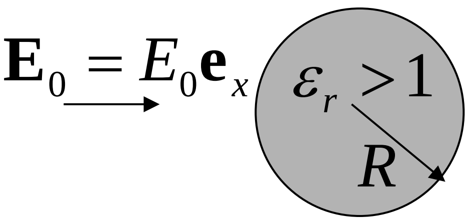
Axisymmetry: $\frac{\partial}{\partial \phi} = 0 \Rightarrow V = C_1 \ln r + C_2 .$

boundary conditions:
$$V_i = C_1 \ln R_i + C_2 ,$$
$$V_a = C_1 \ln R_a + C_2 .$$
$$C_1 = \frac{V_a - V_i}{\ln R_a / R_i}, \; C_2 = \frac{V_i \ln R_a - V_a \ln R_i}{\ln R_a / R_i} .$$

$$V = \frac{V_a \ln r/R_i - V_i \ln r/R_a}{\ln R_a / R_i} .$$

The same solution is obtained by the method of fictitious charges.

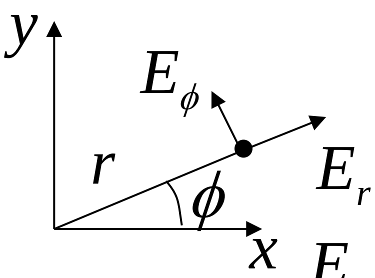
Example: dielectric cylinder in a homogeneous field



homogeneous field: $V = -E_0 r \cos\phi.$

Proof:

$$E_r = -\frac{\partial V}{\partial r} = E_0 \cos\phi, \quad E_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = -E_0 \sin\phi.$$



$$E_x = E_r \cos\phi - E_\phi \sin\phi = E_0 \cos^2\phi + E_0 \sin^2\phi = E_0,$$

$$E_y = E_r \sin\phi + E_\phi \cos\phi = E_0 \cos\phi \sin\phi - E_0 \sin\phi \cos\phi = 0.$$

q. e. d.

$$V(r,\phi) = \begin{cases} V_i(r,\phi), \text{ wenn } r < R, \\ V_a(r,\phi), \text{ wenn } r > R. \end{cases}$$

boundary conditions:

$$\lim_{r\to 0} V_i(r,\phi) < \infty \Rightarrow V_i = \sum_{n\geq 1}\left(A_{in} r^n \cos n\phi + B_{in} r^n \sin n\phi\right),$$

$$\lim_{r\to\infty} V_a(r,\phi) = -E_0 r \cos\phi \Rightarrow$$

$$V_a = -E_0 r \cos\phi + \sum_{n\geq 1}\left(A_{an} r^{-n} \cos n\phi + B_{an} r^{-n} \sin n\phi\right).$$

Boundary and interface conditions:

$$E_{\phi i}(r=R)=E_{\phi a}(r=R)\Rightarrow V_i(R,\phi)=V_a(R,\phi)\Rightarrow$$

$$A_{i1}R=-E_0R+A_{a1}R^{-1},\ A_{in}R^n=A_{an}R^{-n}\ (n>1),\ B_{in}R^n=B_{an}R^{-n}$$

$$D_{ri}(r=R)=D_{ra}(r=R)\Rightarrow\varepsilon_r\frac{\partial V_i(R,\phi)}{\partial r}=\frac{\partial V_a(R,\phi)}{\partial r}\Rightarrow$$

$$\varepsilon_r A_{i1}=-E_0-A_{a1}R^{-2},\ \varepsilon_r A_{in}nR^{n-1}=-A_{an}nR^{-n-1}\ (n>1),$$

$$\varepsilon_r B_{in}nR^{n-1}=-B_{an}nR^{-n-1}.$$

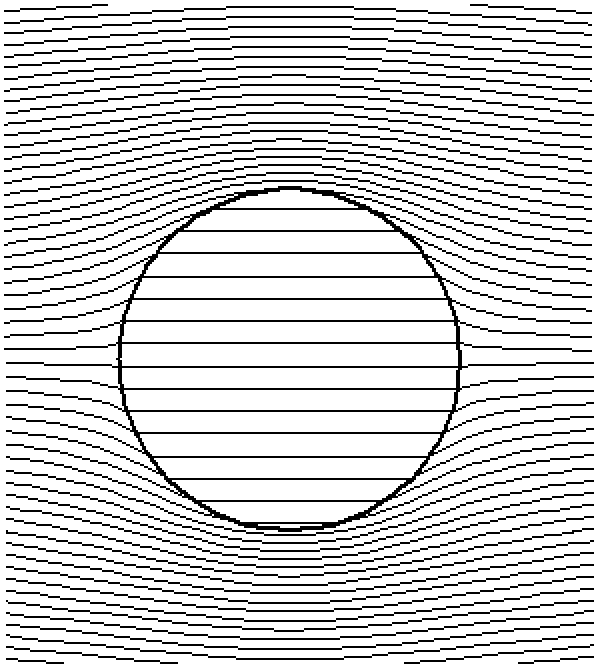
Solution: $$A_{i1}=\frac{-2}{\varepsilon_r+1}E_0,\ A_{a1}=\frac{\varepsilon_r-1}{\varepsilon_r+1}E_0R^2,$$

$$A_{in}=A_{an}=0\quad (n>1),\ \ B_{in}=B_{an}=0.$$

$$V_i(r,\phi) = \frac{-2E_0}{\varepsilon_r + 1} r\cos\phi,$$

$$V_a(r,\phi) = -E_0 r\cos\phi + \frac{\varepsilon_r - 1}{\varepsilon_r + 1} E_0 \frac{R^2}{r}\cos\phi.$$

The field within the cylinder is homogeneous.

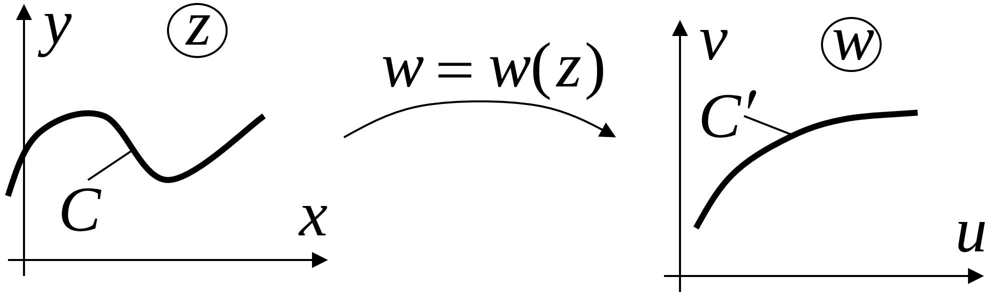


2.2.3 Conformal mapping

Consider an arbitrary regular complex function:

$$w(z) = u(x, y) + jv(x, y), \quad z = x + jy.$$

It realizes a mapping of the *x*-*y* plane to the *u*-*v* plane:



This mapping is conformal, i.e. it preserves angles locally.

Regular complex functions have the following property:

$$w'(z) = \lim_{\Delta x \to 0} \frac{w(z+\Delta x) - w(z)}{\Delta x} = \lim_{\Delta y \to 0} \frac{w(z + j\Delta y) - w(z)}{j\Delta y},$$

$$\lim_{\Delta x \to 0}\left(\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + j\,\frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right) =$$

$$= \lim_{\Delta y \to 0}\left(\frac{u(x, y+\Delta y) - u(x, y)}{j\Delta y} + j\,\frac{v(x, y+\Delta y) - v(x, y)}{j\Delta y} \right),$$

$$\frac{\partial u}{\partial x} + j\,\frac{\partial v}{\partial x} = -j\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

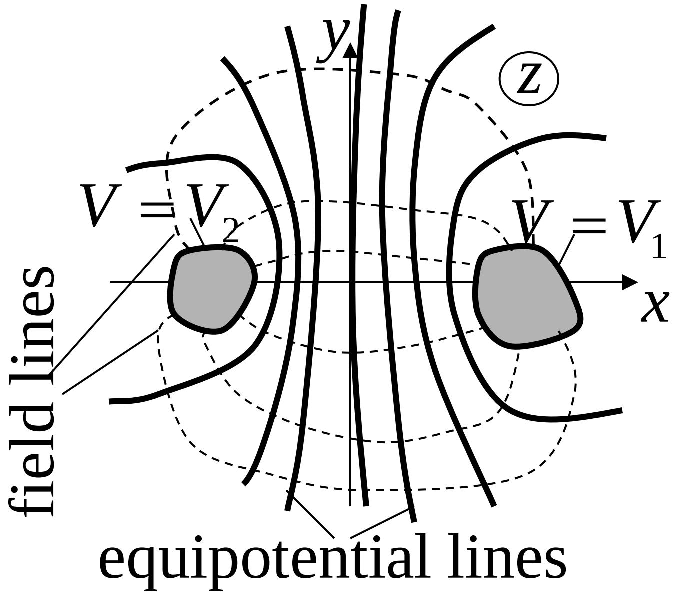
Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$

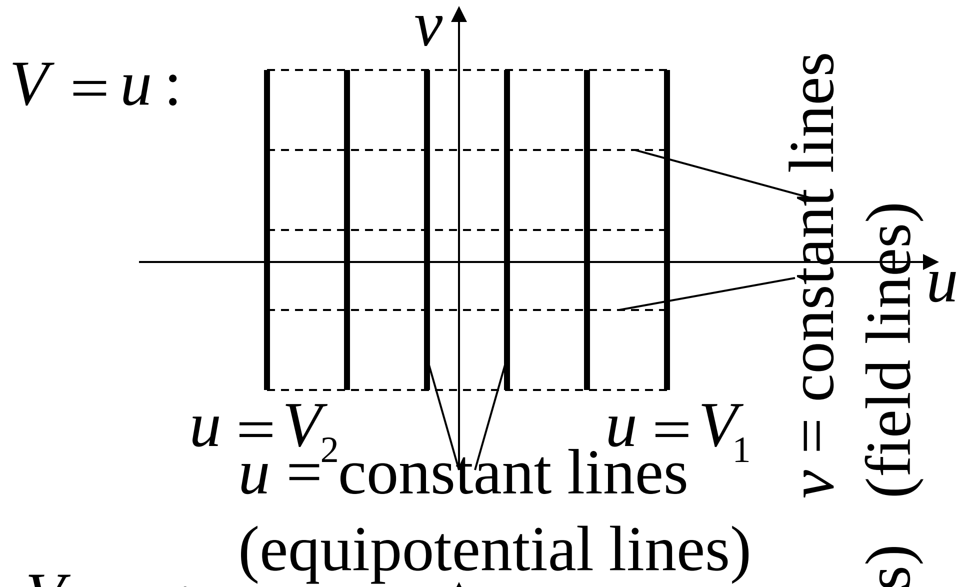
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,}$$

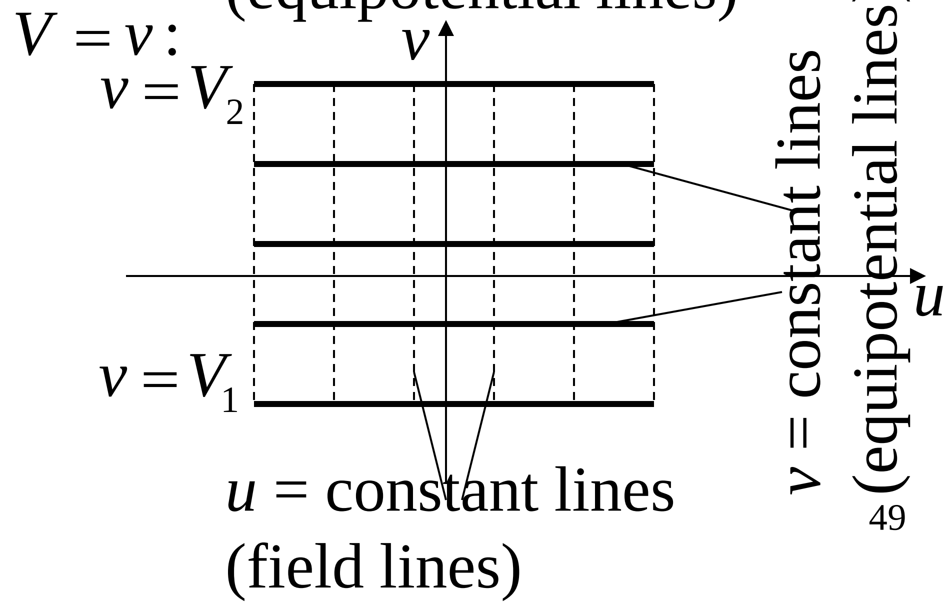
$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.}$$

Both the real part and the imaginary part of a regular complex function satisfy the two-dimensional Laplace equation.

The real part or the imaginary part of a regular complex function is the solution of an electrostatic problem if it is constant along the electrodes.

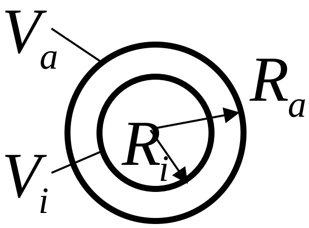






In general: $C_1 w(z) + C_2$.

Example: cylindrical capacitor



$$w(z) = \ln z = \ln re^{j\phi} = \ln r + j\phi,$$

$$u = \ln r = \ln\sqrt{x^2 + y^2}, \quad v = \phi = \arctan\frac{y}{x}.$$

$u = \text{constant} \Rightarrow r = \text{constant}$: circles.

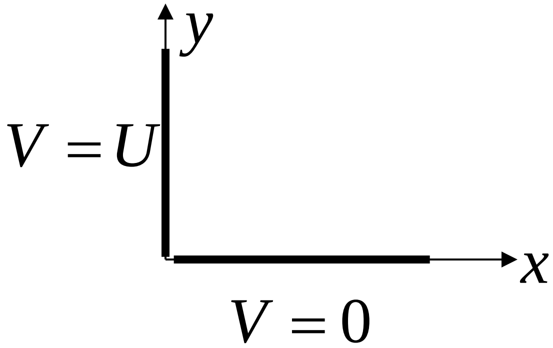
The boundary conditions can be satisfied if $w(z) = C_1 \ln z + C_2$.

$V = u = C_1 \ln r + C_2$. Using the boundary conditions:

$$V = \frac{V_a \ln r/R_i - V_i \ln r/R_a}{\ln R_a/R_i}.$$

The same solution is obtained by the method of fictitious charges or the method of separation of variables.

Example: Two planes normal to each other



$$w(z) = \ln z = \ln re^{j\phi} = \ln r + j\phi,$$

$$u = \ln r = \ln\sqrt{x^2 + y^2}, \quad v = \phi = \arctan\frac{y}{x}.$$

$v = \text{constant} \Rightarrow \varphi = \text{constant}$: radial lines.

The boundary conditions can be satisfied if $w(z) = C_1 \ln z + C_2$.

$$V = v = C_1\phi + C_2.$$

Using the boundary conditions: $V = \dfrac{2U}{\pi}\phi = \dfrac{2U}{\pi}\arctan\dfrac{y}{x}.$

2.3 Numerical methods for solving the boundary value problems for the scalar potential

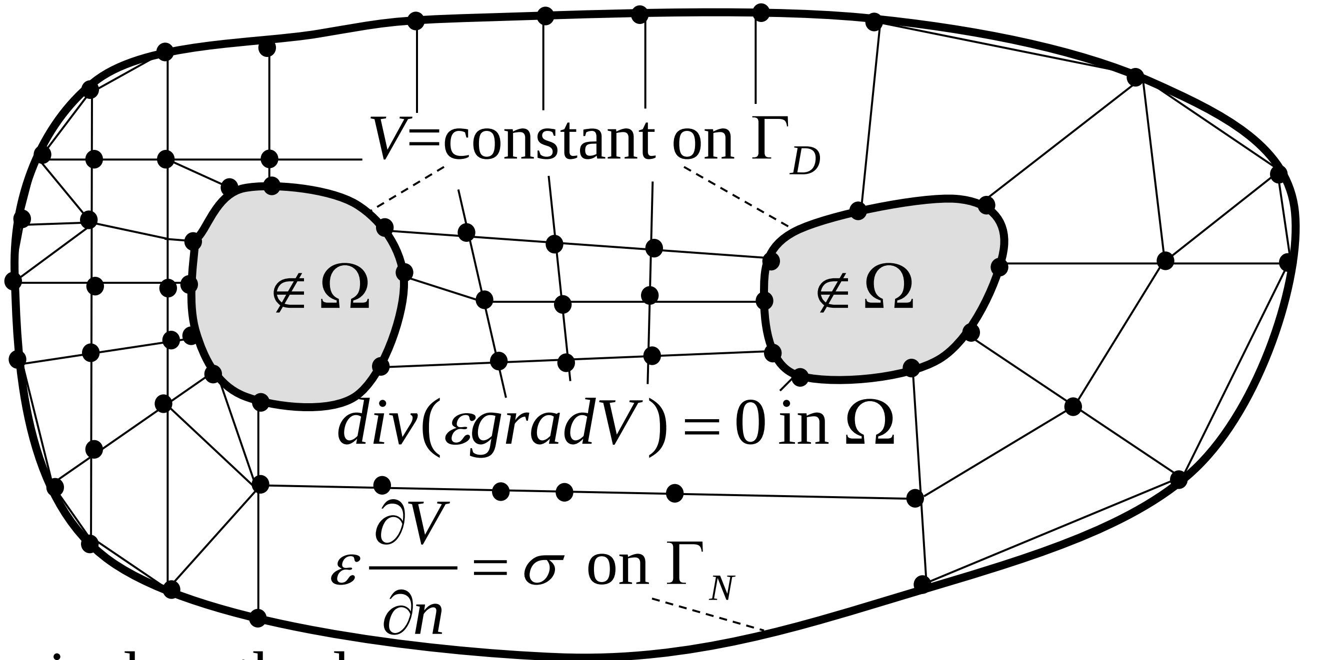
Disadvantages of analytical methods:

- special geometry
- homogeneous materials

Numerical methods:

- geometry discretized
- taking account of non-homogeneous materials

Discretization of geometry:



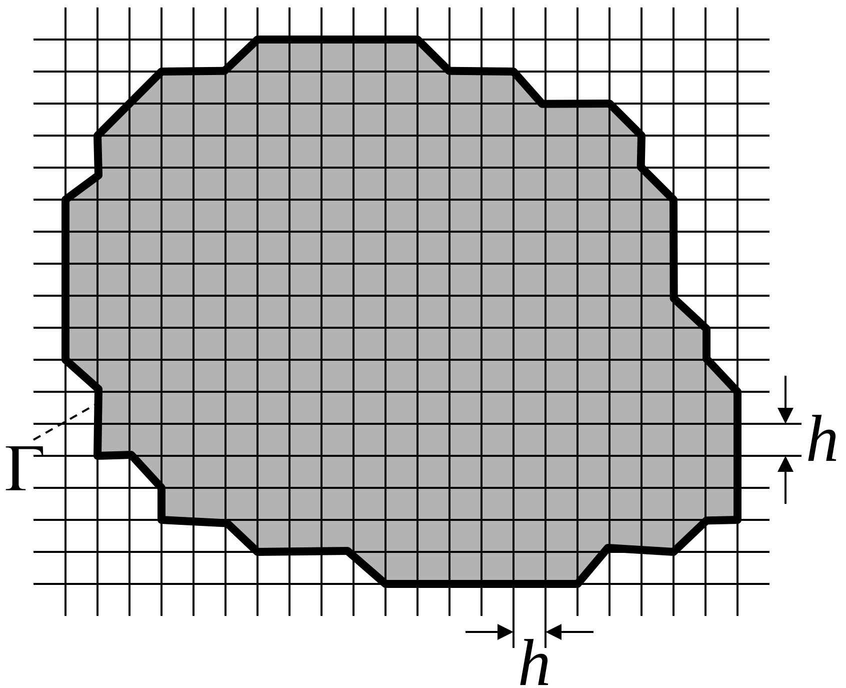
Numerical methods:

The potential is approximated in discrete nodes. The field is computed by numerical differentiation. The Dirichlet boundary conditions in the nodes are satisfied exactly, the Neumann boundary conditions can only be approximately fulfilled.

2.3.1 Method of finite differences

Two-dimensional planar problems are treated only, the generalization to 3D problems is straightforward.

Homogeneous materials are assumed: Laplace equation. Generalization for piecewise homogeneous materials is possible.



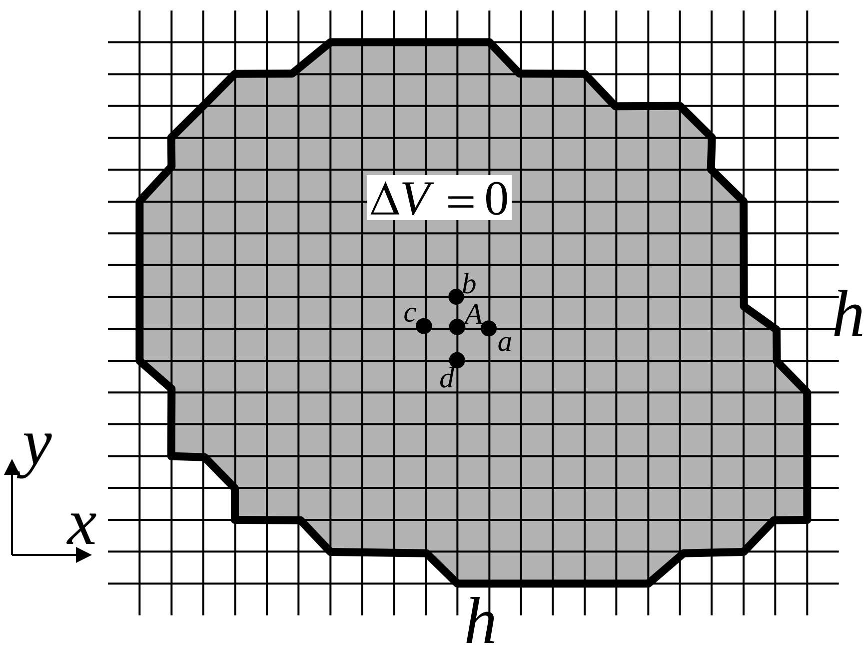
Uniform mesh, generalization for non-uniform mesh is possible.

$$V_a = V_A + \frac{1}{1!}\frac{\partial V}{\partial x}h + \frac{1}{2!}\frac{\partial^2 V}{\partial x^2}h^2 + \frac{1}{3!}\frac{\partial^3 V}{\partial x^3}h^3 + \frac{1}{4!}\frac{\partial^4 V}{\partial x^4}h^4 + \dots$$

$$V_b = V_A + \frac{1}{1!}\frac{\partial V}{\partial y}h + \frac{1}{2!}\frac{\partial^2 V}{\partial y^2}h^2 + \frac{1}{3!}\frac{\partial^3 V}{\partial y^3}h^3 + \frac{1}{4!}\frac{\partial^4 V}{\partial y^4}h^4 + \dots$$

$$V_c = V_A - \frac{1}{1!}\frac{\partial V}{\partial x}h + \frac{1}{2!}\frac{\partial^2 V}{\partial x^2}h^2 - \frac{1}{3!}\frac{\partial^3 V}{\partial x^3}h^3 + \frac{1}{4!}\frac{\partial^4 V}{\partial x^4}h^4 \pm \dots$$

$$+ \quad V_d = V_A - \frac{1}{1!}\frac{\partial V}{\partial y}h + \frac{1}{2!}\frac{\partial^2 V}{\partial y^2}h^2 - \frac{1}{3!}\frac{\partial^3 V}{\partial y^3}h^3 + \frac{1}{4!}\frac{\partial^4 V}{\partial y^4}h^4 \pm \dots$$



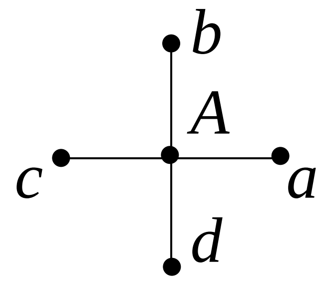
$$V_a + V_b + V_c + V_d = 4V_A +$$

$$+ h^2 \underbrace{\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)}_{=0} + \underbrace{O(h^4)}_{\approx 0}$$

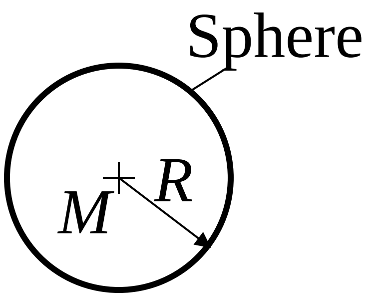
$$4V_A - V_a - V_b - V_c - V_d = 0$$

The equation $4V_A - V_a - V_b - V_c - V_d = 0$, i.e.

