
Robot Vision: Projective Geometry

Prof. Friedrich Fraundorfer

SS 2023

Learning goals

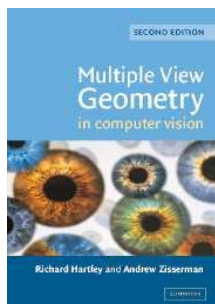
- Understand homogeneous coordinates
- Understand point, line, plane parameters and interpret them geometrically
- Understand point, line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the properties of parallel lines and planes in projective space
- Understand the concept of the line and plane at infinity

Outline

- Differences between euclidean and projective geometry
- 2D projective geometry
 - 1D projective geometry
 - Homogeneous coordinates
 - Points, Lines
 - Duality
- 3D projective geometry
 - Points, Lines, Planes
 - Duality
 - Plane at infinity

Literature

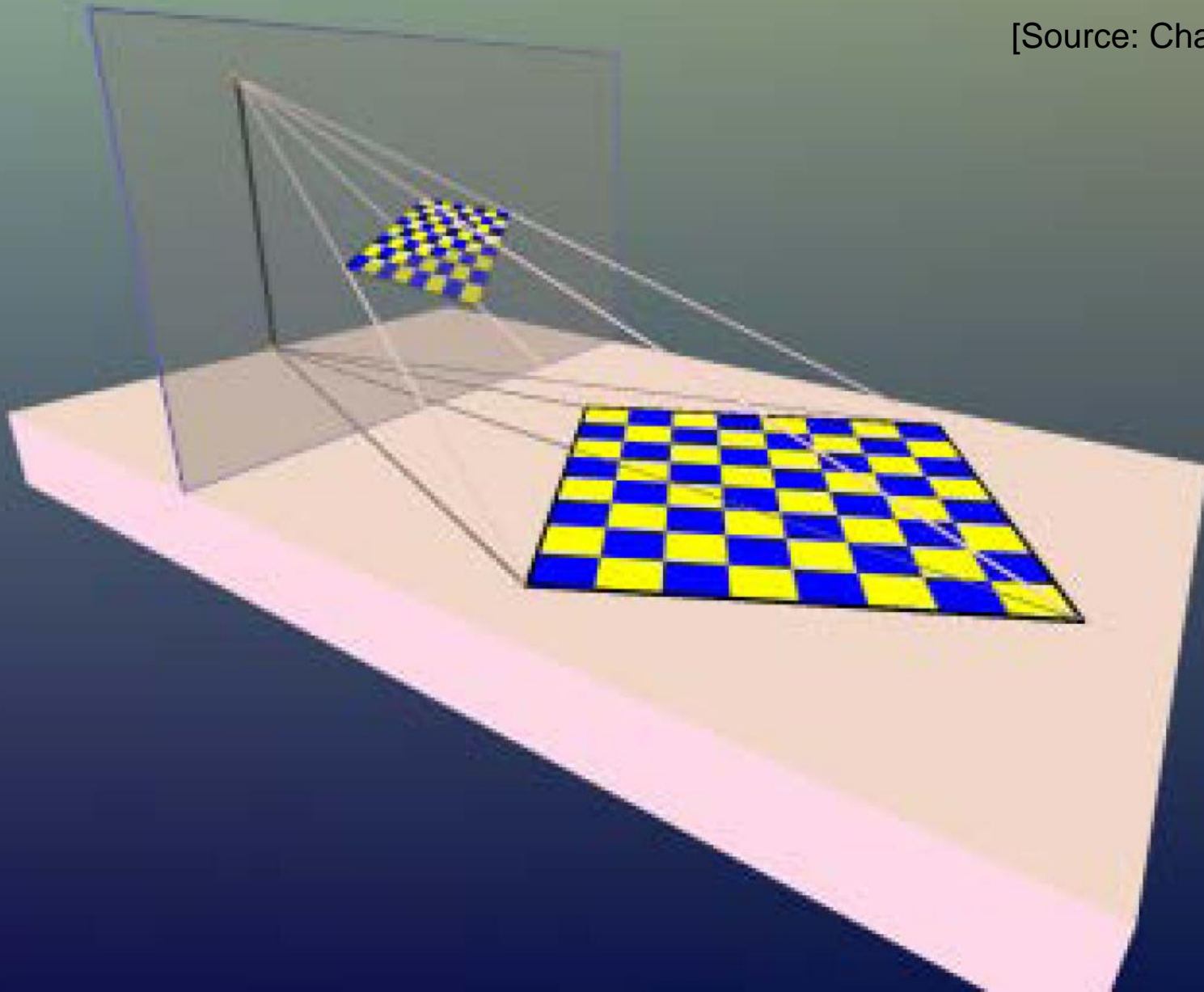
- Multiple View Geometry in Computer Vision. Richard Hartley and Andrew Zisserman. Cambridge University Press, March 2004.