$$V_A = \frac{1}{4}\left(V_a + V_b + V_c + V_d\right)$$



corresponds approximately to the mean value theorem of potential theory (see 2.4.1):

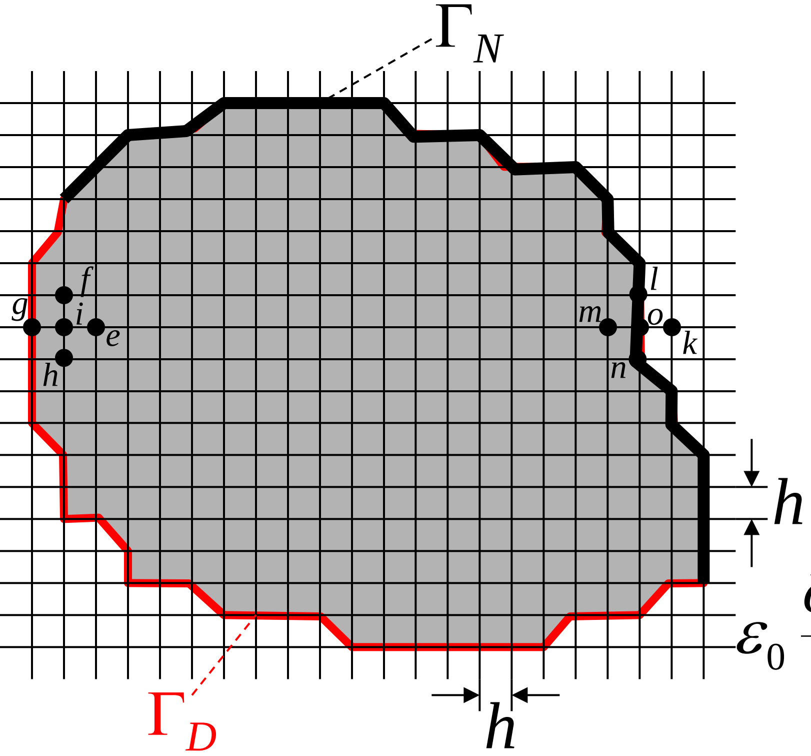


$$V(M) = \frac{1}{4\pi R^2} \oint_{Sphere} V d\Gamma$$

Taking account of boundary conditions:

Dirichlet boundary condition on Γ_D: $V_g = V_0$ (known).

$$4V_i - V_e - V_f - V_g - V_h = 0 \Rightarrow \boxed{4V_i - V_e - V_f - V_h = V_0}$$



Neumann bondary condition on Γ_N:

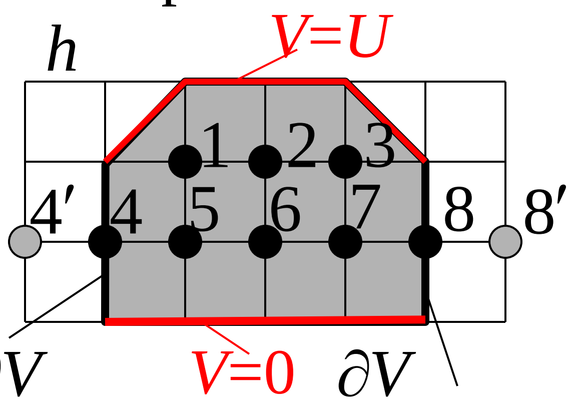
$$\varepsilon_0 \frac{\partial V}{\partial n} = \sigma \text{ at the point } o.$$

k: fictitious node outside of Ω

$$\varepsilon_0 \frac{\partial V}{\partial n}(o) \approx \varepsilon_0 \frac{V_k - V_m}{2h} = \sigma \Rightarrow V_k = V_m + \frac{2h\sigma}{\varepsilon_0}$$

$$4V_o - V_k - V_l - V_m - V_n = 0 \Rightarrow \boxed{4V_o - V_l - 2V_m - V_n = \frac{2h\sigma}{\varepsilon_0}}$$

Example:



$$\varepsilon_0 \frac{\partial V}{\partial n} = \sigma_1 \qquad \varepsilon_0 \frac{\partial V}{\partial n} = \sigma_2$$

$$\varepsilon_0 \frac{V_{4'} - V_5}{2h} = \sigma_1$$

$$\Downarrow$$

$$V_{4'} = V_5 + \frac{2h\sigma_1}{\varepsilon_0}$$

$$\varepsilon_0 \frac{V_{8'} - V_7}{2h} = \sigma_2$$

$$\Downarrow$$

$$V_{8'} = V_7 + \frac{2h\sigma_2}{\varepsilon_0}$$

node 1: $4V_1 - V_2 - V_5 = 2U$

node 2: $-V_1 + 4V_2 - V_3 - V_6 = U$

node 3: $-V_2 + 4V_3 - V_7 = 2U$

node 4: $-V_{4'} + 4V_4 - V_5 = U$

$$4V_4 - 2V_5 = U + \frac{2h\sigma_1}{\varepsilon_0}$$

node 5: $-V_1 - V_4 + 4V_5 - V_6 = 0$

node 6: $-V_2 - V_5 + 4V_6 - V_7 = 0$

node 7: $-V_3 - V_6 + 4V_7 - V_8 = 0$

node 8: $-V_7 + 4V_8 - V_{8'} = U$

$$-2V_7 + 4V_8 = U + \frac{2h\sigma_2}{\varepsilon_0}$$

Equations system in matrix form:

$$\begin{bmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{Bmatrix} = \begin{Bmatrix} 2U \\ U \\ 2U \\ U + 2h\sigma_1 / \varepsilon_0 \\ 0 \\ 0 \\ 0 \\ U + 2h\sigma_2 / \varepsilon_0 \end{Bmatrix}$$

Sparse matrix.

2.3.2 Variational problem of electrostatics

The boundary value problem of electrostatic field

$$-div(\varepsilon grad V) = \rho \text{ in } \Omega,$$

$$V = V_0 \text{ on } \Gamma_D, \ \varepsilon \frac{\partial V}{\partial n} = \sigma \text{ on } \Gamma_N$$

is equivalent to the following variational problem:

Find V (V=V_0 on Γ_D), so that the functional

$$W(V) = \int_\Omega \frac{1}{2} \varepsilon grad^2 V d\Omega - \int_\Omega \rho V d\Omega - \int_{\Gamma_N} \sigma V d\Gamma$$

attains a minimum.

Physical meaning of the functional

$$\int_{\Omega} \frac{1}{2} \varepsilon grad^2 V d\Omega = \int_{\Omega} \frac{1}{2} \mathbf{E} \cdot \mathbf{D} d\Omega : \text{ energy of electrical field: } W_e$$

$$\int_{\Omega} \rho V d\Omega + \int_{\Gamma_N} \sigma V d\Gamma : \text{ potential energy of the charges: } W_p$$

$W = W_e - W_p$ is the action!

The variational problem corresponds to the principle of least action.

2.3.3 Ritz's procedure

Since the variational problem and the boundary value problem are equivalent, an approximate solution of the variational problem is simultaneously an approximate solution of the boundary value problem.

The variational problem can be solved approximately by means of the *Ritz's procedure*.

An approximate solution is sought in the following form:

$$V \approx V^{(n)} = V_D + \sum_{j=1}^{n} V_j w_j,$$

V_D : arbitrary function with $V_D = V_0$ on Γ_D,

$V_j,\ j = 1, 2, ..., n$: numerical parameters,

$w_j,\ j = 1, 2, ..., n$: basis functions with

$$w_j = 0 \text{ on } \Gamma_D.$$

$V^{(n)}$ satisfies the Dirichlet boundary conditions for arbitrary V_j!

The unknown parameters V_j, $j = 1, 2, ..., n$ are determined from the condition that the approximate solution minimizes the functional. The necessary conditions are:

$$\frac{\partial W(V^{(n)})}{\partial V_i} = 0, \quad i = 1, 2, ..., n.$$

These are n equations (the Ritz equations system), allowing the determination of the n unknowns V_j, $j = 1, 2, ..., n$.

$$\frac{\partial W(V^{(n)})}{\partial V_i} = \frac{\partial}{\partial V_i}\int_\Omega \frac{1}{2}\varepsilon\, grad^2 V^{(n)} d\Omega - \frac{\partial}{\partial V_i}\int_\Omega \rho V^{(n)} d\Omega - \frac{\partial}{\partial V_i}\int_{\Gamma_N} \sigma V^{(n)} d\Gamma =$$

$$= \int_\Omega grad\,\frac{\partial V^{(n)}}{\partial V_i}\cdot \varepsilon\, grad V^{(n)} d\Omega - \int_\Omega \frac{\partial V^{(n)}}{\partial V_i}\rho\, d\Omega - \int_{\Gamma_N} \frac{\partial V^{(n)}}{\partial V_i}\sigma\, d\Gamma.$$

$$\frac{\partial V^{(n)}}{\partial V_i} = \frac{\partial}{\partial V_i}(V_D + \sum_{j=1}^{n} V_j w_j) = w_i.$$

The Ritz equations system:

$$\sum_{j=1}^{n} V_j \int_\Omega grad w_i \cdot \varepsilon\, grad w_j d\Omega = -\int_\Omega grad w_i \cdot \varepsilon\, grad V_D d\Omega + \int_\Omega w_i \rho\, d\Omega + \int_{\Gamma_N} w_i \sigma\, d\Gamma,$$

Symmetric matrix!

$$i = 1, 2, ..., n.$$

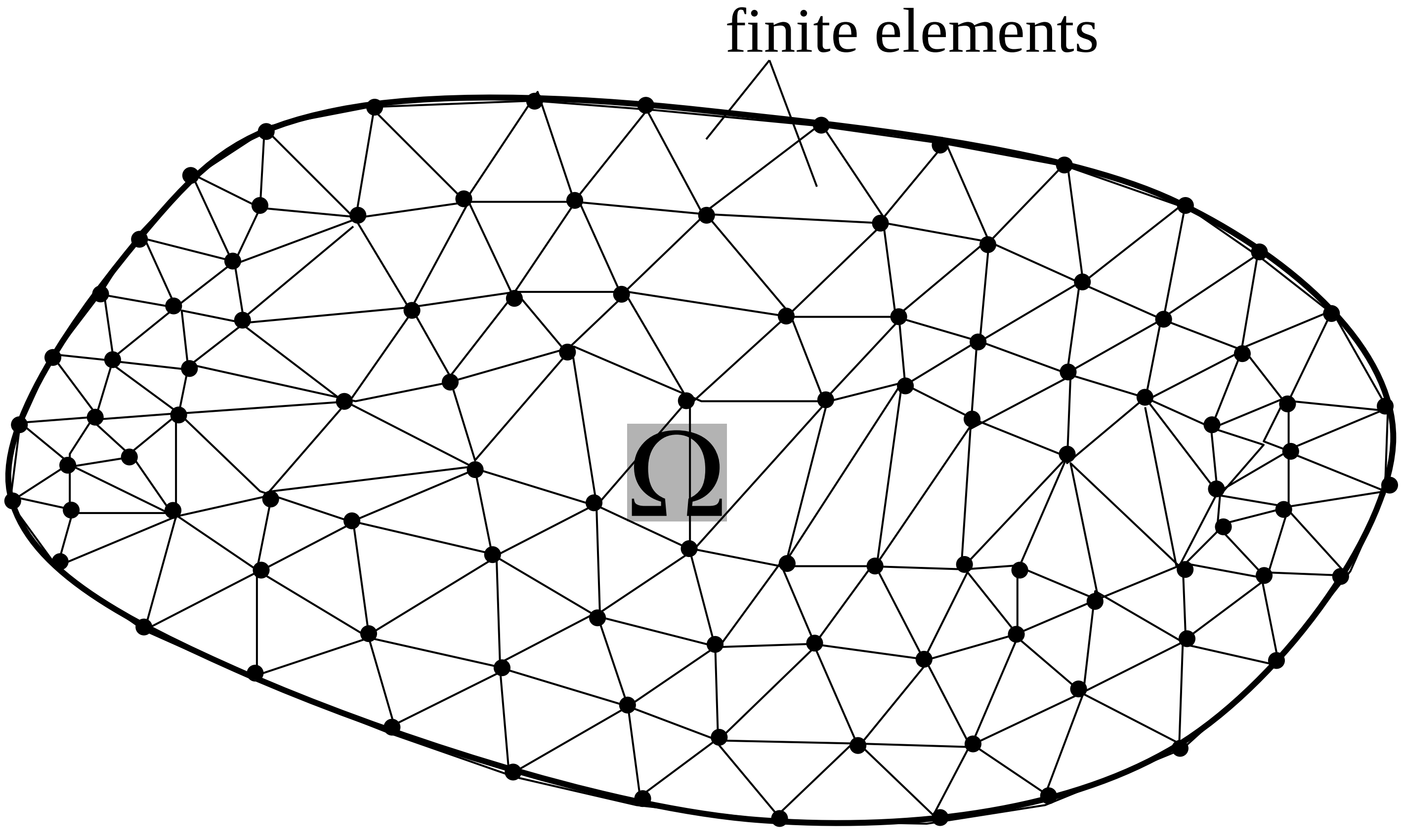
2.3.4 The method of finite elements (Finite Element Method=FEM)

Discretization of the geometry

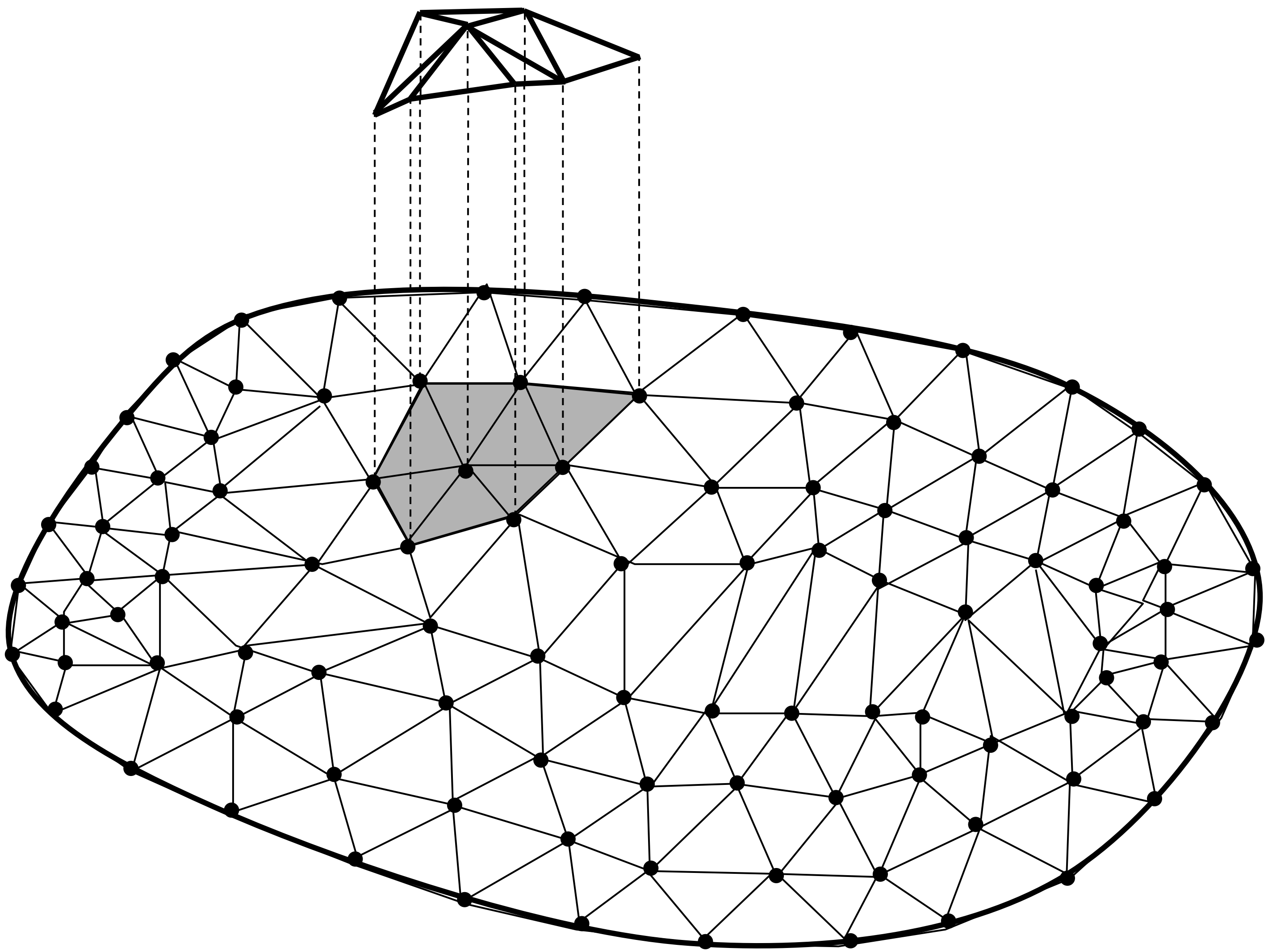
Triangular elements: simplest possible approach

Unknowns: potential values in nodes

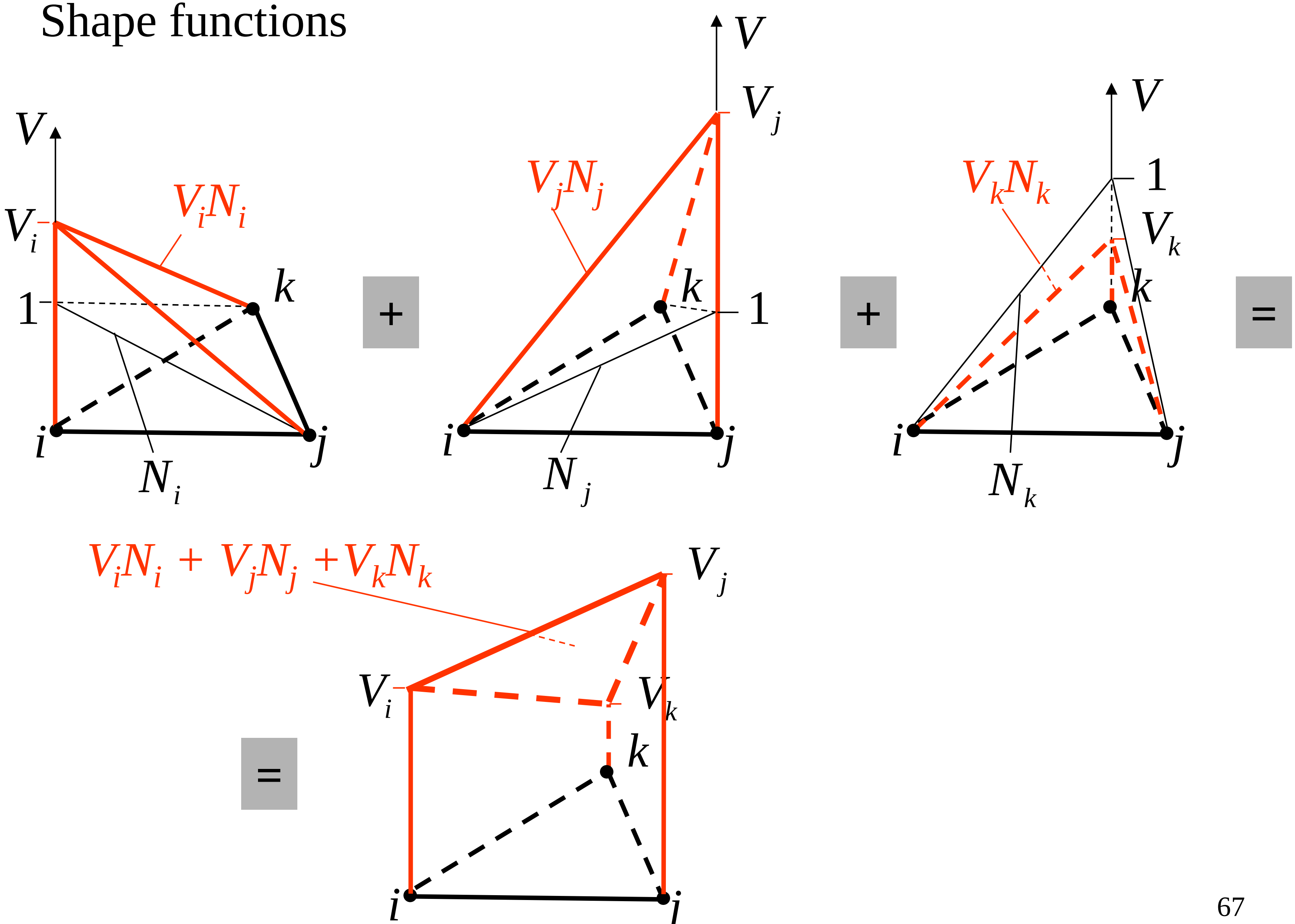
Potential in each element: low order polynomial

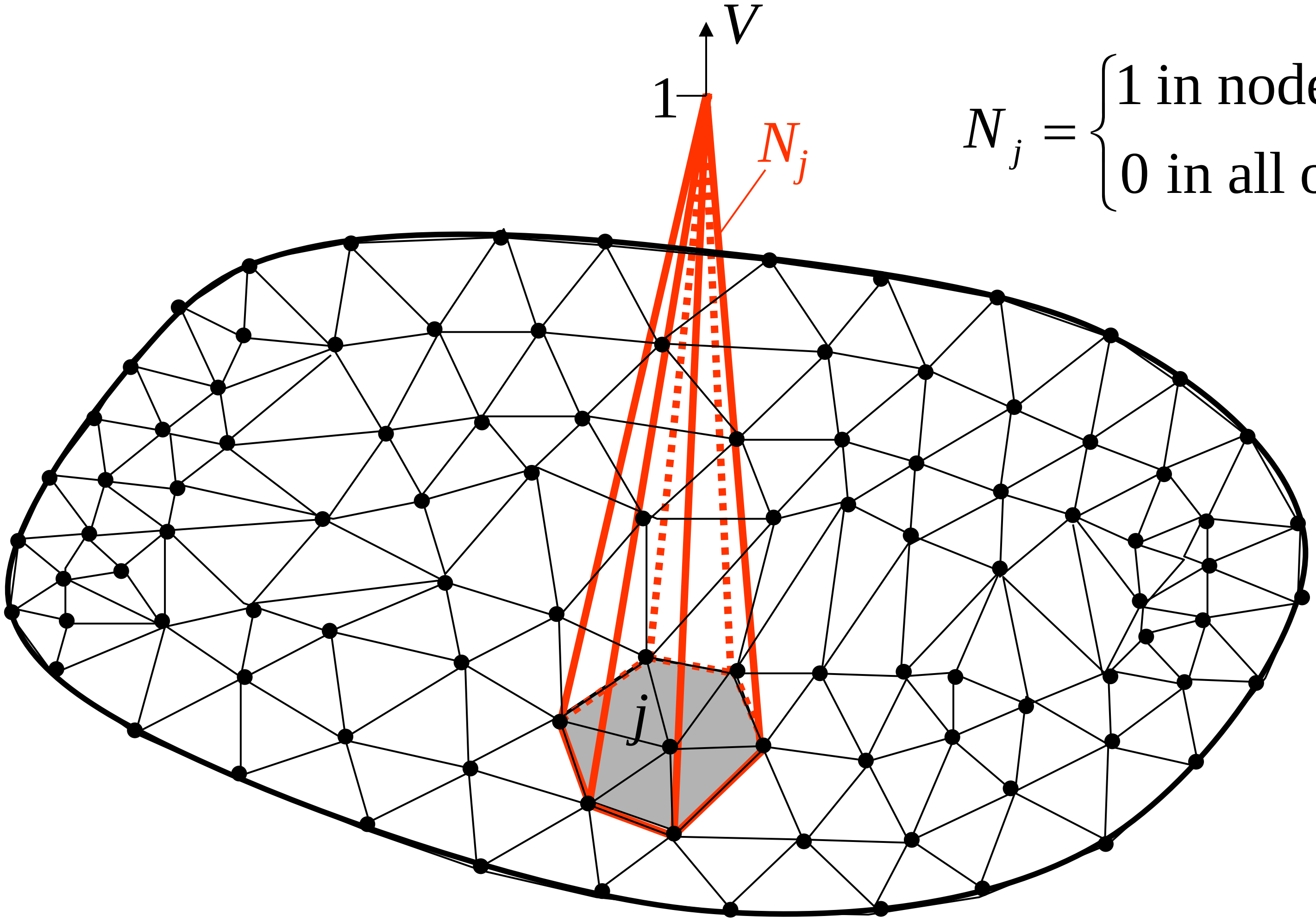


Linear interpolation of the potential function within elements



Shape functions





$$N_j = \begin{cases} 1 \text{ in node } j \\ 0 \text{ in all other nodes} \end{cases}$$

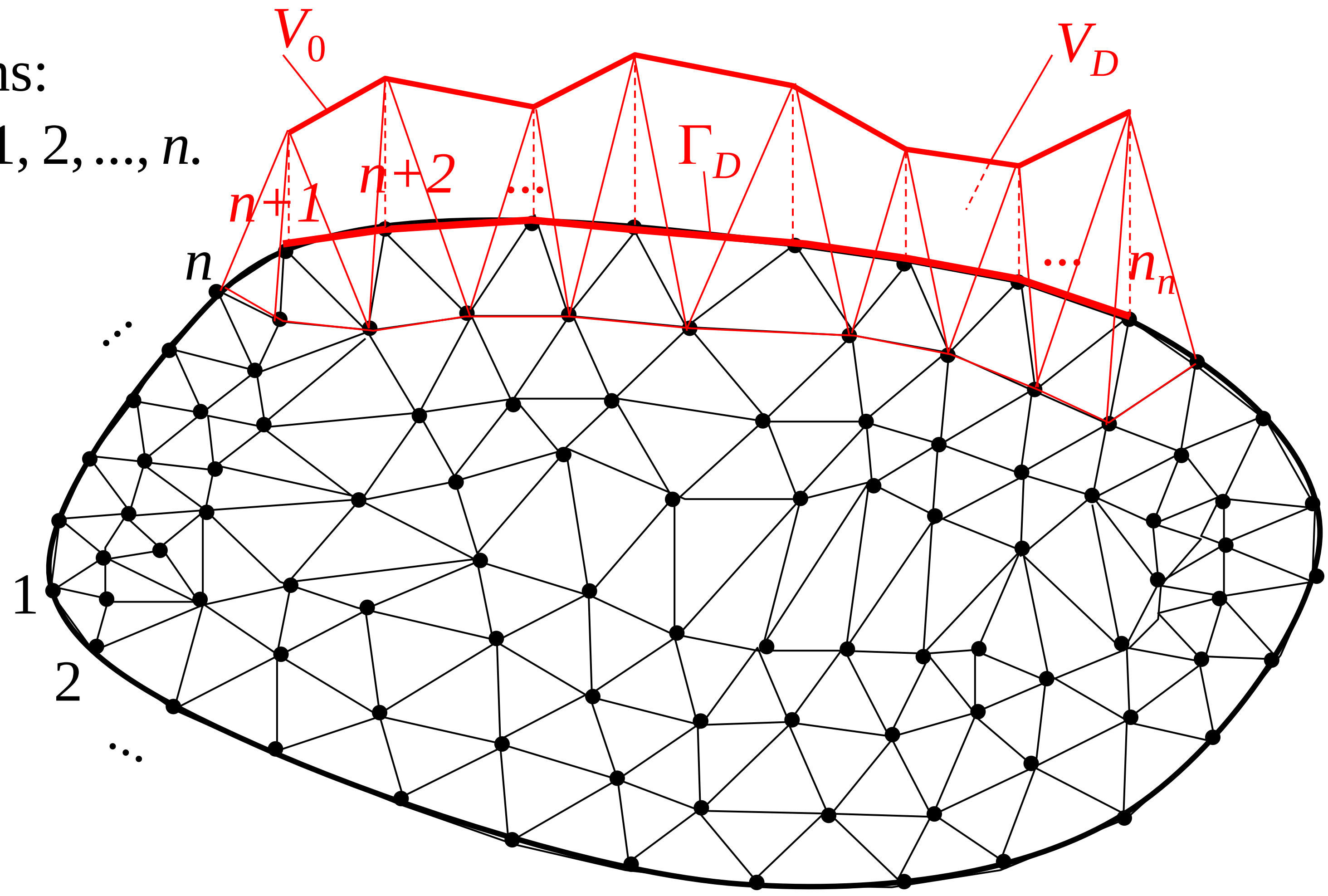
$$V^{(n)} = \sum_{j=1}^{n_n} V_j N_j$$

n_n: number of nodes

V_j: nodal potential values

Basis functions:

$w_j = N_j, \;\; j = 1, 2, ..., n.$



$$V^{(n)} = \sum_{j=1}^{n_n} V_j N_j = \sum_{j=n+1}^{n_n} V_j N_j + \sum_{j=1}^{n} V_j N_j = V_D + \sum_{j=1}^{n} V_j N_j .$$

Ritz equations:

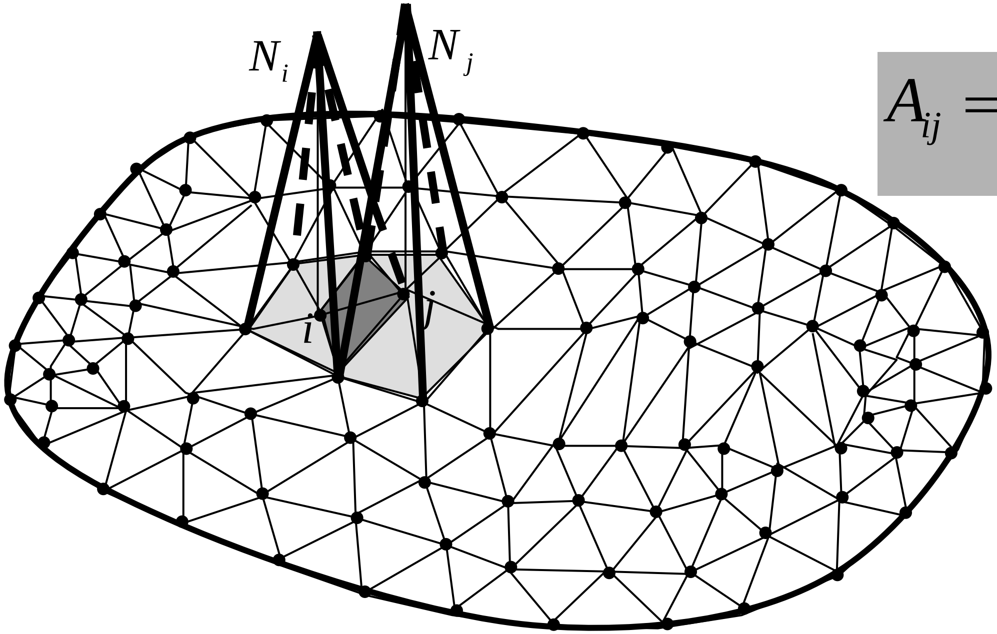
$$\sum_{j=1}^{n} V_j \int_{\Omega} gradN_i \cdot \varepsilon gradN_j d\Omega =$$

$$= -\int_{\Omega} gradN_i \cdot \varepsilon gradV_D d\Omega + \int_{\Omega} N_i \rho d\Omega + \int_{\Gamma_N} N_i \sigma d\Gamma, \qquad i = 1, 2, ..., n.$$

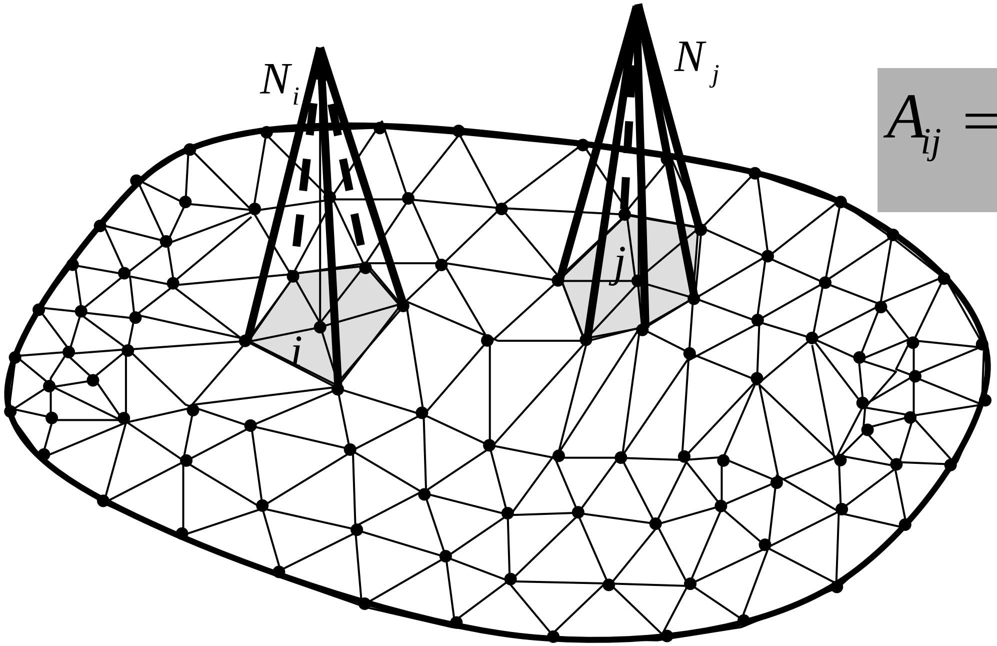
In matrix form: $\lfloor A_{ij} \rfloor \{V_j\} = \{b_i\}.$

$$A_{ij} = \int_{\Omega} gradN_i \cdot \varepsilon gradN_j d\Omega,$$

$$b_i = -\int_{\Omega} gradN_i \cdot \varepsilon gradV_D d\Omega + \int_{\Omega} N_i \rho d\Omega + \int_{\Gamma_N} N_i \sigma d\Gamma.$$



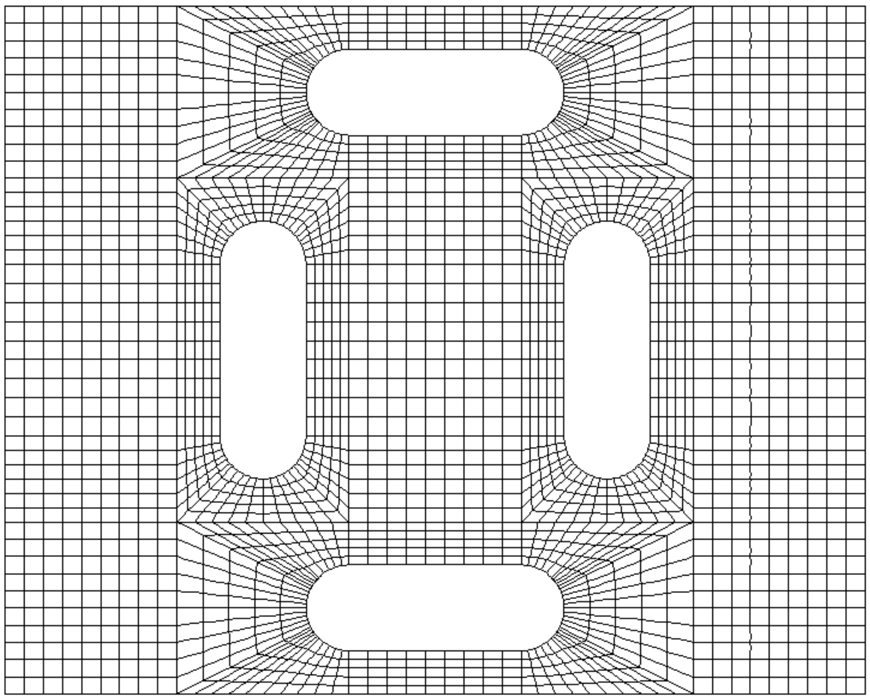
$$A_{ij} = \int_{\Omega} gradN_i \cdot \varepsilon gradN_j d\Omega \neq 0,$$

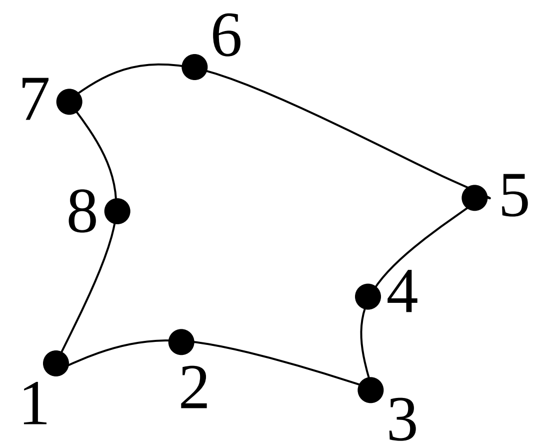


$$A_{ij} = \int_{\Omega} gradN_i \cdot \varepsilon gradN_j d\Omega = 0.$$

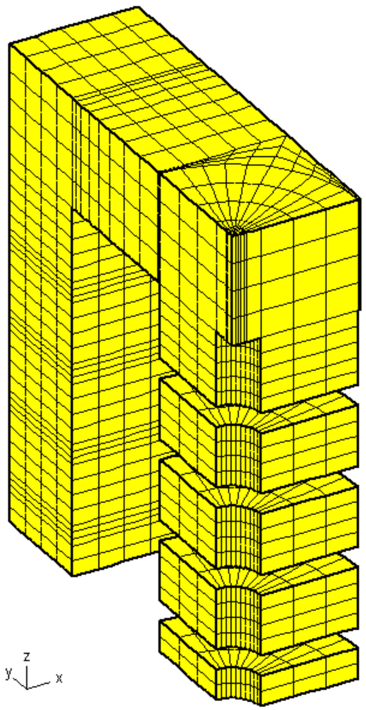
Sparse matrix.

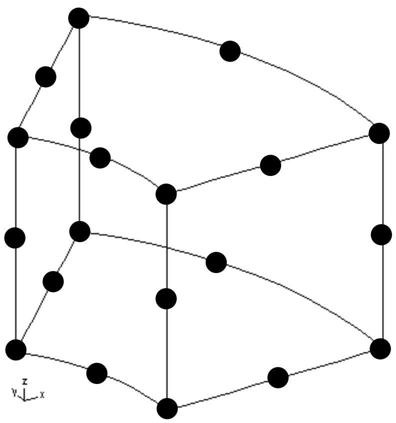
8-node quadrilateral elements



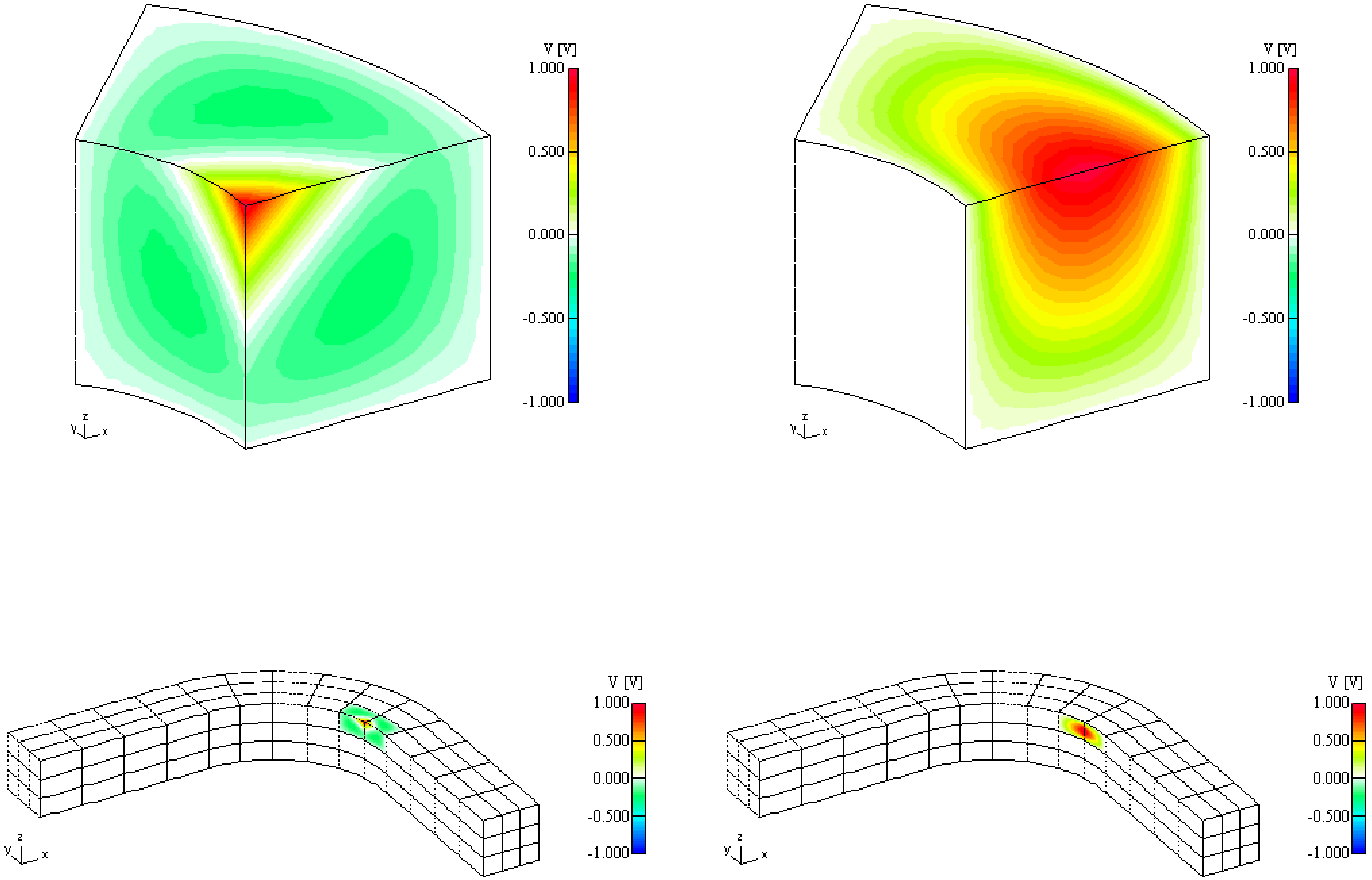


20-node hexahedral elements





Shape and basis functions for 20-node hexahedral elements



2.4 Integral equations for the scalar potential

The boundary value problem for the scalar potential can also be represented by various integral equations.

Integral equations for functions a single variable:

$y(x):$ unknown function, $f(x):$ known function,

$K(x,x'):$ Kernel of the integral equation (given),

$\lambda:$ given konstant.

$$\int_a^b K(x,x')y(x')dx' = f(x)$$

Fredholm integral equation of the first kind

$$y(x) - \lambda \int_a^b K(x,x')y(x')dx' = f(x)$$

Fredholm integral equation of the second kind

Boundary value problem for the electric scalar potential in case of homogeneous medium ($\varepsilon = \varepsilon_0$):

$$divgradV = \Delta V = -\frac{\rho}{\varepsilon_0} \text{ in } \Omega,$$ Laplace-Poisson equation,

$$V = V_0 \text{ auf } \Gamma_D$$ Dirichlet boundary condition,

$$\frac{\partial V}{\partial n} = \frac{\sigma}{\varepsilon_0} \text{ auf } \Gamma_N$$ Neumann boundary condition.

If V and $\frac{\partial V}{\partial n}$ were known on the *entire boundary*, one could compute the potential in any point in Ω with the aid of the *integral representation*.

2.4.1 Fundamentals of potential theory

Green function: solution of the special Laplace-Poisson equation

$$-\Delta G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

in the entire three-dimensional space.

$\delta(\mathbf{r}-\mathbf{r}')$: Dirac impulse function in the point $\mathbf{r}'$, defined by

$$\int_{\Re^3} w(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')d\Omega = w(\mathbf{r}') \text{ or } \int_{\Re^3} w(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d\Omega' = w(\mathbf{r})$$

The solution of the Laplace-Poisson equation $-\Delta V(\mathbf{r}) = \dfrac{\rho(\mathbf{r})}{\varepsilon_0}$

in infinite empty space (ε=ε_0) is:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\Re^3} \frac{\rho(\mathbf{r}'')d\Omega''}{|\mathbf{r}-\mathbf{r}''|}.$$

Therefore, the Green function is $\left(\rho(\mathbf{r}) = \varepsilon_0 \delta(\mathbf{r}-\mathbf{r}')\right)$:

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi} \int_{\Re^3} \frac{\delta(\mathbf{r}''-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}''|} d\Omega'' = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

This is the potential of a point charge of the magnitude ε_0 in the point **r**´.

A point charge of the magnitude Q in the point **r**´ corresponds to the charge density $\rho(\mathbf{r}) = Q\delta(\mathbf{r}-\mathbf{r}').$

Green's theorem:

$$\begin{aligned} & div(\phi grad\,\psi) = grad\,\phi \cdot grad\,\psi + \phi\Delta\psi \\ - \quad & div(\psi grad\,\phi) = grad\,\psi \cdot grad\,\phi + \psi\Delta\phi \end{aligned}$$

$$\phi\Delta\psi - \psi\Delta\phi = div(\phi grad\,\psi) - div(\psi grad\,\phi)$$

$$\int_{\Omega} \left(\phi\Delta\psi - \psi\Delta\phi\right) d\Omega = \oint_{\Gamma} \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n} \right) d\Gamma$$

$$\phi \Leftarrow V(\mathbf{r}'), \psi \Leftarrow G(\mathbf{r},\mathbf{r}'), d\Omega \Leftarrow d\Omega', d\Gamma \Leftarrow d\Gamma'$$

$$\int_\Omega \left(V(\mathbf{r}') \underbrace{\Delta G(\mathbf{r},\mathbf{r}')}_{-\delta(\mathbf{r}-\mathbf{r}')} - G(\mathbf{r},\mathbf{r}') \Delta V(\mathbf{r}') \right) d\Omega' =$$

$$= \oint_\Gamma \left(V(\mathbf{r}') \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}') \frac{\partial V(\mathbf{r}')}{\partial n'} \right) d\Gamma'$$

Integral representation:

$$V(\mathbf{r}) = -\frac{1}{4\pi} \int_\Omega \frac{1}{|\mathbf{r}-\mathbf{r}'|} \Delta V(\mathbf{r}') d\Omega' -$$

$$-\frac{1}{4\pi} \oint_\Gamma V(\mathbf{r}') \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} d\Gamma' + \frac{1}{4\pi} \oint_\Gamma \frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial V(\mathbf{r}')}{\partial n'} d\Gamma'$$

Corollary

Mean value theorem of potential theory :

If $V(\mathbf{r})$ is a solution of von $\Delta V=0$, then the mean value of V, over the surface of a sphere with an arbitrary radius R, equals the value of V in the centre of the sphere.

Proof

Let the centre be $\mathbf{r}$, and the surface Γ be the sphere with the radius R: $\left|\mathbf{r}-\mathbf{r}'\right| = R$

$$V(\mathbf{r}) = -\frac{1}{4\pi R}\int_{\Omega}\underbrace{\Delta V(\mathbf{r}')}_{=0}d\Omega' - \frac{1}{4\pi}\oint_{\Gamma}V(\mathbf{r}')\underbrace{\frac{\partial}{\partial R}\frac{1}{R}}_{-\frac{1}{R^2}}d\Gamma' +$$

$$+\frac{1}{4\pi R}\underbrace{\oint_{\Gamma}\frac{\partial V(\mathbf{r}')}{\partial n'}d\Gamma'}_{=\oint_{\Gamma} gradV\cdot\mathbf{n}d\Gamma'=\int_{\Omega} divgradVd\Omega'=0} = \frac{1}{4\pi R^2}\oint_{\Gamma}V(\mathbf{r}')d\Gamma'$$

cf. method of finite differences

Inserting the quantities known from the boundary value problem:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\Omega}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\Omega' -$$

$$-\frac{1}{4\pi}\int_{\Gamma_D}V_0(\mathbf{r}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\Gamma' + \frac{1}{4\pi\varepsilon_0}\int_{\Gamma_N}\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\Gamma' -$$

known

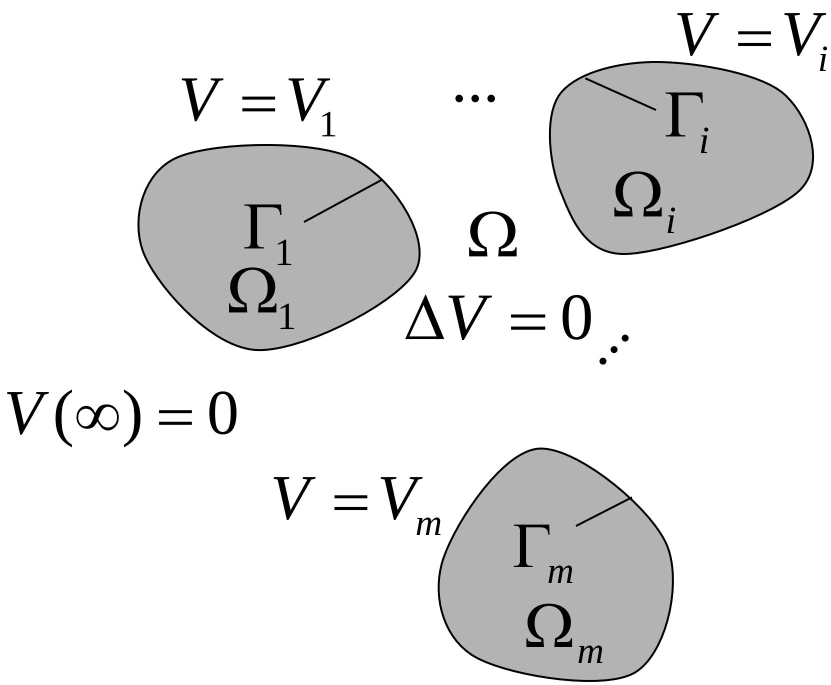
$$-\frac{1}{4\pi}\int_{\Gamma_N}V(\mathbf{r}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\Gamma' + \frac{1}{4\pi\varepsilon_0}\int_{\Gamma_D}\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\Gamma'$$

unknown

Various integral equations can be obtained for the unknown quantities V on Γ_N and $\frac{\partial V}{\partial n}$ on Γ_D.