- Mundy, J.L. and Zisserman, A., Geometric Invariance in Computer Vision, Appendix: Projective Geometry for Machine Vision, MIT Press, Cambridge, MA, 1992
- *Available online: www.cs.cmu.edu/~ph/869/papers/zisser-mundy.pdf*

Motivation – Image formation

[Source: Charles Gunn]

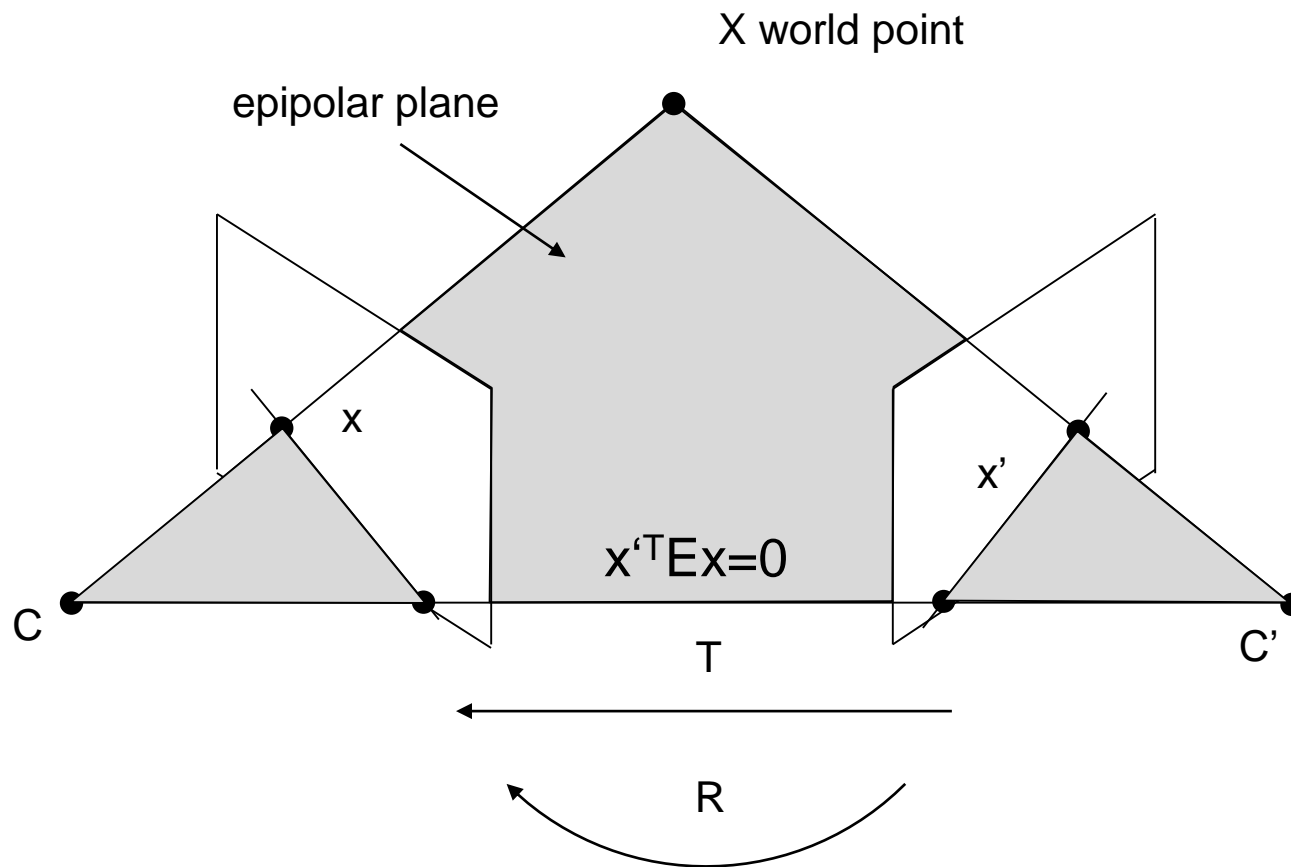


Motivation – Parallel lines



[Source: Flickr]