2.4.2 Integral equation for the surface charge density

Simplest case: electrodes (V_i = constant, i =1, 2, ..., m).



$$\Gamma_N = 0, \Gamma_D = \sum_{i=1}^{n} \Gamma_i, \rho = 0.$$

Integral representation:

$$V(\mathbf{r}) = -\frac{1}{4\pi} \sum_{i=1}^{m} V_i \oint_{\Gamma_i} \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\Gamma' +$$

$$+ \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{m} \oint_{\Gamma_i} \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\Gamma'.$$

$$\oint_{\Gamma_i} \frac{\partial}{\partial n'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\Gamma' = \oint_{\Gamma_i} grad' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \mathbf{n}' d\Gamma' = \int_{\Omega_i} \Delta' \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\Omega' =$$

$$= 0 \text{ falls } \mathbf{r} \notin \Omega_i.$$

Integral equation for the surface charge density on the electrodes:

$$\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{m}\oint_{\Gamma_i}\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\Gamma' = V_i, \text{ for } \mathbf{r}\in\Gamma_i, \; i=1, 2, ..., m.$$

In the 2D case, the Green function is $\frac{1}{2\pi}\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}$.

Let the contours of the electrodes be the curves C_i.

Integral equation for the line charge density on the electrodes:

$$\frac{1}{2\pi\varepsilon_0}\sum_{i=1}^{m}\oint_{C_i}\tau(\mathbf{r}')\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\Gamma' = V_i, \text{ for } \mathbf{r}\in C_i, \; i=1, 2, ..., m.$$

Fredholm integral equations of the first kind.

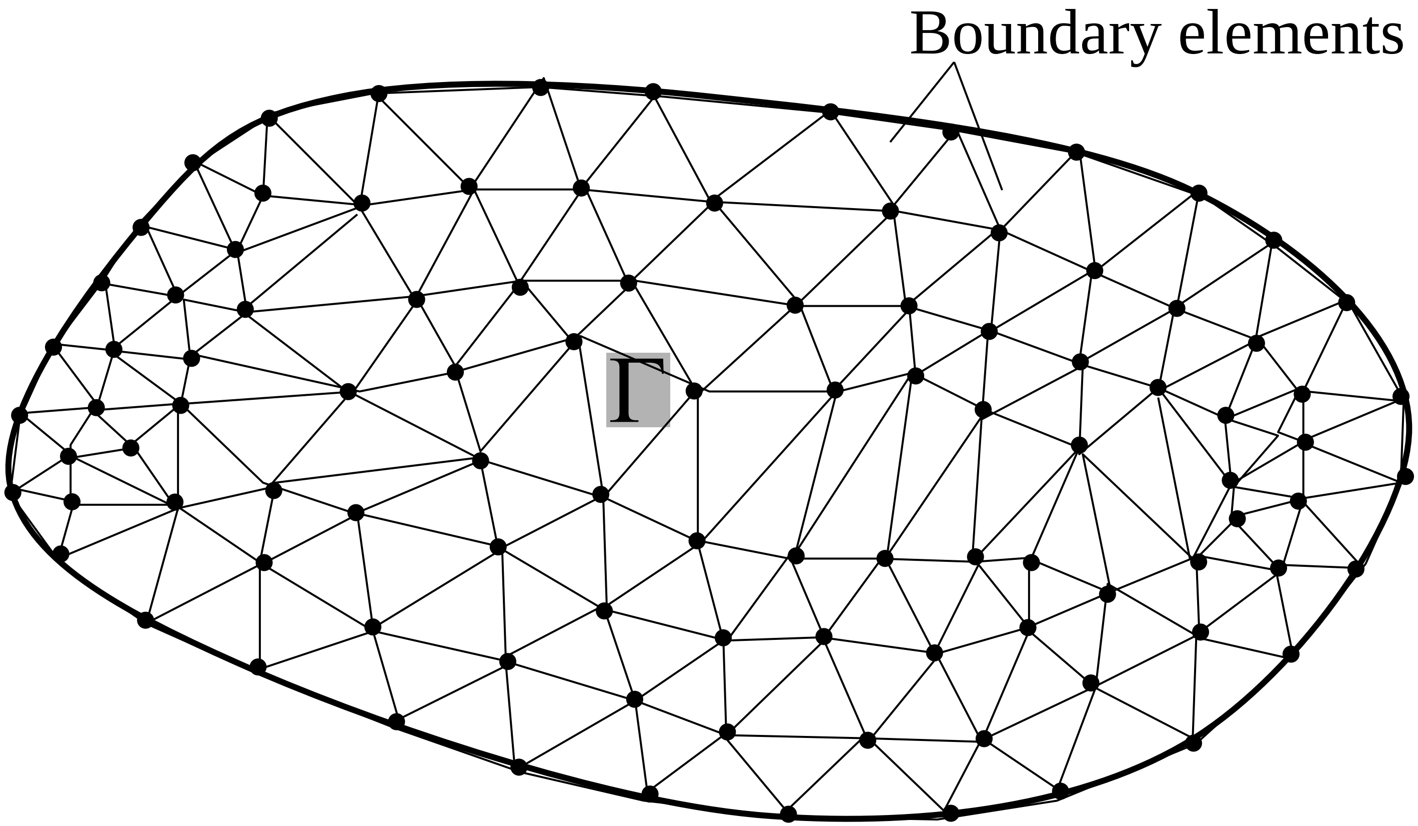
2.4.3 Method of boundary elements (Boundary Element Method = BEM)

Numerical procedure to solve integral equations

Discretization of the electrodes: 3D problems - surfaces

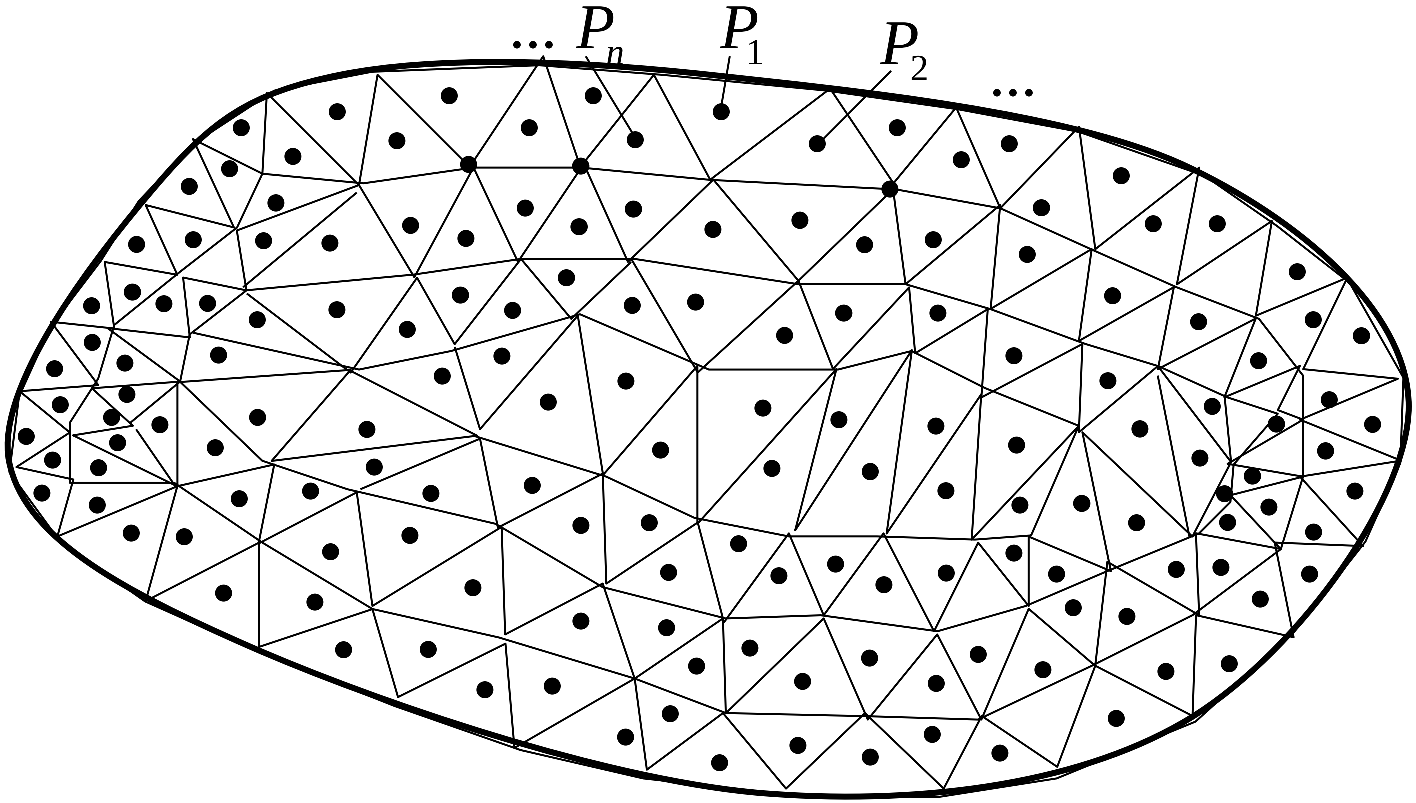
2D problems - curves

Unknowns: surface charge densities in the elements



Simplest assumption: σ is constant in each element.
Let the number of elements be n, the unknowns are σ_j (j = 1, 2, ..., n).

In each element, one test point P_i (i = 1, 2, ..., n) is selected in which the integral equation is required to be satisfied.



Linear equation system for the unknown surface charge densities (Γ_j: j-th element, V_i: potential in P_i):

$$\frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n}\sigma_j\int_{\Gamma_j}\frac{1}{\left|\mathbf{r}_{P_i}-\mathbf{r}'\right|}d\Gamma' = V_i, \quad i = 1, 2, ..., n.$$

In the 2D case, the unknowns are the values of the line charge density (C_j: j-th line segment, V_i: potential in P_i):

$$\frac{1}{2\pi\varepsilon_0}\sum_{j=1}^{n}\tau_j\int_{C_j}\ln\frac{1}{\left|\mathbf{r}_{P_i}-\mathbf{r}'\right|}ds' = V_i, \quad i = 1, 2, ..., n.$$

Non-symmetric full matrix.

Having solved the equation system, the potential can be computed in any point by evaluating the integral representation.

2.5 Boundary value problems for the vector potential

Magnetostatic field:

$$div\mathbf{B} = \mathbf{0} \Rightarrow \mathbf{B} = curl\mathbf{A},$$ **A**: magnetic vector potential

Static current field:

$$div\mathbf{J} = \mathbf{0} \Rightarrow \mathbf{J} = curl\mathbf{T},$$ **T**: current vector potential

Differential equations:

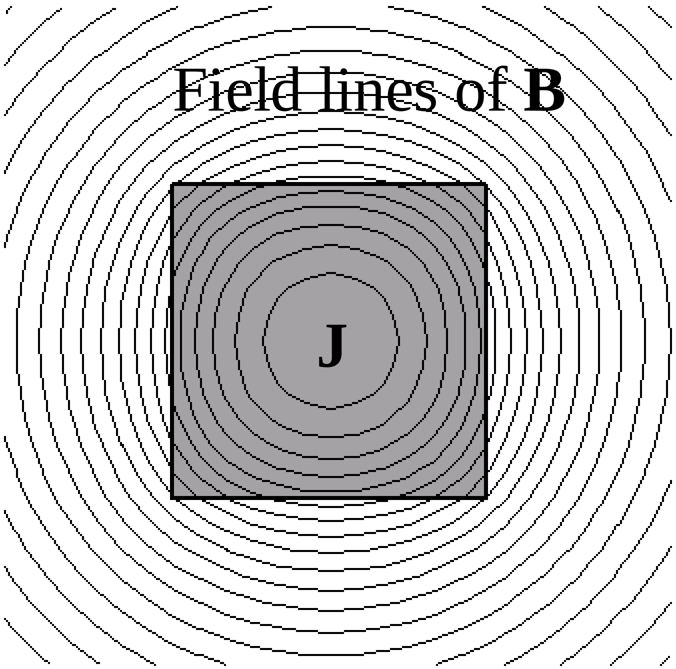
$$curl\mathbf{H} = \mathbf{J} \Rightarrow curl(\frac{1}{\mu}curl\mathbf{A}) = \mathbf{J},$$

$$curl\mathbf{E} = \mathbf{0} \Rightarrow curl(\frac{1}{\gamma}curl\mathbf{T}) = \mathbf{0}.$$

2.5.1 Planar 2D problems

Magnetostatic field:

$$\frac{\partial}{\partial z}=0: \mathbf{J}=J(x,y)\mathbf{e}_z,\ \mathbf{B}=B_x(x,y)\mathbf{e}_x+B_y(x,y)\mathbf{e}_y.$$



$$\mathbf{A}=A(x,y)\mathbf{e}_z$$

$$\mathbf{B}=curl\mathbf{A}=\begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0\\ 0 & 0 & A\end{vmatrix}=\frac{\partial A}{\partial y}\mathbf{e}_x-\frac{\partial A}{\partial x}\mathbf{e}_y.$$

Differential equation for the single component vector potential in planar 2D case:

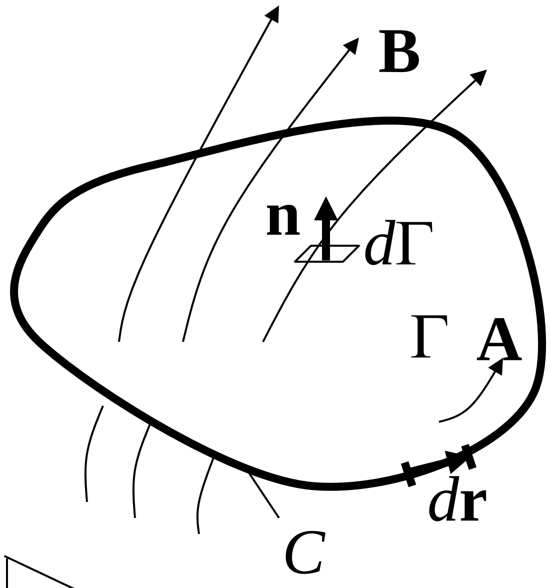
$$curl[\frac{1}{\mu}curl(A\mathbf{e}_z)] = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ \frac{1}{\mu}\frac{\partial A}{\partial y} & -\frac{1}{\mu}\frac{\partial A}{\partial x} & 0 \end{vmatrix} =$$

$$= -\left[\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial A}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mu}\frac{\partial A}{\partial y}\right)\right]\mathbf{e}_z = -\mathbf{e}_z div(\frac{1}{\mu} gradA).$$

$$-div(\frac{1}{\mu} gradA) = J,$$ generalized Laplace-Poisson equation.

Magnetic flux by means of **A**:

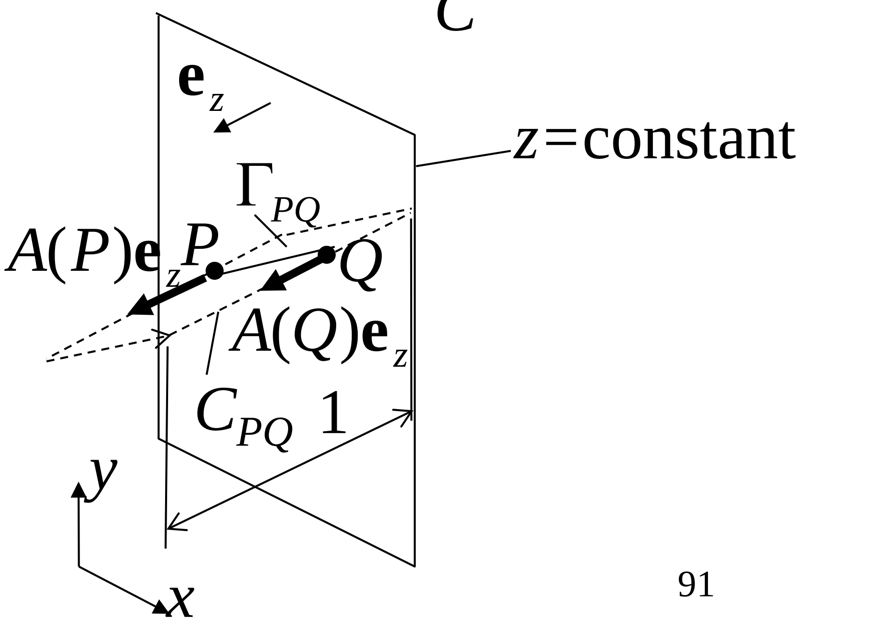
$$\Phi = \int_\Gamma \mathbf{B}\cdot\mathbf{n}\,d\Gamma = \int_\Gamma curl\mathbf{A}\cdot\mathbf{n}\,d\Gamma = \oint_C \mathbf{A}\cdot d\mathbf{r}.$$



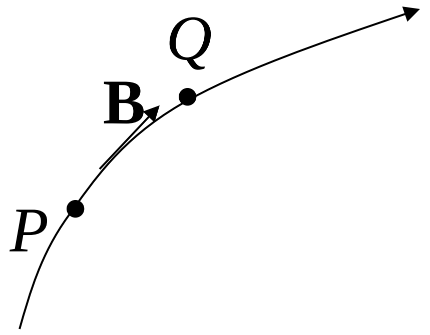
Planar 2D case:

Γ_{PQ}: surface of unit length through the points P and Q

$$\Phi_{PQ} = \oint_{C_{PQ}} \mathbf{A}\cdot d\mathbf{r} = A(P) - A(Q)$$



Flux lines (magnetic field lines) are parallel to **B**.



For any two points P and Q on a flux line one has $\Phi_{PQ} = A(P) - A(Q) = 0$.

$$A(P) = A(Q) \Rightarrow A = \text{constant along flux lines!}$$

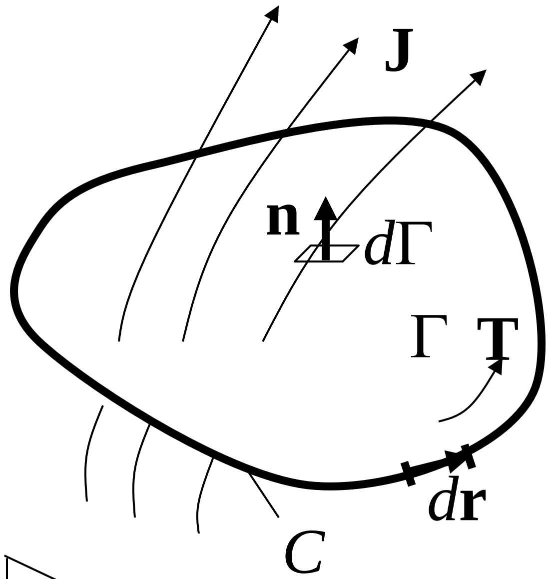
$$dA(P) = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = -B_y dx + B_x dy = 0 \Rightarrow \frac{dy}{dx} = \frac{B_y}{B_x}.$$

Flux lines: lines of constant $A(x, y)$.

If flux lines are drawn so that the difference of the vector potential between any two neighboring flux lines is constant, then the density of the lines is proportional to the magnitude of **B**.

Current by means of **T**:

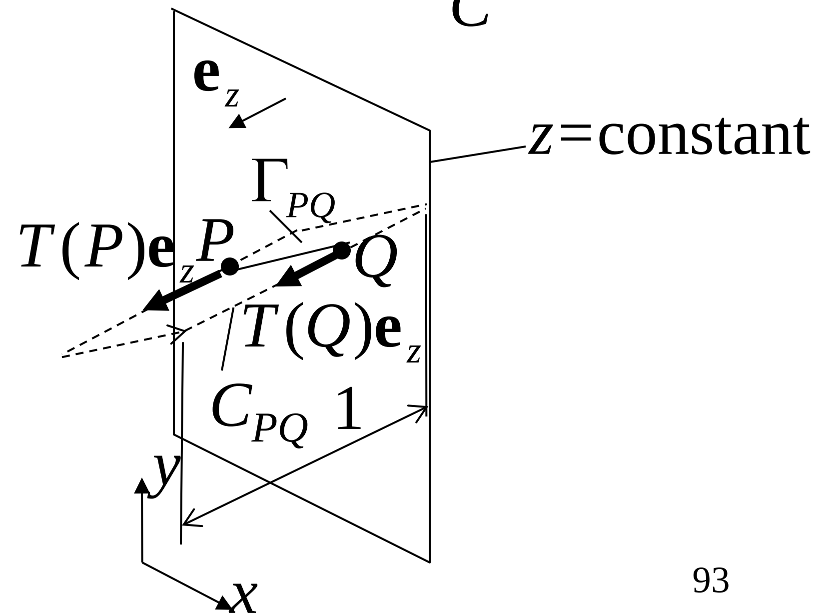
$$I = \int_{\Gamma} \mathbf{J}\cdot\mathbf{n}\,d\Gamma = \int_{\Gamma} curl\mathbf{T}\cdot\mathbf{n}\,d\Gamma = \oint_{C} \mathbf{T}\cdot d\mathbf{r}.$$



Planar 2D case:

Γ_{PQ}: surface of unit length through the points P and Q

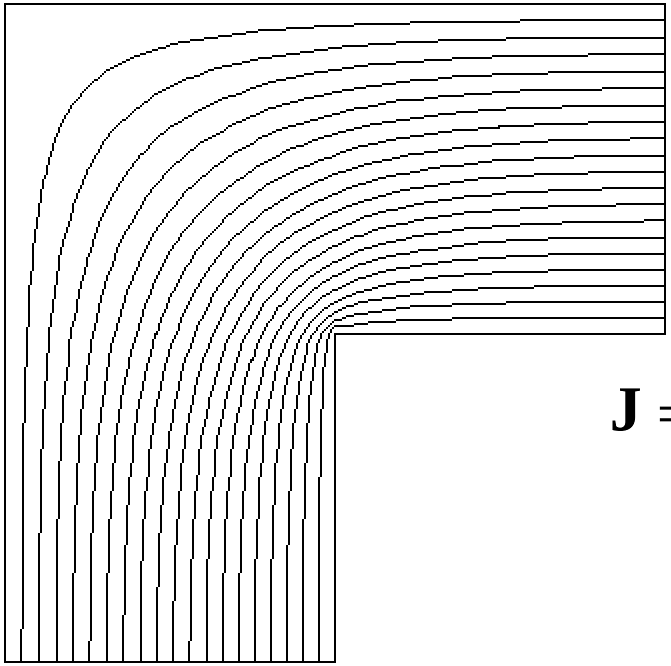
$$I_{PQ} = \oint_{C_{PQ}} \mathbf{T}\cdot d\mathbf{r} = T(P) - T(Q)$$



Current lines:
Lines of constant T(x,y).

Static current field:

$$\frac{\partial}{\partial z} = 0 : \mathbf{J} = J_x(x, y)\mathbf{e}_x + J_y(x, y)\mathbf{e}_y .$$



$$\mathbf{T} = T(x, y)\mathbf{e}_z$$

$$\mathbf{J} = curl\mathbf{T} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & T \end{vmatrix} = \frac{\partial T}{\partial y}\mathbf{e}_x - \frac{\partial T}{\partial x}\mathbf{e}_y .$$

$$-div(\frac{1}{\gamma} gradT) = 0,$$ generalized Laplace equation.

Boundary conditions:

Dirichlet boundary condition: $A = A_0$ (known) on Γ_D,

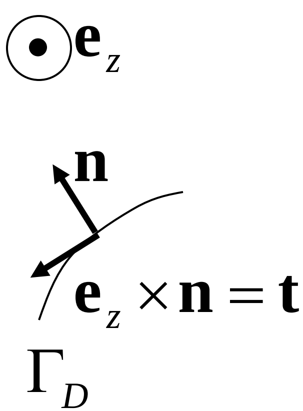
$T = T_0$ (known) on Γ_D.

This means the prescription of B_n or J_n:

e.g. for the magnetic field:

$$B_n = \mathbf{n} \cdot curl\mathbf{A} = \mathbf{n} \cdot curl(A\mathbf{e}_z) = \mathbf{n} \cdot (gradA \times \mathbf{e}_z) = gradA \cdot (\mathbf{e}_z \times \mathbf{n}) =$$

$$= \mathbf{t} \cdot gradA = \frac{\partial A}{\partial t}: \text{ tangential derivative of } A!$$



In most cases A_0=constant or T_0=constant:
Then, the section of Γ_D with a plane z=constant is a flux line or a current line. The differences in the values of A_0 or T_0 yield the flux or current per unit length between the lines.

Neumann boundary condition: $\frac{1}{\mu}\frac{\partial A}{\partial n} = \alpha$ (known) on Γ_N,

$$\frac{1}{\gamma}\frac{\partial T}{\partial n} = e \text{ (known) on } \Gamma_N .$$

This means the prescription of H_t or E_t:

e.g. for the magnetic field:

$$H_t = (\mathbf{e}_z \times \mathbf{n}) \cdot \frac{1}{\mu} curl(A\mathbf{e}_z) = (\mathbf{e}_z \times \mathbf{n}) \cdot \frac{1}{\mu}(gradA \times \mathbf{e}_z) =$$

$$= \mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{n}) \cdot \frac{1}{\mu} gradA = -\mathbf{n} \cdot \frac{1}{\mu} gradA = -\frac{1}{\mu}\frac{\partial A}{\partial n}.$$

On interfaces to highly permeable regions or electrodes, the Neumann boundary condition is

homogeneous: $\frac{1}{\mu}\frac{\partial A}{\partial n} = 0$ or $\frac{1}{\gamma}\frac{\partial T}{\partial n} = 0.$

Boundary value problem for the single component vector potential functions in the planar 2D case:

magnetostatic field:

$$-div\left(\frac{1}{\mu} gradA\right) = J \text{ in } \Omega,$$

$$A = A_0 \text{ on } \Gamma_D, \ \frac{1}{\mu}\frac{\partial A}{\partial n} = \alpha \text{ on } \Gamma_N.$$

Static current field:

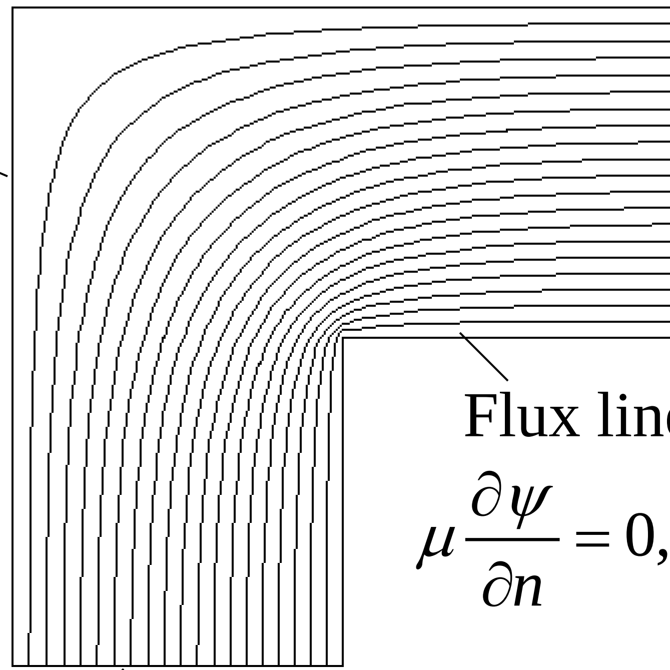
$$-div\left(\frac{1}{\gamma} gradT\right) = 0 \text{ in } \Omega,$$

$$T = T_0 \text{ on } \Gamma_D, \ \frac{1}{\gamma}\frac{\partial T}{\partial n} = 0 \text{ on } \Gamma_N.$$

Similar boundary value problem to the ones for the scalar potential functions.

Duality between the boundary conditions for the scalar potential and the single component vector potential in the planar 2D case

Flux line



Magnetic wall:

$\psi =$ constant,

$$\frac{1}{\mu}\frac{\partial A}{\partial n} = 0.$$

Flux line:

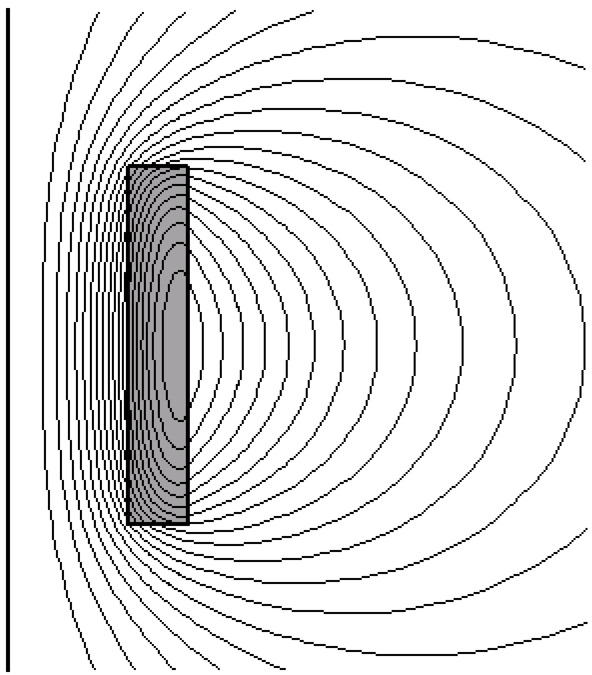
$$\mu\frac{\partial \psi}{\partial n} = 0, \;\; A = \text{ constant.}$$

Magnetic wall

2.5.2 Axisymmetric 2D problems

Magnetostatic field:

$$\frac{\partial}{\partial \phi} = 0: \mathbf{J} = J(r,z)\mathbf{e}_\phi, \mathbf{B} = B_r(r,z)\mathbf{e}_r + B_z(r,z)\mathbf{e}_z.$$



$$\mathbf{A} = A(r,z)\mathbf{e}_\phi$$

$$\mathbf{B} = curl\mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{e}_r & \mathbf{e}_\phi & \frac{1}{r}\mathbf{e}_z \\ \frac{\partial}{\partial r} & 0 & \frac{\partial}{\partial z} \\ 0 & rA & 0 \end{vmatrix} =$$

$$= -\frac{\partial A}{\partial z}\mathbf{e}_r + \frac{1}{r}\frac{\partial (rA)}{\partial r}\mathbf{e}_z.$$

Differential equation for the single component vector potential in the axisymmetric case:

$$curl[\frac{1}{\mu}curl(A\mathbf{e}_\phi)] = \begin{vmatrix} \frac{1}{r}\mathbf{e}_r & \mathbf{e}_\phi & \frac{1}{r}\mathbf{e}_z \\ \frac{\partial}{\partial r} & 0 & \frac{\partial}{\partial z} \\ -\frac{1}{\mu}\frac{\partial A}{\partial z} & 0 & \frac{1}{\mu r}\frac{\partial (rA)}{\partial r} \end{vmatrix} =$$

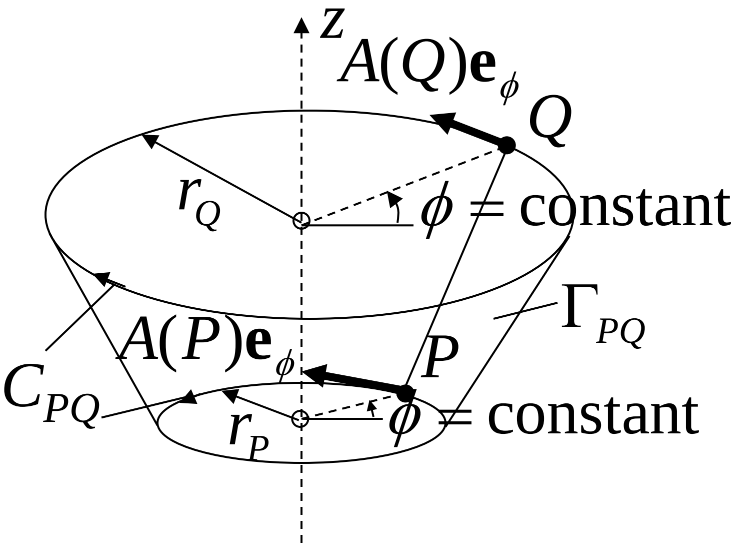
$$= -\left[\frac{\partial}{\partial r}\left(\frac{1}{\mu r}\frac{\partial (rA)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu}\frac{\partial A}{\partial z}\right)\right]\mathbf{e}_\varphi \neq -\mathbf{e}_\varphi div(\frac{1}{\mu}gradA).$$

$$\left(div(\frac{1}{\mu}gradA) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{\mu}\frac{\partial A}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu}\frac{\partial A}{\partial z}\right)\right)$$

$$-\left[\frac{\partial}{\partial r}\left(\frac{1}{\mu r}\frac{\partial (rA)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu}\frac{\partial A}{\partial z}\right)\right] = J$$

Flux in axisymmetric 2D case:

Γ_{PQ}: Conical surface through the points P and Q



$$\Phi_{PQ} = \oint_{C_{PQ}} \mathbf{A} \cdot d\mathbf{r} = 2\pi[r_P A(P) - r_Q A(Q)].$$

Flux lines: Lines of constant $rA(r, z)$.

Current field: Lines of constant $rT(r, z)$ are the current lines.

2.5.3 3D problems

The vector potential functions are not unique:

$\mathbf{B} = curl\mathbf{A} = curl(\mathbf{A} + grad u)$ u is an arbitrary scalar function,

$\mathbf{J} = curl\mathbf{T} = curl(\mathbf{T} + grad u)$ u is an arbitrary scalar function,

Differential equations:

$$curl\mathbf{H} = \mathbf{J} \Rightarrow curl(\frac{1}{\mu} curl\mathbf{A}) = \mathbf{J},$$

$$curl\mathbf{E} = 0 \Rightarrow curl(\frac{1}{\gamma} curl\mathbf{T}) = 0,$$

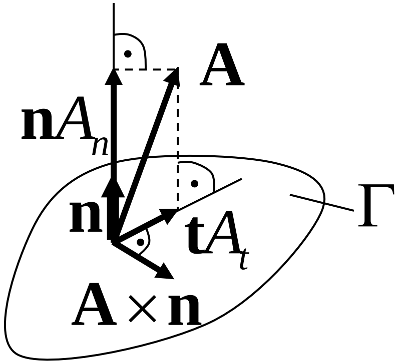
Boundary conditions:

Prescription of B_n or of H_t.

Prescription of J_n or of E_t.

In case of 3D problems, the boundary conditions are specified for the normal and tangential components of the field quantities or vector potentials.

$$\mathbf{A} = \mathbf{n}A_n + \mathbf{t}A_t.$$



$$\mathbf{n}A_n = \mathbf{n}(\mathbf{A} \cdot \mathbf{n})$$

$$\mathbf{t}A_t = \mathbf{n} \times (\mathbf{A} \times \mathbf{n}) = \mathbf{A}\underbrace{(\mathbf{n} \cdot \mathbf{n})}_{1} - \mathbf{n}\underbrace{(\mathbf{A} \cdot \mathbf{n})}_{A_n} = \mathbf{A} - \mathbf{n}A_n$$

Instead of $\mathbf{t}A_t$ we use $\mathbf{A} \times \mathbf{n}$.

Specification of B_n or of J_n with the aid of the vector potentials:

$$div(\mathbf{A}\times\mathbf{n}) = \mathbf{n}\cdot curl\mathbf{A} - \mathbf{A}\cdot\underbrace{curl\mathbf{n}}_{0} = \mathbf{n}\cdot\mathbf{B} = B_n, \quad div(\mathbf{T}\times\mathbf{n}) = J_n.$$

$$\left(div\mathbf{v}(x_1, x_2, x_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(v_1 h_2 h_3\right) + \frac{\partial}{\partial x_2}\left(v_2 h_1 h_3\right) + \frac{\partial}{\partial x_3}\left(v_3 h_1 h_2\right)\right]\right)$$

$\mathbf{A}\times\mathbf{n}$ and $\mathbf{T}\times\mathbf{n}$ have no normal component

$\Downarrow$

When building the divergence, no differentiation in the normal direction occurs.

By specifying $\mathbf{A}\times\mathbf{n}$ or $\mathbf{T}\times\mathbf{n}$, B_n or J_n are determined: Dirichlet boundary condition.

Specification of H_t or of E_t with the aid of the vector potentials:

$\mathbf{H}\times\mathbf{n}$ or $\mathbf{E}\times\mathbf{n}$ are prescribed:

$$\mathbf{H}\times\mathbf{n}=\frac{1}{\mu}curl\mathbf{A}\times\mathbf{n},\ \mathbf{E}\times\mathbf{n}=\frac{1}{\gamma}curl\mathbf{T}\times\mathbf{n}:$$ Neumann boundary condition.

Boundary value problems for the vector potential functions in 3D case:

Static magnetic field:

$$curl(\frac{1}{\mu} curl\mathbf{A}) = \mathbf{J} \text{ in } \Omega,$$

$$\mathbf{A} \times \mathbf{n} = \mathbf{a} \text{ on } \Gamma_D, \ \frac{1}{\mu} rot\mathbf{A} \times \mathbf{n} = \boldsymbol{\alpha} \text{ on } \Gamma_N.$$

Static current field:

$$curl(\frac{1}{\gamma} curl\mathbf{T}) = \mathbf{0} \text{ in } \Omega,$$

$$\mathbf{T} \times \mathbf{n} = \boldsymbol{\tau} \text{ on } \Gamma_D, \ \frac{1}{\gamma} curl\mathbf{T} \times \mathbf{n} = \mathbf{e} \text{ on } \Gamma_N.$$

The solution of the boundary value problems is not unique.

Special case: vector potential due to a given current density in infinite free space $(\mu = \mu_0$ and $\Gamma \to \infty)$:

$$\boxed{\begin{aligned} & curl\,curl\,\mathbf{A} = \mu_0 \mathbf{J}, \\ & \mathbf{A}(\infty) = 0 \end{aligned}} \quad (\Rightarrow \mathbf{A}(\infty) \times \mathbf{n} = \mathbf{0} \text{ or } curl\,\mathbf{A}(\infty) \times \mathbf{n} = \mathbf{0}).$$

Only **B**=*curl***A** is defined uniquely, but not **A**.

A becomes unique if *div***A** is additionally defined: <u>gauging</u>.

The choice *div***A** = 0 is the <u>Coulomb gauge</u>.

The gauge makes **A** unique:

$$\left.\begin{aligned} div\mathbf{A} = div(\mathbf{A} + gradu) &\Rightarrow \Delta u = 0, \\ \mathbf{A}(\infty) = \mathbf{A}(\infty) + gradu(\infty) &\Rightarrow gradu(\infty) = \mathbf{0}, \end{aligned}\right\} \Rightarrow gradu = \mathbf{0}.$$

Boundary value problem for **A**:

$$\left.\begin{aligned} & curlcurl\mathbf{A} = \mu_0 \mathbf{J}, \\ & div\mathbf{A} = 0, \\ & \mathbf{A}(\infty) = \mathbf{0}. \end{aligned}\right\} \Rightarrow \quad \begin{aligned} & curlcurl\mathbf{A} - graddiv\mathbf{A} = -\Delta\mathbf{A} = \mu_0 \mathbf{J}, \\ & \mathbf{A}(\infty) = \mathbf{0}. \end{aligned}$$

$$\Downarrow$$

Vector Laplace-Poisson differential equation.

The Coulomb gauge follows from the vector Laplace-Poisson differential equation:

$$\left.\begin{aligned} &curl\,curl\mathbf{A} - grad\,div\mathbf{A} = -\Delta\mathbf{A} = \mu_0\mathbf{J}, \\ &\mathbf{A}(\infty) = \mathbf{0}. \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} &\underbrace{div(curl\,curl\mathbf{A} - grad\,div\mathbf{A})}_{\Delta(div\mathbf{A})} = \underbrace{\mu_0 div\mathbf{J}}_{0} \\ &div\mathbf{A}(\infty) = 0, \end{aligned}\right\} \Rightarrow div\mathbf{A} = 0.$$

Solution of

$$\boxed{\begin{aligned} &-\Delta\mathbf{A} = \mu_0\mathbf{J}, \quad \mathbf{A}(\infty) = \mathbf{0}: \\ &\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}')d\Omega'}{|\mathbf{r} - \mathbf{r}'|}. \end{aligned}}$$

Biot-Savart's law:

$$\mathbf{B}(\mathbf{r}) = curl\mathbf{A} = \frac{\mu_0}{4\pi}\int_\Omega curl\,\frac{\mathbf{J}(\mathbf{r}')d\Omega'}{|\mathbf{r}-\mathbf{r}'|} =$$

$$= \frac{\mu_0}{4\pi}\int_\Omega (grad\,\frac{1}{|\mathbf{r}-\mathbf{r}'|})\times\mathbf{J}(\mathbf{r}')d\Omega' = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\Omega',$$

$$\boxed{\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}')\times\mathbf{e}_{\mathbf{r}'\to\mathbf{r}}}{|\mathbf{r}-\mathbf{r}'|^2}d\Omega'.}$$

3. Quasi-static fields

Maxwell's equations $\left(|\mathbf{J}| >> \left| \frac{\partial \mathbf{D}}{\partial t} \right| \right)$:

In conducting media (Ω_l):

$$curl\mathbf{H} = \mathbf{J},$$

$$curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$div\mathbf{B} = 0,$$

$$\mathbf{B} = \mu\mathbf{H}, \mathbf{J} = \gamma\mathbf{E} \quad (\mathbf{E}_e = \mathbf{0}).$$

J is unknown:
time dependent
quasi-static field

In non-conducting media (Ω_i):

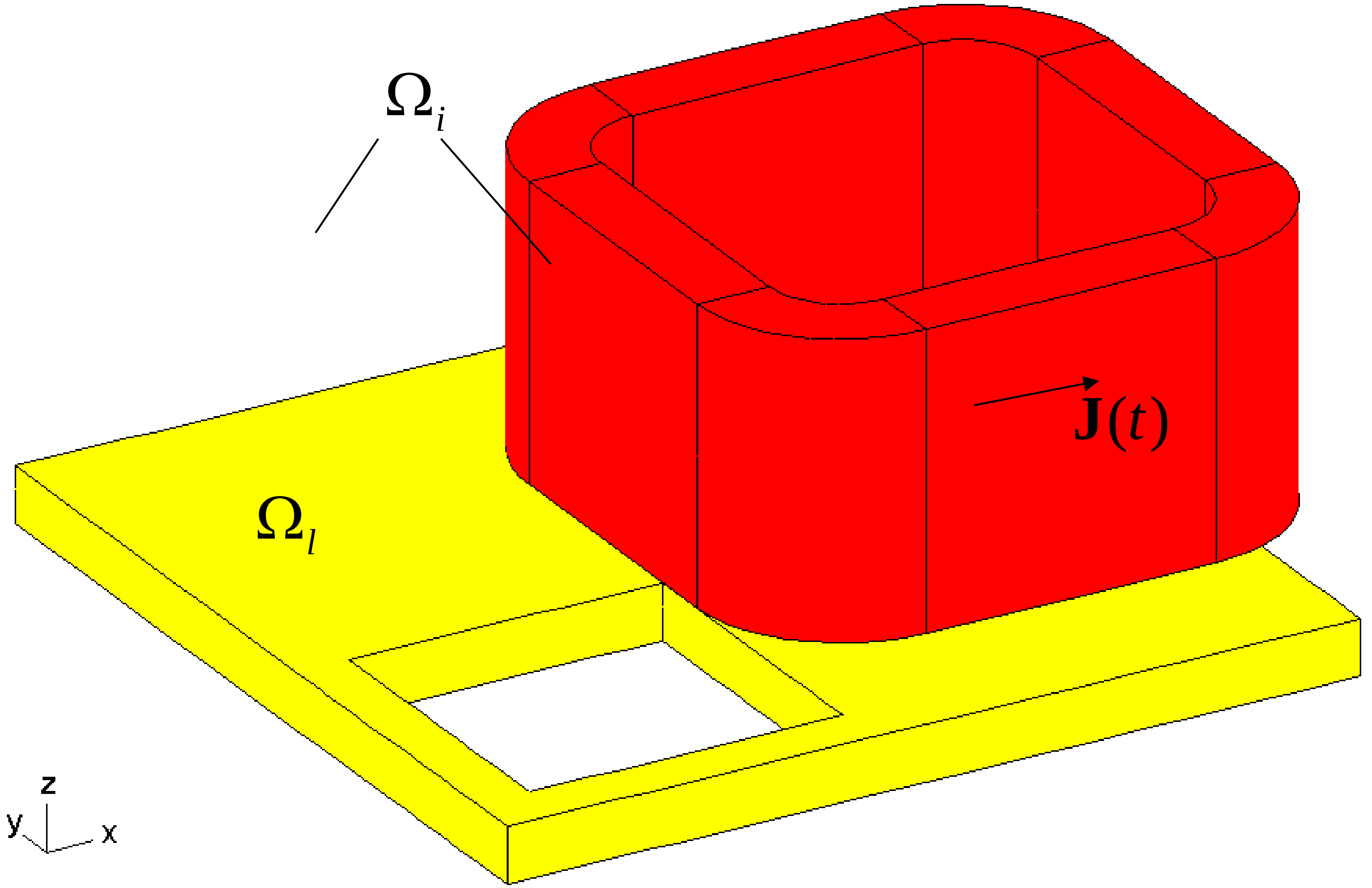
$$curl\mathbf{H} = \mathbf{J},$$

$$div\mathbf{B} = 0,$$

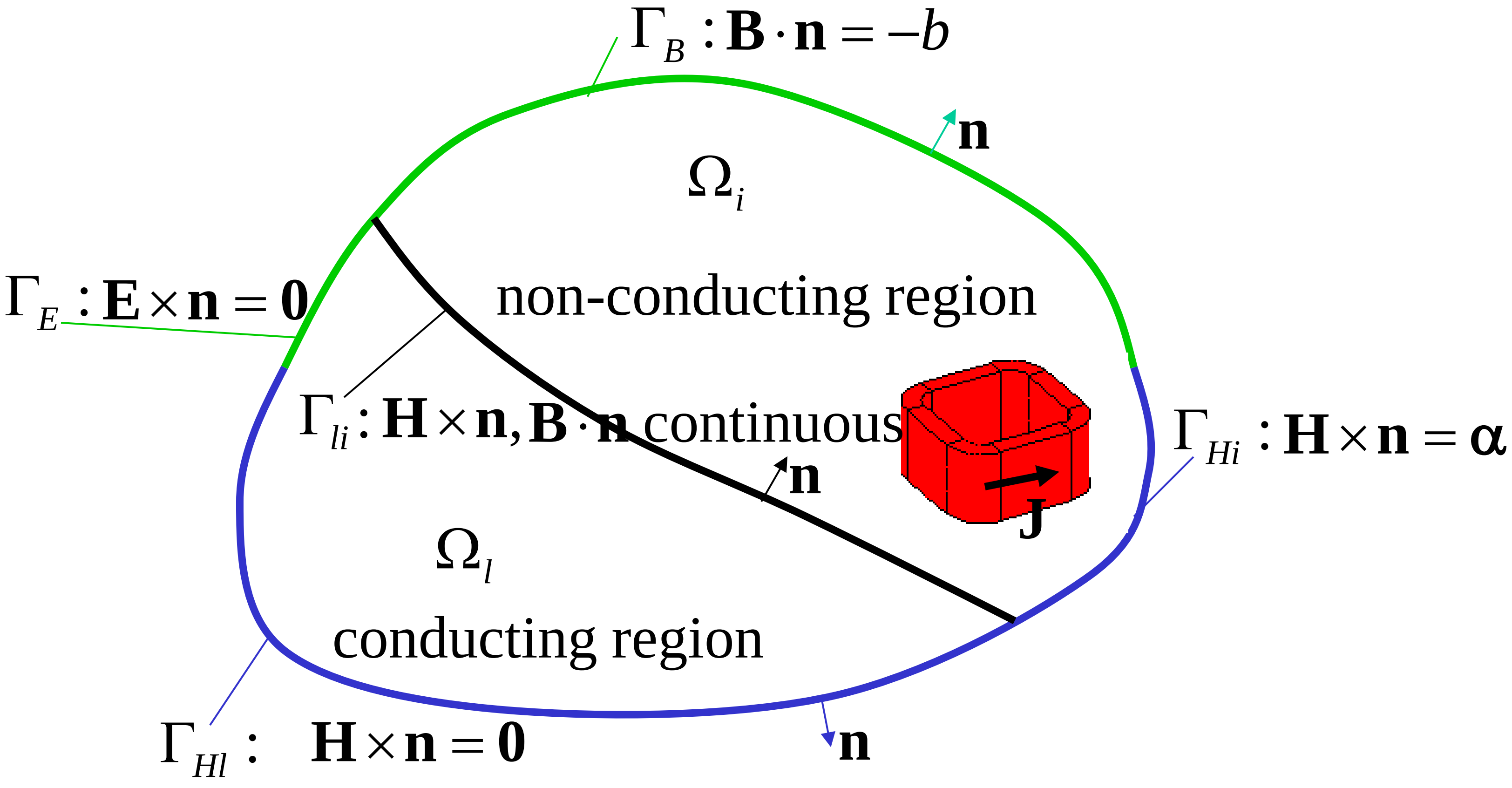
$$\mathbf{B} = \mu\mathbf{H}.$$

J is known:
time dependent
magnetostatic field

Example:



Boundary and interface conditions:



Summary

Differential equations in Ω_l (eddy current region):

$$curl\mathbf{H}_l = \mathbf{J}_l$$

$$curl\mathbf{E}_l = -\frac{\partial \mathbf{B}_l}{\partial t}$$

$$div\mathbf{B}_l = 0$$

$$\mathbf{B}_l = \mu\mathbf{H}_l, \mathbf{H}_l = \nu\mathbf{B}_l, \mathbf{J}_l = \gamma\mathbf{E}_l$$

Differential equations in Ω_i (eddy current free region):

$$curl\mathbf{H}_i = \mathbf{J}_i$$

$$div\mathbf{B}_i = 0$$

$$\mathbf{B}_i = \mu\mathbf{H}_i, \mathbf{H}_i = \nu\mathbf{B}_i$$

boundary conditions:

$$\mathbf{H}_l \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{Hl},$$
$$\mathbf{E}_l \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_E,$$
$$\mathbf{H}_i \times \mathbf{n} = \mathbf{K} \text{ on } \Gamma_{Hi},$$
$$\mathbf{B}_i \cdot \mathbf{n} = -b \text{ on } \Gamma_B.$$

interface conditions on Γ_{li}:

$$\mathbf{H}_l \times \mathbf{n}_l + \mathbf{H}_i \times \mathbf{n}_i = \mathbf{0}$$

$$\mathbf{B}_l \cdot \mathbf{n}_l + \mathbf{B}_i \cdot \mathbf{n}_i = 0$$

initial conditions at *t=0*: $\mathbf{B}_l = \mathbf{B}_{l0}$ in Ω_l, $\mathbf{B}_i = \mathbf{B}_{i0}$ in Ω_i

Complex notation for time harmonic quantities:

Time function : $B_x(\mathbf{r},t) = \hat{B}_x(\mathbf{r})\cos(\omega t + \varphi_x(\mathbf{r}))$

$$B_y(\mathbf{r},t) = \hat{B}_y(\mathbf{r})\cos(\omega t + \varphi_y(\mathbf{r}))$$

$$B_z(\mathbf{r},t) = \hat{B}_z(\mathbf{r})\cos(\omega t + \varphi_z(\mathbf{r}))$$

specially for linear polarization: $\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{B}}(\mathbf{r})\cos(\omega t + \varphi(\mathbf{r}))$

Complex amplitude: $\mathbf{B}(\mathbf{r}) = \hat{\mathbf{B}}(\mathbf{r})e^{j\varphi(\mathbf{r})}$

Time derivative:

$\frac{\partial}{\partial t}$ in time domain $\rightarrow$ multiplication by $j\omega$ in frequency domain

Maxwell's equations for complex amplitudes (quasi-static case):

$$curl\mathbf{H} = \mathbf{J}, \quad curl\mathbf{E} = -j\omega\mathbf{B}, \quad div\mathbf{B} = 0.$$

Poynting's theorem for complex amplitudes in quasi-static case:

$$\frac{1}{2}\mathbf{E}\cdot curl\mathbf{H}^* - \frac{1}{2}\mathbf{H}^*\cdot curl\mathbf{E} = -\frac{1}{2}div\left(\mathbf{E}\times\mathbf{H}^*\right) = \frac{1}{2}\mathbf{E}\cdot\mathbf{J}^* + \frac{1}{2}j\omega\mathbf{B}\cdot\mathbf{H}^*$$

$$\frac{1}{2}\int_\Omega \mathbf{E}\cdot\mathbf{J}^* d\Omega + j\omega\frac{1}{2}\int_\Omega \mathbf{B}\cdot\mathbf{H}^* d\Omega = -\frac{1}{2}\oint_\Gamma \left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\mathbf{n}\,d\Gamma = \underline{S}$$

$\underline{S}$: complex power flowing into the region Ω through the boundary Γ.

Proof:

Time function of Poynting's vector:

$$\mathbf{S}(t)=\mathbf{E}(t)\times\mathbf{H}(t)=\hat{\mathbf{E}}\cos(\omega t+\varphi_E)\times\hat{\mathbf{H}}\cos(\omega t+\varphi_H)=$$

$$=\frac{1}{2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}\left[\cos(\varphi_E-\varphi_H)+\cos(2\omega t+\varphi_E+\varphi_H)\right]=$$

$$=\underbrace{\frac{1}{2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}\cos(\varphi_E-\varphi_H)\left[1+\cos 2(\omega t+\varphi_E)\right]}_{\text{Effective part}}+$$

$$+\underbrace{\frac{1}{2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}\sin(\varphi_E-\varphi_H)\left[\sin 2(\omega t+\varphi_E)\right]}_{\text{Reactive part}}$$

Effective power: $P=-\oint_\Gamma \frac{1}{2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}\cos(\varphi_E-\varphi_H)\cdot\mathbf{n}d\Gamma$

Reactive power: $Q=-\oint_\Gamma \frac{1}{2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}\sin(\varphi_E-\varphi_H)\cdot\mathbf{n}d\Gamma$

$$\underline{S} = P + jQ = -\oint_{\Gamma} \frac{1}{2} \hat{\mathbf{E}} \times \hat{\mathbf{H}} [\cos(\varphi_E - \varphi_H) + j\sin(\varphi_E - \varphi_H)] \cdot \mathbf{n} d\Gamma =$$

$$= -\oint_{\Gamma} \frac{1}{2} \hat{\mathbf{E}} \times \hat{\mathbf{H}} e^{j(\varphi_E - \varphi_H)} \cdot \mathbf{n} d\Gamma = -\oint_{\Gamma} \frac{1}{2} \hat{\mathbf{E}} e^{j\varphi_E} \times \hat{\mathbf{H}} e^{-j\varphi_H} \cdot \mathbf{n} d\Gamma =$$

$$= -\frac{1}{2} \oint_{\Gamma} \left(\mathbf{E} \times \mathbf{H}^* \right) \cdot \mathbf{n} d\Gamma.$$

q. e. d.

Complex Poynting's vector: $$\boxed{\underline{\mathbf{S}} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^*}$$

Effective power: $$P = \frac{1}{2} \int_{\Omega} \mathbf{E} \cdot \mathbf{J}^* d\Omega = \frac{1}{2} \int_{\Omega} \frac{|\mathbf{J}|^2}{\gamma} d\Omega, \quad (\mathbf{E}_e = \mathbf{0}),$$

Reactive power: $$Q = \omega \frac{1}{2} \int_{\Omega} \mathbf{B} \cdot \mathbf{H}^* d\Omega = \frac{1}{2} \omega \int_{\Omega} \mu |\mathbf{H}|^2 d\Omega.$$

3.1 Some analytical solutions of the boundary value problem for the magnetic vector potential

In conducting region (Ω_l):

$$div\mathbf{B} = 0 \Rightarrow \boxed{\mathbf{B} = curl\mathbf{A},}$$

$$curl\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = curl\mathbf{E} + \frac{\partial curl\mathbf{A}}{\partial t} = curl(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \Rightarrow \boxed{\mathbf{E} = -gradV - \frac{\partial \mathbf{A}}{\partial t}.}$$

Differential equations:

$$curl\mathbf{H} - \mathbf{J} = \mathbf{0} \Rightarrow \boxed{curl(\frac{1}{\mu} curl\mathbf{A}) + \gamma \frac{\partial \mathbf{A}}{\partial t} + \gamma gradV = \mathbf{0},}$$

$$\left(div\mathbf{J} = 0 \Rightarrow -div(\gamma gradV) - div(\gamma \frac{\partial \mathbf{A}}{\partial t}) = 0. \right)$$

boundary conditions: $\mathbf{A}(\infty) = 0, V(\infty) = 0.$

Special case: μ = constant, γ = constant, time harmonic case.

Coulomb gauge: $div\mathbf{A}=0$

$$\underbrace{-div(\gamma\, gradV) - j\omega div(\gamma\mathbf{A}) = 0 \;\Rightarrow -\Delta V = j\omega div\mathbf{A} = 0}$$

$$\Downarrow$$

$$\left.\begin{cases} \Delta V = 0 \\ V(\infty) = 0 \end{cases}\right\} \Rightarrow V = 0$$

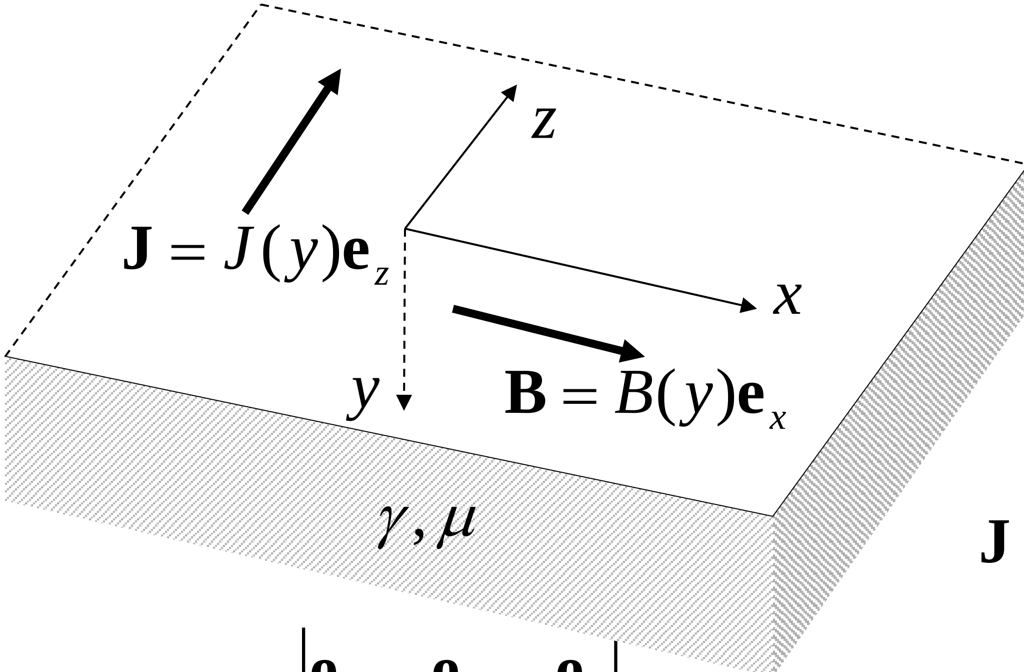
Differential equation in Ω_l:

$$\boxed{-\Delta\mathbf{A} + j\omega\mu\gamma\mathbf{A} = \mathbf{0}}$$ vector diffusion equation.

Planar 2D problems:

$$\boxed{-\Delta A(x,y) + j\omega\mu\gamma A(x,y) = 0}$$ scalar diffusion equation.

3.1.1 Current flow in an infinite conducting half space



$$\frac{\partial}{\partial z} = 0, \frac{\partial}{\partial x} = 0 :$$

$$\mathbf{A} = A(y)\mathbf{e}_z,$$

$$\mathbf{E} = -j\omega A(y)\mathbf{e}_z = E(y)\mathbf{e}_z,$$

$$\mathbf{J} = \gamma\mathbf{E} = -j\omega\gamma A(y)\mathbf{e}_z = J(y)\mathbf{e}_z,$$

$$\mathbf{B} = curl\mathbf{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & A \end{vmatrix} = \frac{dA(y)}{dy}\mathbf{e}_x = B(y)\mathbf{e}_x,$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu}\frac{dA(y)}{dy}\mathbf{e}_x = H(y)\mathbf{e}_x.$$

Diffusion equation: $$-\frac{d^2 A(y)}{dy^2} + j\omega\mu\gamma A(y) = 0.$$

$$j\omega\mu\gamma = p^2,\ \ p = \sqrt{j\omega\mu\gamma} = \frac{1+j}{\sqrt{2/\omega\mu\gamma}} = \frac{1+j}{\delta},$$

δ: penetration depth.

$$\frac{d^2 A(y)}{dy^2} = p^2 A(y)$$

$$A(y) = A_1 e^{-py} + A_2 e^{py}$$

$$\lim_{y\to\infty} |A(y)| < \infty \Rightarrow A_2 = 0:$$

$$A(y) = A_1 e^{-py} = A_1 e^{-\frac{y}{\delta}} e^{-j\frac{y}{\delta}}$$

Field quantities:

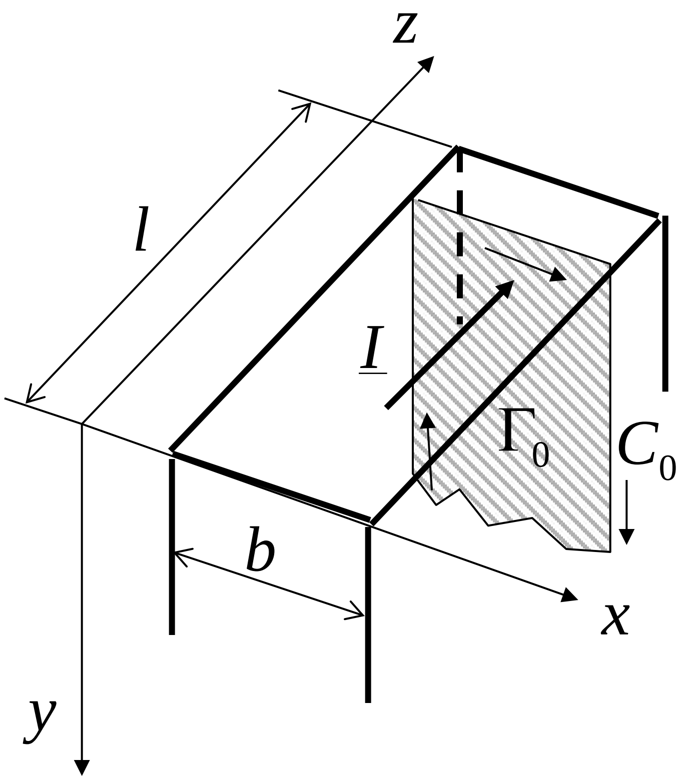
$$E(y) = -j\omega A(y) = -j\omega A_1 e^{-py},$$

$$J(y) = -j\omega\gamma A(y) = -j\omega\gamma A_1 e^{-py},$$

$$B(y) = \frac{dA(y)}{dy} = -pA_1 e^{-py},$$

$$H(y) = \frac{1}{\mu}\frac{dA(y)}{dy} = -\frac{p}{\mu} A_1 e^{-py}.$$

Determination of the constant A_1: the current through a conductor of width b is assumed to be given.



$$\int_{\Gamma_0} \mathbf{J}\cdot\mathbf{n}\,d\Gamma = \oint_{C_0} \mathbf{H}\cdot d\mathbf{r} = H(y=0)b = \underline{I}$$

$$-\frac{pb}{\mu}A_1 = \underline{I} \Rightarrow A_1 = -\frac{\mu \underline{I}}{pb}$$

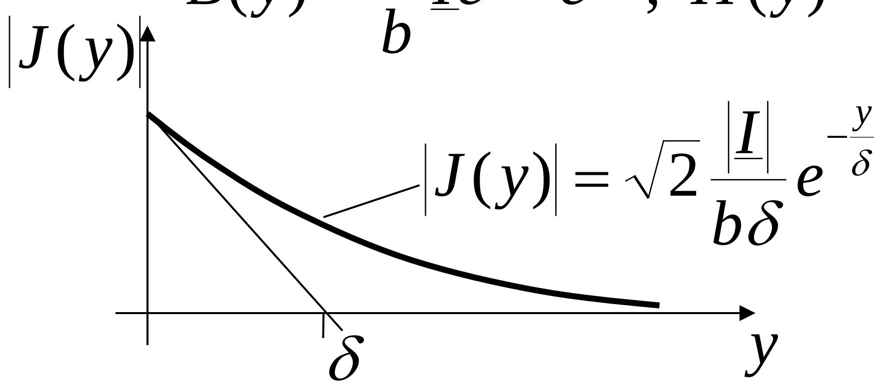
$$E(y) = \frac{j\omega\mu}{pb}\underline{I}e^{-jpy} = \frac{p}{\gamma b}\underline{I}e^{-j\frac{y}{\delta}}e^{-\frac{y}{\delta}},$$

$$J(y) = \gamma E(y) = \frac{p}{b}\underline{I}e^{-j\frac{y}{\delta}}e^{-\frac{y}{\delta}},$$

Skin effect:

$$B(y) = \frac{\mu}{b}\underline{I}e^{-j\frac{y}{\delta}}e^{-\frac{y}{\delta}},\ H(y) = \frac{\underline{I}}{b}e^{-j\frac{y}{\delta}}e^{-\frac{y}{\delta}}.$$

Magnitude of the current density decays exponentially



Impedance of a conductor of width *b* and length *l*:

$$\underline{S} = \frac{1}{2}\left|\underline{I}\right|^2 \underline{Z} = -\frac{1}{2}\oint_\Gamma \left(\mathbf{E}\times\mathbf{H}^*\right)\cdot \mathbf{n}\,d\Gamma$$

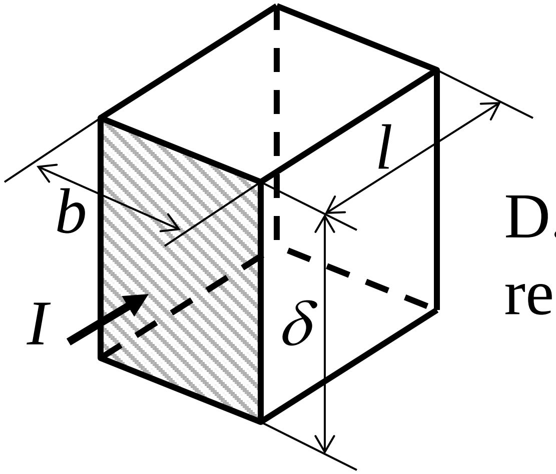
$$\mathbf{E}\times\mathbf{H}^* = E(y)\mathbf{e}_z \times H^*(y)\mathbf{e}_x = E(y)H^*(y)\mathbf{e}_y$$

$$\mathbf{n} = -\mathbf{e}_y \text{ for } y = 0, \text{ otherwise } \mathbf{n}\perp\mathbf{e}_y.$$

$$\frac{1}{2}\left|\underline{I}\right|^2 \underline{Z} = \frac{1}{2}E(y=0)H^*(y=0)bl = \frac{1}{2}\frac{p}{\gamma b}\underline{I}\,\frac{\underline{I}^*}{b}bl = \frac{1}{2}\left|\underline{I}\right|^2\frac{pl}{\gamma b}.$$

$$\underline{Z} = R + jX = (1+j)\frac{l}{\gamma\delta b},$$

$$R = \frac{l}{\gamma\delta b}, \quad X = \frac{l}{\gamma\delta b}.$$



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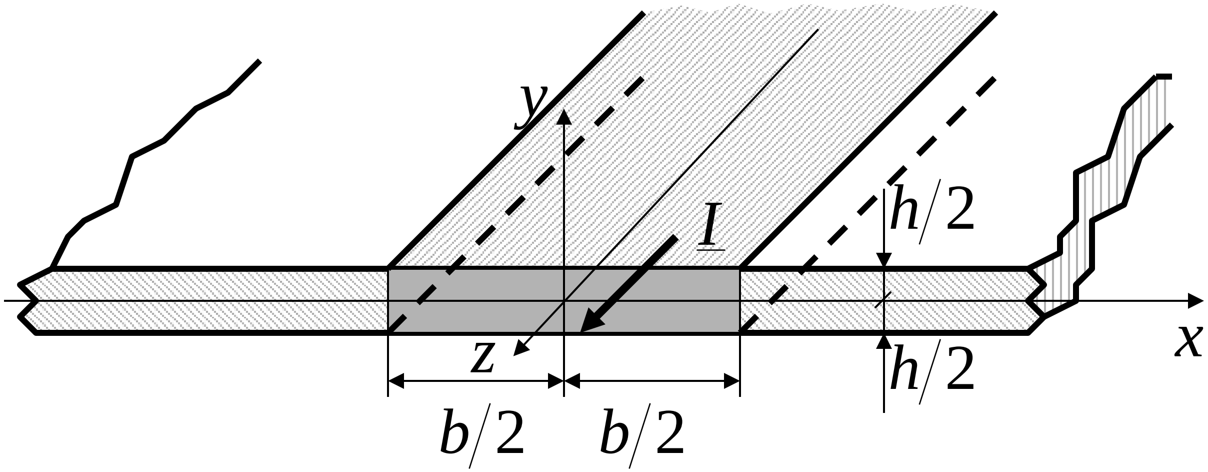
3.1.2 Current flow in an infinite conducting plate



$$\mathbf{E} = -j\omega A(y)\mathbf{e}_z = E(y)\mathbf{e}_z, \quad \mathbf{J} = \gamma\mathbf{E} = -j\omega\gamma A(y)\mathbf{e}_z = J(y)\mathbf{e}_z,$$

$$\mathbf{B} = \frac{dA(y)}{dy}\mathbf{e}_x = B(y)\mathbf{e}_x, \qquad \mathbf{H} = \frac{1}{\mu}\frac{dA(y)}{dy}\mathbf{e}_x = H(y)\mathbf{e}_x.$$

Diffusion equation:

$$\boxed{\frac{d^2A(y)}{dy^2} = p^2A(y),} \qquad p = \sqrt{j\omega\mu\gamma} = \frac{1+j}{\sqrt{2/\omega\mu\gamma}} = \frac{1+j}{\delta}.$$

Solution of the diffusion equation:

$$A(y) = A_1 e^{-py} + A_2 e^{py} = C_1 \cosh(py) + C_2 \sinh(py).$$

The current density $J(y) = -j\omega\gamma A(y)$ has to be an even function ($J(y) = J(-y)$): $C_2 = 0.$

$$\boxed{A(y) = C_1 \cosh(py).}$$

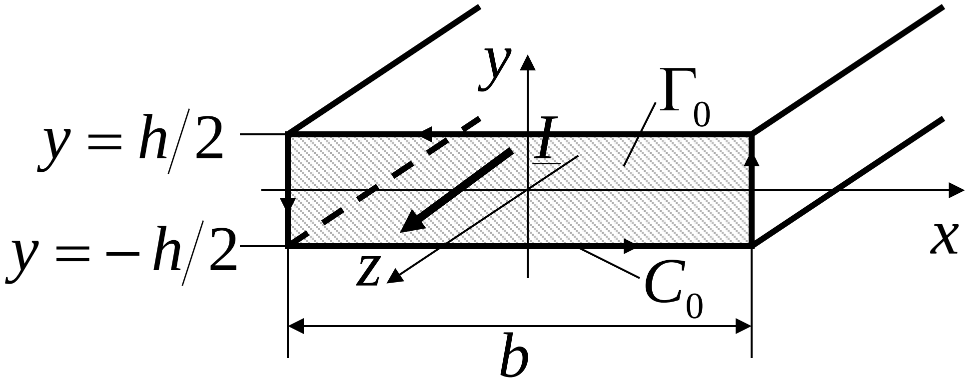
Field quantities: $E(y) = -j\omega A(y) = -j\omega C_1 \cosh(py)$

$$J(y) = -j\omega\gamma A(y) = -j\omega\gamma C_1 \cosh(py)$$

$$B(y) = \frac{dA(y)}{dy} = pC_1 \sinh(py)$$

$$H(y) = \frac{1}{\mu}\frac{dA(y)}{dy} = \frac{p}{\mu} C_1 \sinh(py)$$

Determination of the constant C_1: Current through a conductor of width b is assumed to be given.



$$\int_{\Gamma_0} \mathbf{J}\cdot\mathbf{n}\,d\Gamma = \oint_{C_0} \mathbf{H}\cdot d\mathbf{r} = -H(y=h/2)b + H(y=-h/2)b =$$

$$= -2H(y=h/2)b = -\frac{2pb}{\mu} C_1 \sinh(\frac{ph}{2}) = \underline{I},$$

$$C_1 = -\frac{\mu \underline{I}}{2pb\sinh(\frac{ph}{2})}.$$

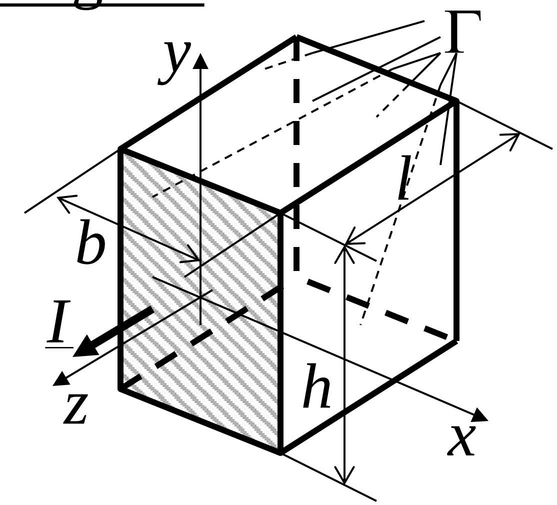
$$E(y) = \frac{j\omega\mu \underline{I}}{2pb\sinh(\frac{ph}{2})}\cosh(py) = \frac{p\underline{I}}{2\gamma b\sinh(\frac{ph}{2})}\cosh(py),$$

$$J(y) = \gamma E(y) = \frac{p\underline{I}}{2b\sinh(\frac{ph}{2})}\cosh(py),$$

$$B(y) = -\frac{\mu\underline{I}}{2b\sinh(\frac{ph}{2})}\sinh(py),$$

$$H(y) = -\frac{\underline{I}}{2b\sinh(\frac{ph}{2})}\sinh(py).$$

Impedance of a conductor of width b and length l:



$$\underline{S} = \frac{1}{2}|\underline{I}|^2 \underline{Z} = -\frac{1}{2}\oint_\Gamma \left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\mathbf{n}\,d\Gamma$$

$$\mathbf{E}\times\mathbf{H}^* = E(y)\mathbf{e}_z \times H^*(y)\mathbf{e}_x = E(y)H^*(y)\mathbf{e}_y$$

$\mathbf{n} = \mathbf{e}_y$ for $y = h/2$, $\mathbf{n} = -\mathbf{e}_y$ for $y = -h/2$, otherwise $\mathbf{n} \perp \mathbf{e}_y$.

$$\frac{1}{2}|\underline{I}|^2 \underline{Z} = -\frac{1}{2}E(y=\frac{h}{2})H^*(y=\frac{h}{2})bl + \frac{1}{2}E(y=-\frac{h}{2})H^*(y=-\frac{h}{2})bl =$$

$$= -E(y=\frac{h}{2})H^*(y=\frac{h}{2})bl = \frac{p\underline{I}\cosh(\frac{ph}{2})}{2\gamma b\sinh(\frac{ph}{2})}\frac{\underline{I}^*}{2b}bl.$$

$$\underline{Z} = \frac{pl\cosh(\frac{ph}{2})}{2\gamma b\sinh(\frac{ph}{2})}, \qquad \frac{ph}{2} = \frac{1+j}{2}\frac{h}{\delta} = \frac{1+j}{2}h\sqrt{\frac{\omega\mu\gamma}{2}}.$$

Low frequency: $\left|\frac{ph}{2}\right| << 1 \Rightarrow \cosh(\frac{ph}{2}) \approx 1, \sinh(\frac{ph}{2}) \approx \frac{ph}{2}.$

$\underline{Z} \approx \frac{l}{\gamma bh}$: like D. C. in the entire height h.

High frequency: $\left|\frac{ph}{2}\right| >> 1 \Rightarrow \cosh(\frac{ph}{2}) \approx \sinh(\frac{ph}{2}).$

$\underline{Z} \approx \frac{pl}{2\gamma b} = (1+j)\frac{l}{2\gamma\delta b}$: D. C. resistance of two layers each of thickness δ, reactance same as resistance.

4. Electromagnetic waves

The full set of Maxwell's equations:

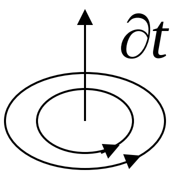
$$curl\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},\ curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$div\mathbf{B} = 0,\ div\mathbf{D} = \rho, \qquad \mathbf{D} = \varepsilon\mathbf{E},\ \mathbf{B} = \mu\mathbf{H},\ \mathbf{J} = \gamma(\mathbf{E} + \mathbf{E}_e)$$

describes electromagnetic waves.

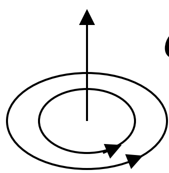
in vacuum:

$$\frac{\partial \mathbf{D}}{\partial t}$$



$$\mathbf{H}(t)$$

$$-\frac{\partial \mathbf{B}}{\partial t}$$



$$\mathbf{E}(t)$$

The time varying electric and magnetic fields mutually sustain each other:
electromagnetic field

4.1 Planar waves

Maxwell's equations in vacuum ($\mu = \mu_0$, $\varepsilon = \varepsilon_0$), in absence of charges and currents ($\rho = 0$, $\mathbf{J} = \mathbf{0}$):

$$curl\mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad curl\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t},$$

$$div\mathbf{H} = 0, \qquad div\mathbf{E} = 0.$$

$$curlcurl\mathbf{H} = \underbrace{graddiv\mathbf{H}}_{=\mathbf{0}} - \Delta\mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t} curl\mathbf{E} = -\varepsilon_0\mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2},$$

$$\Delta\mathbf{H} = \varepsilon_0\mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2},$$

Similarly: $$\Delta\mathbf{E} = \varepsilon_0\mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

} *vector 3D wave equation*

Assumption: $\frac{\partial}{\partial x} = 0, \frac{\partial}{\partial y} = 0.$ The electromagnetic field is constant in any plane z = constant:

Planar waves:

$$\frac{\partial^2 \mathbf{H}}{\partial z^2} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2}, \qquad \frac{\partial^2 \mathbf{E}}{\partial z^2} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}:$$

Vector 1D wave equation.

All components E_x, E_y, E_z, H_x, H_y, H_z of the electromagnetic field satisfy the scalar 1D wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \varepsilon_0 \mu_0 \frac{\partial^2 f}{\partial t^2}. \qquad \text{Solution: } f(t \mp \frac{z}{v}),\ v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c.$$

Waves propagating with light velocity:

$$E_x(z,t) = E_x(t \mp \frac{z}{c}), \quad E_y(z,t) = E_y(t \mp \frac{z}{c}), \quad E_z(z,t) = E_z(t \mp \frac{z}{c}),$$

$$H_x(z,t) = H_x(t \mp \frac{z}{c}), \quad H_y(z,t) = H_y(t \mp \frac{z}{c}), \quad H_z(z,t) = H_z(t \mp \frac{z}{c}).$$

Relationship between **E** and **H**:

$$curl\mathbf{H} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

Planar waves: $\frac{\partial}{\partial x} = 0, \; \frac{\partial}{\partial y} = 0, \; \frac{\partial}{\partial z} = \mp \frac{1}{c}\frac{\partial}{\partial t}.$

$$curl\mathbf{H} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & \mp\frac{1}{c}\frac{\partial}{\partial t} \\ H_x & H_y & H_z \end{vmatrix} = \mp\frac{1}{c}\frac{\partial}{\partial t}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ H_x & H_y & H_z \end{vmatrix} = \mp\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{e}_z \times \mathbf{H}).$$

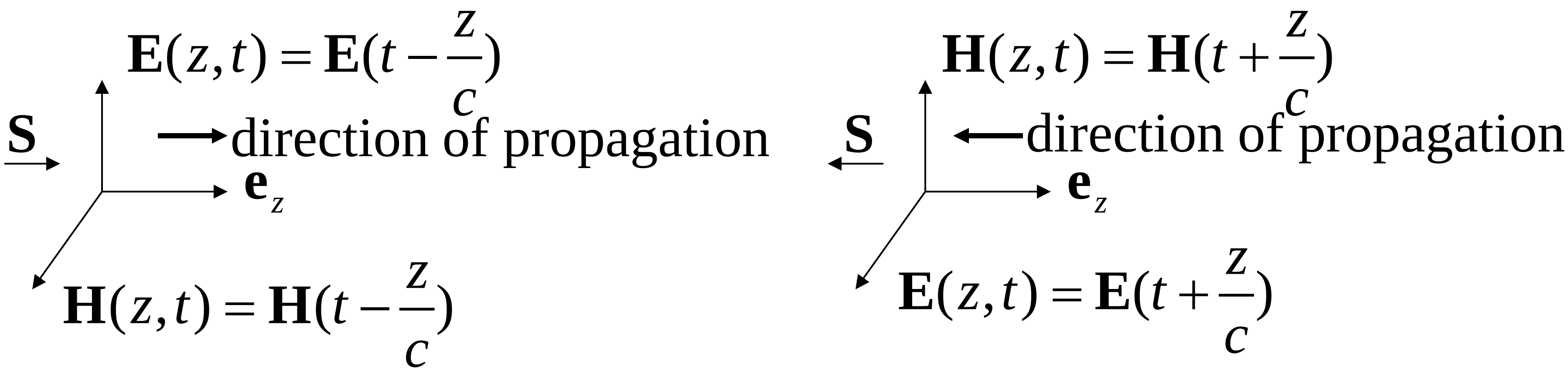
$$\mp\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{e}_z \times \mathbf{H}) = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \Rightarrow \mp\frac{1}{c}(\mathbf{e}_z \times \mathbf{H}) = \varepsilon_0\mathbf{E}; \quad \boxed{(\mathbf{e}_z \times \mathbf{H}) = \mp\sqrt{\frac{\varepsilon_0}{\mu_0}}\mathbf{E}.}$$

Similarly, from Faraday's law:

$$\boxed{(\mathbf{e}_z \times \mathbf{E}) = \pm\sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{H}.}$$

Sign above: propagation in the positive z-direction,

Sign below: propagation in the negative z-direction.



The direction of Poynting's vector coincides with the direction of propagation.

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\mp\sqrt{\frac{\mu_0}{\varepsilon_0}}(\mathbf{e}_z\times\mathbf{H})\times\mathbf{H}=\pm\mathbf{e}_z\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\mathbf{H}\right|^2=\pm\mathbf{e}_z\frac{1}{\sqrt{\mu_0\varepsilon_0}}\mu_0\left|\mathbf{H}\right|^2,$$

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\mathbf{E}\times[\pm\sqrt{\frac{\varepsilon_0}{\mu_0}}(\mathbf{e}_z\times\mathbf{E})]=\pm\mathbf{e}_z\sqrt{\frac{\varepsilon_0}{\mu_0}}\left|\mathbf{E}\right|^2=\pm\mathbf{e}_z\frac{1}{\sqrt{\mu_0\varepsilon_0}}\varepsilon_0\left|\mathbf{E}\right|^2,$$

$$\boxed{\mathbf{S}=\pm\mathbf{e}_z c\frac{1}{2}(\varepsilon_0\left|\mathbf{E}\right|^2+\mu_0\left|\mathbf{H}\right|^2).}$$

More general material properties: $\mu, \varepsilon, \gamma =$ konstant.

Maxwell's equations in absence of charges ($\rho = 0$):

$$curl\mathbf{H} = \gamma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}, \quad curl\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$

$$div\mathbf{H} = 0, \qquad div\mathbf{E} = 0.$$

$$curlcurl\mathbf{H} = \underbrace{graddiv\mathbf{H}}_{=\mathbf{0}} - \Delta\mathbf{H} = \gamma curl\mathbf{E} + \varepsilon\frac{\partial}{\partial t}curl\mathbf{E} = -\gamma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2},$$

$$\Delta\mathbf{H} - \gamma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2} = \mathbf{0},$$

Similarly: $$\Delta\mathbf{E} - \gamma\mu\frac{\partial\mathbf{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} = \mathbf{0}.$$

For planar waves: $\frac{\partial}{\partial x}=0, \frac{\partial}{\partial y}=0 \Rightarrow \Delta=\frac{\partial^2}{\partial z^2}.$

$$\frac{\partial^2\mathbf{E}}{\partial z^2}=\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}+\mu\gamma\frac{\partial\mathbf{E}}{\partial t}, \qquad \frac{\partial^2\mathbf{H}}{\partial z^2}=\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}+\mu\gamma\frac{\partial\mathbf{H}}{\partial t}.$$

Total analogy with the transmission line equations:

$$\frac{\partial^2 u}{\partial z^2}=LC\frac{\partial^2 u}{\partial t^2}+(RC+LG)\frac{\partial u}{\partial t}+RGu,$$

$$\frac{\partial^2 i}{\partial z^2}=LC\frac{\partial^2 i}{\partial t^2}+(RC+LG)\frac{\partial i}{\partial t}+RGi.$$

$$\boxed{u \Leftrightarrow \mathbf{E},\ i \Leftrightarrow \mathbf{H},\ R \Leftrightarrow 0,\ L \Leftrightarrow \mu,\ G \Leftrightarrow \gamma,\ C \Leftrightarrow \varepsilon.}$$

Time harmonic case, complex notation

$\mathbf{E}(z), \mathbf{H}(z):$ complex amplitudes.

Solution (due to analogy):

$$\mathbf{E}(z) = \mathbf{E}^{+}e^{-pz} + \mathbf{E}^{-}e^{pz}, \quad \mathbf{H}(z) = \frac{\mathbf{E}^{+}}{Z_0}e^{-pz} - \frac{\mathbf{E}^{-}}{Z_0}e^{pz}.$$

Propagation coefficient: $p = \sqrt{j\omega\mu(\gamma + j\omega\varepsilon)} = \alpha + j\beta,$

Wave impedance: $Z_0 = \sqrt{\dfrac{j\omega\mu}{\gamma + j\omega\varepsilon}}.$

Attenuated waves propagating in the positive and negative z-direction.

Lossless medium: $\gamma = 0.$

$$p = \sqrt{(j\omega\mu)(j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon} \Rightarrow \alpha = 0,\ \beta = \omega\sqrt{\mu\varepsilon}.$$

$$Z_0 = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}.$$

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_r\varepsilon_r}}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \le c.$$

Optics: $v = \dfrac{c}{n}$, n: refraction index

Maxwell's relationship: $\boxed{n = \sqrt{\varepsilon_r};\ n^2 = \varepsilon_r,}$

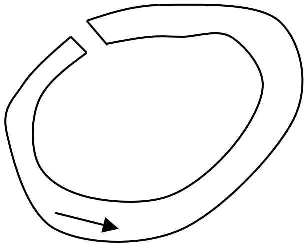
since for optically transparent media $\mu_r = 1$.

4.2 Electromagnetic waves in homogeneous, infinite space

Assumptions:

- current density **J** and charge density ρ are known everywhere at any time instant: $\mathbf{J}(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$ are given,
- material properties μ and ε are constant everywhere, e. g. $\mu = \mu_0$, $\varepsilon = \varepsilon_0$,
- lossless medium: $\gamma = 0$.

e. g. antenna:



$\mathbf{J}(\mathbf{r},t)$

homogeneous medium
(e. g. vacuum or air: μ_0, ε_0)

electromagnetic field: $\mathbf{E}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$

4.2.1. Solution of Maxwell's equations with retarded potentials

Maxwell's equations:

$$curl\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$div\mathbf{B} = 0, \qquad div\mathbf{D} = \rho, \qquad \mathbf{B} = \mu_0\mathbf{H},\ \mathbf{D} = \varepsilon_0\mathbf{E}.$$

Potentials: $\mathbf{B} = curl\mathbf{A},\ \mathbf{H} = \frac{1}{\mu_0} curl\mathbf{A},$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - gradV,\ \mathbf{D} = -\varepsilon_0 \frac{\partial \mathbf{A}}{\partial t} - \varepsilon_0 gradV.$$

$$curlcurl\mathbf{A} = graddiv\mathbf{A} - \Delta\mathbf{A} = \mu_0\mathbf{J} - \mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \mu_0\varepsilon_0 grad\frac{\partial V}{\partial t}.$$

The divergence of **A** can be freely chosen:

$$div\mathbf{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}: \text{ Lorenz gauge.}$$

$$\boxed{-\Delta\mathbf{A} + \mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}.}$$

$$div(-gradV - \frac{\partial\mathbf{A}}{\partial t}) = -\Delta V - \frac{\partial div\mathbf{A}}{\partial t} = \frac{\rho}{\varepsilon_0}.$$

Non-homogeneous 3D wave equations

Using the Lorenz gauge:

$$\boxed{-\Delta V + \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\varepsilon_0}.}$$

In the static case $(\frac{\partial}{\partial t} = 0)$ these equations reduce to the Laplace-Poisson equations $-\Delta \mathbf{A} = \mu_0 \mathbf{J}$, $-\Delta V = \frac{\rho}{\varepsilon_0}$

whose solutions are known to be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_\Omega \frac{\mathbf{J}(\mathbf{r}')d\Omega'}{|\mathbf{r} - \mathbf{r}'|}, \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_\Omega \frac{\rho(\mathbf{r}')d\Omega'}{|\mathbf{r} - \mathbf{r}'|}.$$

In regions with vanishing current density and charge density, one obtains the wave equations

$$-\Delta \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{0}, \qquad -\Delta V + \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = 0.$$

The solutions of the non-homogeneous wave equations in infinite free space are

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})d\Omega'}{|\mathbf{r}-\mathbf{r}'|},$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})d\Omega'}{|\mathbf{r}-\mathbf{r}'|}, \qquad (c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}).$$

$\mathbf{A}(\mathbf{r},t)$ and $V(\mathbf{r},t)$ are the *retarded* potentials.

Time harmonic case:

$\mathbf{A}(\mathbf{r}),\ V(\mathbf{r}),\ \mathbf{J}(\mathbf{r}')$ and $\rho(\mathbf{r}')$ are complex amplitudes.

In time domain: $\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) = \hat{\mathbf{J}}(\mathbf{r}')\cos[\omega(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) + \varphi(\mathbf{r}')]$.

In frequency domain: $\hat{\mathbf{J}}(\mathbf{r}')e^{j\varphi(\mathbf{r}')}e^{-j\omega\frac{|\mathbf{r}-\mathbf{r}'|}{c}} = \mathbf{J}(\mathbf{r}')e^{-jk_0|\mathbf{r}-\mathbf{r}'|}$,

$k_0 = \frac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0}$: wave number (phase factor).

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\mathbf{J}(\mathbf{r}')e^{-jk_0|\mathbf{r}-\mathbf{r}'|}d\Omega'}{|\mathbf{r}-\mathbf{r}'|}, \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{\rho(\mathbf{r}')e^{-jk_0|\mathbf{r}-\mathbf{r}'|}d\Omega'}{|\mathbf{r}-\mathbf{r}'|}.$$

Using the Lorenz gauge, *V* can be eliminated:

$$div\mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$$ is in the frequency domain $div\mathbf{A} = -j\omega\mu_0\varepsilon_0 V.$

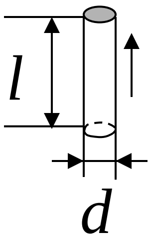
$$V = -\frac{1}{j\omega\mu_0\varepsilon_0} div\mathbf{A}.$$

$$\mathbf{E} = -j\omega\mathbf{A} - gradV = -j\omega\mathbf{A} + \frac{1}{j\omega\mu_0\varepsilon_0} graddiv\mathbf{A} =$$

$$= \frac{1}{j\omega\mu_0\varepsilon_0}(\omega^2\mu_0\varepsilon_0\mathbf{A} + graddiv\mathbf{A}).$$

$$\mathbf{B} = curl\mathbf{A}, \quad \mathbf{E} = \frac{1}{j\omega\mu_0\varepsilon_0}(k_0^2\mathbf{A} + graddiv\mathbf{A}).$$

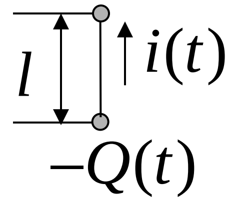
4.2.2. Hertz dipole



$$i(t) = \hat{I}\cos(\omega t) \quad l \ll \lambda = \frac{2\pi}{k_0}, \quad d \ll l.$$

$$Q(t) = \hat{Q}\sin(\omega t)$$

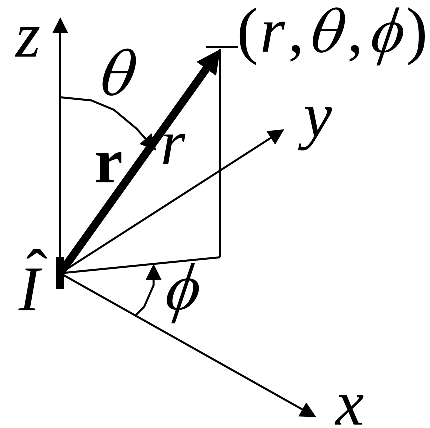
Equivalent to a dipole:



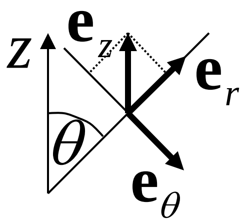
$\hat{I}$ can be considered to be the complex amplitude of the current.

$$\hat{I} = j\omega\hat{Q}$$

Spherical coordinate system:



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\Omega}\frac{\mathbf{J}(\mathbf{r}')e^{-jk_0|\mathbf{r}-\mathbf{r}'|}d\Omega'}{|\mathbf{r}-\mathbf{r}'|}, \quad \mathbf{r}'=\mathbf{0},\ \mathbf{J}(\mathbf{0})d\Omega' = \hat{I}l\mathbf{e}_z,\ |\mathbf{r}| = r.$$



$$\mathbf{e}_z = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta.$$

No integration is necessary, since the integrand is constant:

$$\mathbf{A}(r,\theta,\phi) = \frac{\mu_0\hat{I}l}{4\pi}\frac{e^{-jk_0r}}{r}(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta).$$

Magnetic field:

$$\mathbf{H} = \frac{1}{\mu_0} rot\mathbf{A} = \frac{1}{\mu_0}\begin{vmatrix} \frac{1}{r^2 \sin\theta}\mathbf{e}_r & \frac{1}{r\sin\theta}\mathbf{e}_\theta & \frac{1}{r}\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\ A_r & rA_\theta & 0 \end{vmatrix} = \mathbf{e}_\phi \frac{1}{\mu_0 r}[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}].$$

$$\mathbf{H} = \frac{\hat{I}l}{4\pi} e^{-jk_0 r}(\frac{1}{r^2} + j\frac{k_0}{r})\sin\theta\mathbf{e}_\phi .$$

$$H_r = 0, \quad H_\theta = 0, \quad H_\phi = \frac{\hat{I}l}{4\pi} jk_0(1 + \frac{1}{jk_0 r})\frac{e^{-jk_0 r}}{r}\sin\theta.$$

Electric field: $$\mathbf{E} = \frac{1}{j\omega\mu_0\varepsilon_0}(k_0^2\mathbf{A} + graddiv\mathbf{A}).$$

$$div\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} = \frac{\mu_0 \hat{I}l}{4\pi}e^{-jk_0 r}(-\frac{1}{r^2} - \frac{jk_0}{r})\cos\theta,$$

$$\left(graddiv\mathbf{A}\right)_r = \frac{\partial div\mathbf{A}}{\partial r} = \frac{\mu_0 \hat{I}l}{4\pi}e^{-jk_0 r}(\frac{2}{r^3} + 2\frac{jk_0}{r^2} - \frac{k_0^2}{r})\cos\theta,$$

$$\left(graddiv\mathbf{A}\right)_\theta = \frac{1}{r}\frac{\partial div\mathbf{A}}{\partial\theta} = \frac{\mu_0 \hat{I}l}{4\pi}e^{-jk_0 r}(\frac{1}{r^3} + \frac{jk_0}{r^2})\sin\theta,$$

$$E_r = \frac{\hat{I}l}{2\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}(1 + \frac{1}{jk_0 r})\frac{e^{-jk_0 r}}{r^2}\cos\theta,$$

$$E_\theta = \frac{\hat{I}l}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}jk_0[1 + \frac{1}{jk_0 r} + \frac{1}{(jk_0 r)^2}]\frac{e^{-jk_0 r}}{r}\sin\theta, \quad E_\phi = 0.$$

Near field: $k_0 r = 2\pi \frac{r}{\lambda} \ll 1.$

$$H_\phi = \frac{\hat{I}l}{4\pi} jk_0 (1 + \frac{1}{jk_0 r}) \frac{e^{-jk_0 r}}{r} \sin\theta \approx \frac{\hat{I}l}{4\pi} \frac{e^{-jk_0 r}}{r^2} \sin\theta.$$

Biot-Savart: $\mathbf{H}(\mathbf{r}) = \frac{1}{4\pi} \int_\Omega \frac{\mathbf{J}(\mathbf{r}') \times \mathbf{e}_{\mathbf{r}'\to\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|^2} d\Omega' \Rightarrow \mathbf{H}(\mathbf{r}) = \frac{\hat{I}l}{4\pi} \frac{\overbrace{\mathbf{e}_z \times \mathbf{e}_r}^{\mathbf{e}_\phi}}{r^2}.$

$$E_r = \frac{\hat{I}l}{2\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} (1 + \frac{1}{jk_0 r}) \frac{e^{-jk_0 r}}{r^2} \cos\theta \approx \frac{\hat{Q}l}{2\pi\varepsilon_0} \frac{e^{-jk_0 r}}{r^3} \cos\theta,$$

$$E_\theta = \frac{\hat{I}l}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} jk_0 [1 + \frac{1}{jk_0 r} + \frac{1}{(jk_0 r)^2}] \frac{e^{-jk_0 r}}{r} \sin\theta \approx \frac{\hat{Q}l}{4\pi\varepsilon_0} \frac{e^{-jk_0 r}}{r^3} \sin\theta.$$

Static dipole field: $E_r = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}.$

$$\text{Far field: } k_0 r = 2\pi \frac{r}{\lambda} \gg 1.$$

$$H_\phi = \frac{\hat{I}l}{4\pi} jk_0 (1 + \frac{1}{jk_0 r}) \frac{e^{-jk_0 r}}{r} \sin\theta \approx \frac{\hat{I}l}{4\pi} j\beta \frac{e^{-jk_0 r}}{r} \sin\theta = \boxed{\frac{j\hat{I}}{2} \frac{l}{\lambda} \frac{e^{-jk_0 r}}{r} \sin\theta.}$$

$$E_r = \frac{\hat{I}l}{2\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} (1 + \frac{1}{jk_0 r}) \frac{e^{-jk_0 r}}{r^2} \cos\theta \approx 0,$$

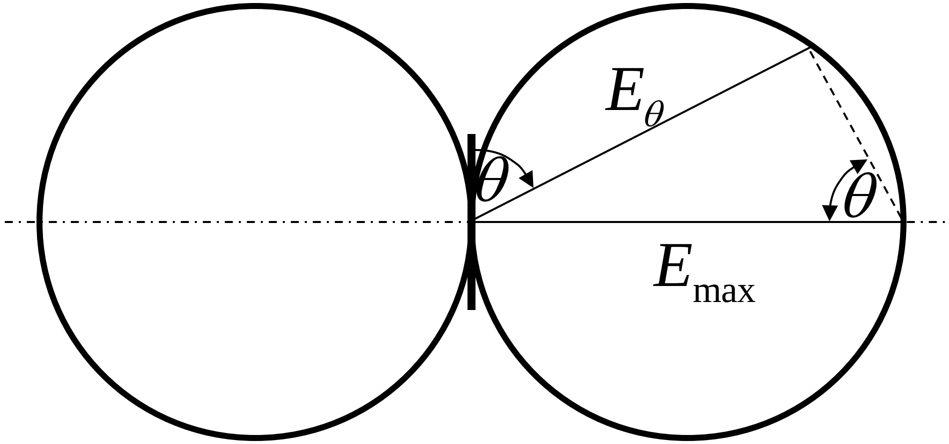
$$E_\theta = \frac{\hat{I}l}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} jk_0 [1 + \frac{1}{jk_0 r} + \frac{1}{(jk_0 r)^2}] \frac{e^{-jk_0 r}}{r} \sin\theta \approx \frac{\hat{I}l}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} jk_0 \frac{e^{-jk_0 r}}{r} \sin\theta =$$

$$= \boxed{\frac{j\hat{I}}{2} \frac{l}{\lambda} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{e^{-jk_0 r}}{r} \sin\theta.}$$

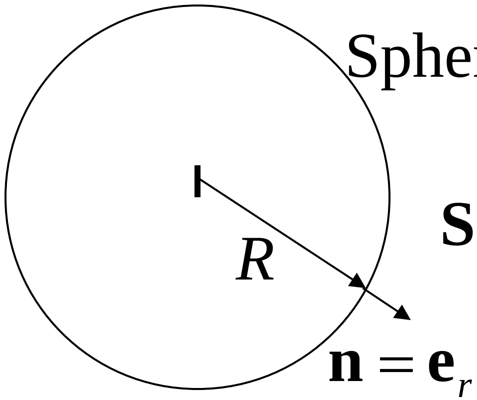
$$\frac{E_\theta^{(Fern)}}{H_\phi^{(Fern)}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 \approx 120\pi\,\Omega \approx 377\,\Omega.$$

$$E_\theta^{(Fern)}(\theta) = E_0(r)\sin\theta,$$

$$\frac{E_\theta^{(Fern)}(\theta)}{E_\theta^{(Fern)}(\theta_{\max})} = \sin\theta: \text{ radiation pattern.}$$



Radiated power:



Sphere $R \gg \lambda$: far field.

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{1}{2}E_\theta H_\phi^* \mathbf{e}_\theta \times \mathbf{e}_\phi = \frac{1}{2}E_\theta H_\phi^* \mathbf{e}_r .$$

$$\frac{1}{2}E_\theta H_\phi^* = \frac{\left|\hat{I}\right|^2}{8}\left(\frac{l}{\lambda}\right)^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{R^2}\sin^2\theta.$$

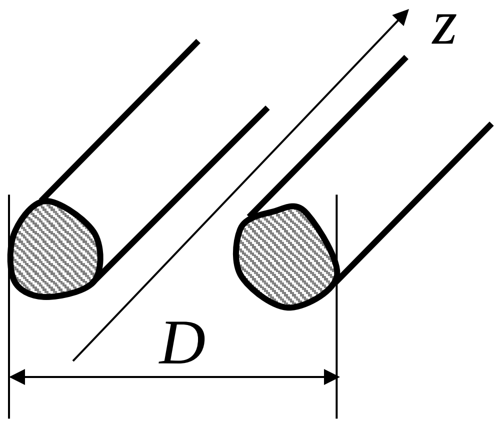
$$P = \oint_{Kugel} \mathbf{S}\cdot\mathbf{n}\,d\Gamma. \quad d\Gamma = R d\theta R \sin\theta d\phi = R^2 \sin\theta d\theta d\phi.$$

$$P = \frac{\left|\hat{I}\right|^2}{8}\left(\frac{l}{\lambda}\right)^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \int_0^{2\pi}\int_0^{\pi} \sin^3\theta d\theta d\phi = \frac{\left|\hat{I}\right|^2 \pi}{4}\left(\frac{l}{\lambda}\right)^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \underbrace{\int_0^{\pi} \sin^3\theta d\theta}_{4/3} =$$

$$= \frac{\left|\hat{I}\right|^2 \pi}{3}\left(\frac{l}{\lambda}\right)^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{2}R_s\left|\hat{I}\right|^2. \quad R_s \approx 80\pi^2\left(\frac{l}{\lambda}\right)^2 : \text{ Radiation resistance.}$$

4.3 Guided waves

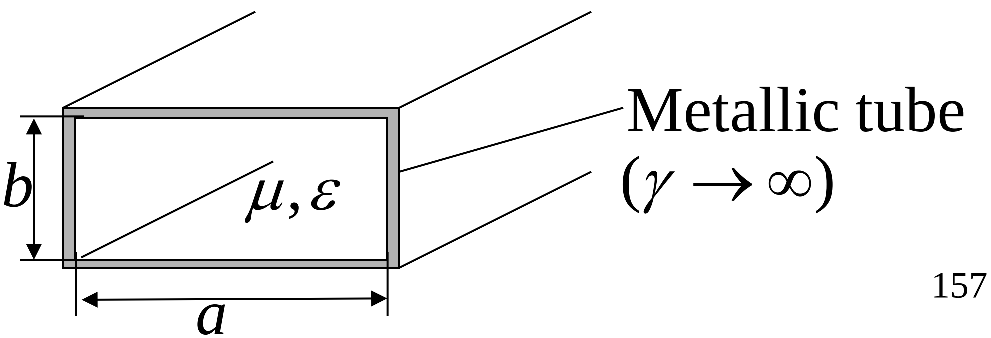
Transmission lines: transversal (x, y) dimensions are much smaller than the wave length.



$$D \ll \lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{2\pi}{k} = \frac{1}{f\sqrt{\mu\varepsilon}} = \frac{v}{f}.$$

If this condition is not fulfilled: *waveguide*.

e. g.: cylindrical waveguide



Assumptions:

- sinusoidal time dependence: *all quantities are complex amplitudes*,
- material properties are constant: μ, ε = constant,
- Lossless medium: $\gamma = 0$,
- No free charges: $\rho = 0$.

4.4.1 TM and TE waves

Wave equations for the potentials **A** and *V*:

In time domain:

$$-\Delta \mathbf{A} + \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{0}, \qquad -\Delta V + \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = 0.$$

In frequency domain:

$$-\Delta \mathbf{A} - \omega^2 \mu\varepsilon \mathbf{A} = \mathbf{0}, \qquad -\Delta V - \omega^2 \mu\varepsilon V = \mathbf{0},$$

$$k = \omega\sqrt{\mu\varepsilon}: \qquad \boxed{-\Delta \mathbf{A} - k^2 \mathbf{A} = \mathbf{0}.}$$

Electromagnetic field in case $\mathbf{A}(x,y,z) = A(x,y,z)\mathbf{e}_z$:

$$\mathbf{H} = \frac{1}{\mu} curl(A\mathbf{e}_z) = \frac{1}{\mu}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A \end{vmatrix} = \frac{1}{\mu}\frac{\partial A}{\partial y}\mathbf{e}_x - \frac{1}{\mu}\frac{\partial A}{\partial x}\mathbf{e}_y,$$

$$\mathbf{E} = \frac{1}{j\omega\mu\varepsilon}[k^2 A\mathbf{e}_z + graddiv(A\mathbf{e}_z)] =$$

$$= \frac{1}{j\omega\mu\varepsilon}[\frac{\partial^2 A}{\partial x\partial z}\mathbf{e}_x + \frac{\partial^2 A}{\partial y\partial z}\mathbf{e}_y + (\frac{\partial^2 A}{\partial z^2} + k^2 A)\mathbf{e}_z].$$

$H_z = 0$: the longitudinal component of the magnetic field is zero.

The magnetic field is transversal: *TM waves*.

Alternative to the potentials **A** and *V*:

$$\mathbf{D} = curl\mathbf{F},\ \mathbf{E} = \frac{1}{\varepsilon} curl\mathbf{F},$$

$$curl\mathbf{H} - j\omega\mathbf{D} = curl(\mathbf{H} - j\omega\mathbf{F}) = \mathbf{0} \Rightarrow \begin{array}{l} \mathbf{H} = j\omega\mathbf{F} - grad\psi, \\ \mathbf{B} = j\omega\mu\mathbf{F} - \mu grad\psi. \end{array}$$

$\mathbf{F}$: electric vector potential, ψ: magnetic scalar potential.

$$curl\mathbf{E} = -j\omega\mathbf{B}:\ curlcurl\mathbf{F} = graddiv\mathbf{F} - \Delta\mathbf{F} = -j\omega\mu\varepsilon(j\omega\mathbf{F} - grad\psi).$$

Lorenz gauge: $div\mathbf{F} = j\omega\mu\varepsilon\psi.$

$$-\Delta\mathbf{F} - \omega^2\mu\varepsilon\mathbf{F} = \mathbf{0}, \quad div\mathbf{B} = 0 \Rightarrow -\Delta\psi - \omega^2\mu\varepsilon\psi = 0.$$

$$-\Delta\mathbf{F} - k^2\mathbf{F} = \mathbf{0}:$$ wave equation.

Using the Lorenz gauge, ψ can be eliminated:

$$div\mathbf{F} = j\omega\mu\varepsilon\psi \Rightarrow \boxed{\psi = \frac{1}{j\omega\mu\varepsilon} div\mathbf{F}.}$$

$$\mathbf{H} = j\omega\mathbf{F} - grad\psi = j\omega\mathbf{F} - \frac{1}{j\omega\mu\varepsilon} graddiv\mathbf{F} =$$

$$= -\frac{1}{j\omega\mu\varepsilon}(\omega^2\mu\varepsilon\mathbf{F} + graddiv\mathbf{F}).$$

$$\boxed{\mathbf{D} = curl\mathbf{F}, \quad \mathbf{H} = -\frac{1}{j\omega\mu\varepsilon}(k^2\mathbf{F} + graddiv\mathbf{F}).}$$

Electromagnetic field in case $\mathbf{F}(x,y,z) = F(x,y,z)\mathbf{e}_z$:

$$\mathbf{E} = \frac{1}{\varepsilon} curl(F\mathbf{e}_z) = \frac{1}{\varepsilon}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & F \end{vmatrix} = \frac{1}{\varepsilon}\frac{\partial F}{\partial y}\mathbf{e}_x - \frac{1}{\varepsilon}\frac{\partial F}{\partial x}\mathbf{e}_y,$$

$$\mathbf{H} = -\frac{1}{j\omega\mu\varepsilon}[k^2 F\mathbf{e}_z + graddiv(F\mathbf{e}_z)] =$$

$$= -\frac{1}{j\omega\mu\varepsilon}[\frac{\partial^2 F}{\partial x \partial z}\mathbf{e}_x + \frac{\partial^2 F}{\partial y \partial z}\mathbf{e}_y + (\frac{\partial^2 F}{\partial z^2} + k^2 F)\mathbf{e}_z].$$

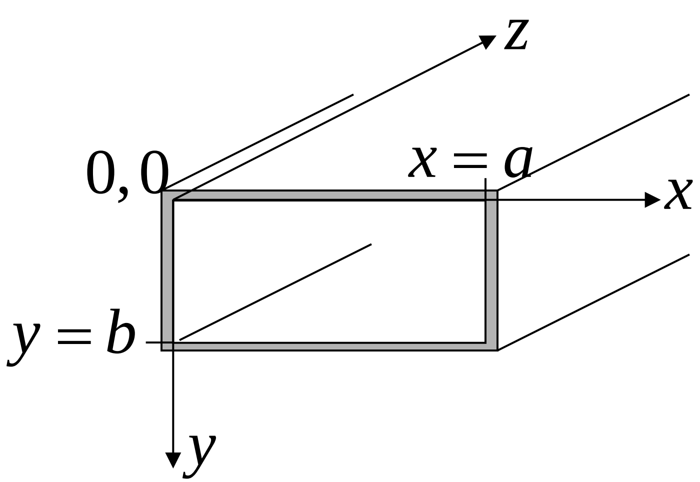
$E_z = 0$: the longitudinal component of the electric field is zero.

The electric field is transversal: *TE waves*.

The general solution of Maxwell's equations in homogeneous media can be written as the superposition of TM and TE waves.

Hence, the single component vector potentials allow the description of the electromagnetic field by means of two scalar functions.

4.4.2 Waves in rectangular waveguides



Assumption:
Wave propagation in the positive z-direction.

TM waves: $\mathbf{A}(x,y,z) = A(x,y,z)\mathbf{e}_z = A(x,y)e^{-j\beta z}\mathbf{e}_z,$

TE waves: $\mathbf{F}(x,y,z) = F(x,y,z)\mathbf{e}_z = F(x,y)e^{-j\beta z}\mathbf{e}_z.$

Boundary conditions: $\mathbf{E}\times\mathbf{n} = \mathbf{0}$ on the walls of the waveguide.

$$E_y = 0, E_z = 0 \text{ at } x = 0 \text{ and } x = a,$$

$$E_x = 0, E_z = 0 \text{ at } y = 0 \text{ and } y = b.$$

Boundary conditions for the potentials: $\frac{\partial}{\partial z} = -j\beta,\ \frac{\partial^2}{\partial z^2} = -\beta^2.$

TM waves: $$\mathbf{E} = \frac{1}{j\omega\mu\varepsilon}[\frac{\partial^2 A}{\partial x \partial z}\mathbf{e}_x + \frac{\partial^2 A}{\partial y \partial z}\mathbf{e}_y + (\frac{\partial^2 A}{\partial z^2} + k^2 A)\mathbf{e}_z] =$$

$$= -\frac{1}{j\omega\mu\varepsilon}[j\beta\frac{\partial A}{\partial x}\mathbf{e}_x + j\beta\frac{\partial A}{\partial y}\mathbf{e}_y + (\beta^2 - k^2)A\mathbf{e}_z].$$

$A = 0$ at $x = 0,\ x = a,\ y = 0$ and $y = b,$

Dirichlet B.C.

TE waves: $$\mathbf{E} = \frac{1}{\varepsilon}\frac{\partial F}{\partial y}\mathbf{e}_x - \frac{1}{\varepsilon}\frac{\partial F}{\partial x}\mathbf{e}_y.$$

$\frac{\partial F}{\partial x} = 0$ at $x = 0$ and $x = a,$

$\frac{\partial F}{\partial y} = 0$ at $y = 0$ and $y = b,$

Neumann B.C.

TM waves: $-\Delta \mathbf{A} - k^2 \mathbf{A} = \mathbf{0}, \qquad k = \omega\sqrt{\mu\varepsilon}.$

$\mathbf{A}(x, y, z) = A(x, y) e^{-j\beta z} \mathbf{e}_z$:

$$-\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 A}{\partial y^2} + \beta^2 A - k^2 A = 0.$$

Solution by separation: $A(x, y) = X(x)Y(y).$

$$-Y(y)\frac{d^2 X(x)}{dx^2} - X(x)\frac{d^2 Y(y)}{dy^2} + (\beta^2 - k^2)X(x)Y(y) = 0,$$

$$-\underbrace{\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2}}_{f(x)} - \underbrace{\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2}}_{g(y)} + \beta^2 - k^2 = 0.$$

This is only possible if $f(x) = \text{constant}, \; g(y) = \text{constant}.$

The following ordinary differential equations are obtained:

$$\frac{d^2X(x)}{dx^2} = fX(x),\ \frac{d^2Y(y)}{dy^2} = gY(y).$$

Solution:

$$f = -k_x^2,\ g = -k_y^2:\quad X(x) = C_{1x}\cos k_x x + C_{2x}\sin k_x x,$$

$$Y(y) = C_{1y}\cos k_y y + C_{2y}\sin k_y y.$$

boundary conditions: $X(0) = X(a) = 0 \Rightarrow C_{1x} = 0,\ k_x = m\frac{\pi}{a},\ m = 1, 2, \ldots ,$

$$Y(0) = Y(b) = 0 \Rightarrow C_{1y} = 0,\ k_y = n\frac{\pi}{b},\ n = 1, 2, \ldots .$$

TM$_{mn}$ waves:

$$C = C_{2x}C_{2y}: \quad A(x,y,z) = C\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-j\beta z}.$$

$$\mathbf{H} = \frac{1}{\mu}\frac{\partial A}{\partial y}\mathbf{e}_x - \frac{1}{\mu}\frac{\partial A}{\partial x}\mathbf{e}_y: \quad H_x = \frac{C}{\mu}\frac{n\pi}{b}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$H_y = -\frac{C}{\mu}\frac{m\pi}{a}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$H_z = 0.$$

$$\mathbf{E} = -\frac{1}{j\omega\mu\varepsilon}[j\beta\frac{\partial A}{\partial x}\mathbf{e}_x + j\beta\frac{\partial A}{\partial y}\mathbf{e}_y + (\beta^2 - k^2)A\mathbf{e}_z]:$$

$$E_x = -\frac{C\beta}{\omega\mu\varepsilon}\frac{m\pi}{a}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$E_y = -\frac{C\beta}{\omega\mu\varepsilon}\frac{n\pi}{b}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$E_z = \frac{C(k^2 - \beta^2)}{j\omega\mu\varepsilon}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-j\beta z}.$$

TE waves: $-\Delta\mathbf{F} - k^2\mathbf{F} = \mathbf{0}, \qquad k = \omega\sqrt{\mu\varepsilon}.$

$\mathbf{F}(x, y, z) = F(x, y)e^{-j\beta z}\mathbf{e}_z :$

$$-\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} + \beta^2 F - k^2 F = 0.$$

Solution by separation: $F(x, y) = X(x)Y(y).$

$$-Y(y)\frac{d^2X(x)}{dx^2} - X(x)\frac{d^2Y(y)}{dy^2} + (\beta^2 - k^2)X(x)Y(y) = 0,$$

$$-\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{f(x)} - \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{g(y)} + \beta^2 - k^2 = 0.$$

This is only possible if $f(x) = \text{constant},\ g(y) = \text{constant}.$

The following ordinary differential equations are obtained:

$$\frac{d^2 X(x)}{dx^2} = fX(x), \ \frac{d^2 Y(y)}{dy^2} = gY(y).$$

Solution:

$$f = -k_x^2, \ g = -k_y^2: \quad X(x) = C_{1x} \cos k_x x + C_{2x} \sin k_x x,$$

$$Y(y) = C_{1y} \cos k_y y + C_{2y} \sin k_y y.$$

boundary conditions:

$$\frac{dX(0)}{dx} = \frac{dX(a)}{dx} = 0 \Rightarrow C_{2x} = 0, \ k_x = m\frac{\pi}{a}, \ m = 0, 1, 2, \dots ,$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \Rightarrow C_{2y} = 0, \ k_y = n\frac{\pi}{b}, \ n = 0, 1, 2, \dots .$$

TE_{mn} waves:

$$C = C_{1x}C_{1y}: \quad F(x,y,z) = C\cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-j\beta z}.$$

$$\mathbf{E} = \frac{1}{\varepsilon}\frac{\partial F}{\partial y}\mathbf{e}_x - \frac{1}{\varepsilon}\frac{\partial F}{\partial x}\mathbf{e}_y : E_x = -\frac{C}{\varepsilon}\frac{n\pi}{b}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$E_y = \frac{C}{\varepsilon}\frac{m\pi}{a}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$E_z = 0.$$

$$\mathbf{H} = \frac{1}{j\omega\mu\varepsilon}[j\beta\frac{\partial F}{\partial x}\mathbf{e}_x + j\beta\frac{\partial F}{\partial y}\mathbf{e}_y + (\beta^2 - k^2)F\mathbf{e}_z]:$$

$$H_x = -\frac{C\beta}{\omega\mu\varepsilon}\frac{m\pi}{a}\sin(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$H_y = -\frac{C\beta}{\omega\mu\varepsilon}\frac{n\pi}{b}\cos(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)e^{-j\beta z},$$

$$H_z = \frac{C(\beta^2 - k^2)}{j\omega\mu\varepsilon}\cos(\frac{m\pi}{a}x)\cos(\frac{n\pi}{b}y)e^{-j\beta z}.$$

Satisfaction of the separation equation: $-f-g+\beta^2-k^2=0.$

$$k_x^2+k_y^2+\beta^2-k^2=0,\quad k_x=\frac{m\pi}{a},\ k_y=\frac{n\pi}{b},\ k^2=\omega^2\mu\varepsilon.$$

$$\boxed{(\frac{m\pi}{a})^2+(\frac{n\pi}{b})^2+\beta^2-\omega^2\mu\varepsilon=0,}$$ m, n = (0), 1, 2, ...

At given values of n and m (wave modes), the angular frequency is not arbitrary: β^2 cannot be negative!

If $\beta^2<0,$ one had $\beta=\pm j\alpha,-j\beta=\mp\alpha\Rightarrow e^{-j\beta z}=e^{\mp\alpha z}$:

attenuation only, no wave propagation.

$$\beta^2 = \omega^2 \mu\varepsilon - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2 \geq 0 \Rightarrow \omega \geq \underbrace{\frac{\pi}{\sqrt{\mu\varepsilon}} \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}}_{2\pi f_g}.$$

$$f \geq f_g = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}.$$

f_g: *cut-off frequency*.

Any particular mode is only propagable above the cut-off frequency.