Motivation – Epipolar constraint

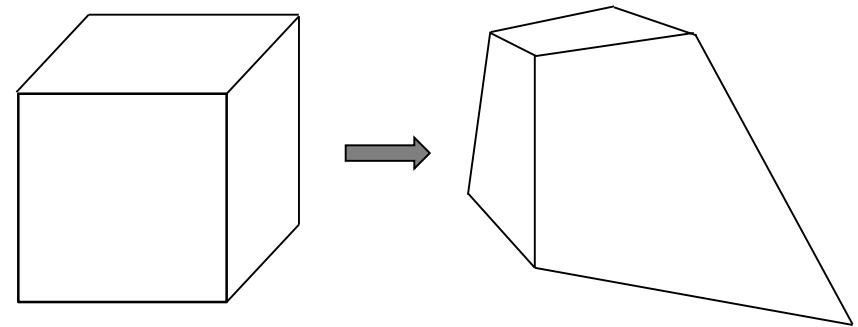
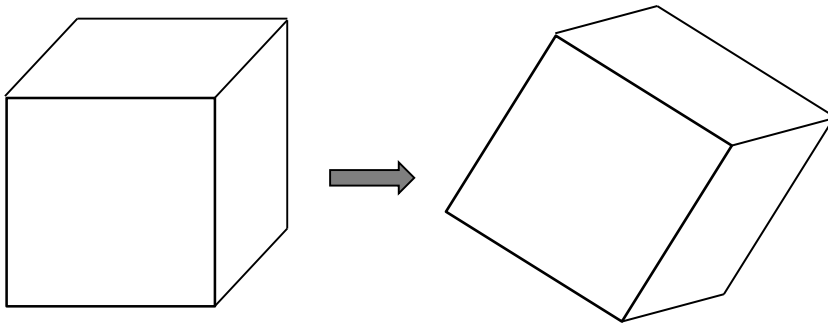
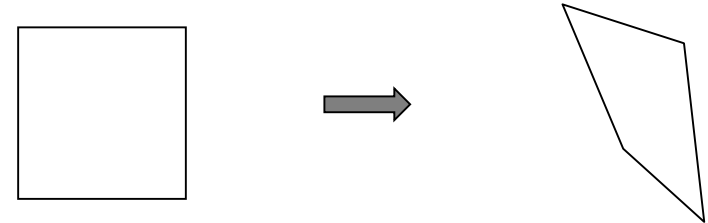
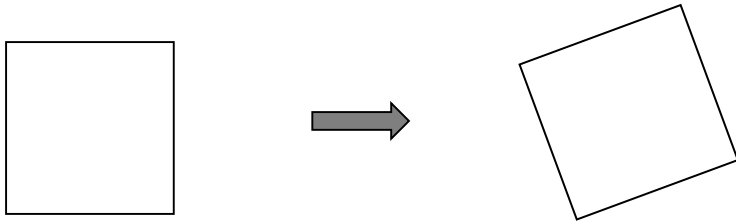


Euclidean geometry vs. projective geometry

Definitions:

- **Geometry** is the teaching of points, lines, planes and their relationships and properties (angles)
- Geometries are defined based on **invariances** (what is changing if you transform a configuration of points, lines etc.)
- Geometric transformations of Euclidean geometry **preserve** distances
- Geometric transformations of projective geometry do **NOT preserve** distances
- Projective geometry was developed to explain the perspective changes of three-dimensional objects when projected to a plane.

Difference between Euclidean and projective transformation



Euclidean transformation

Projective transformation

Projective Geometry

2D Projective Geometry

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2D projective geometry

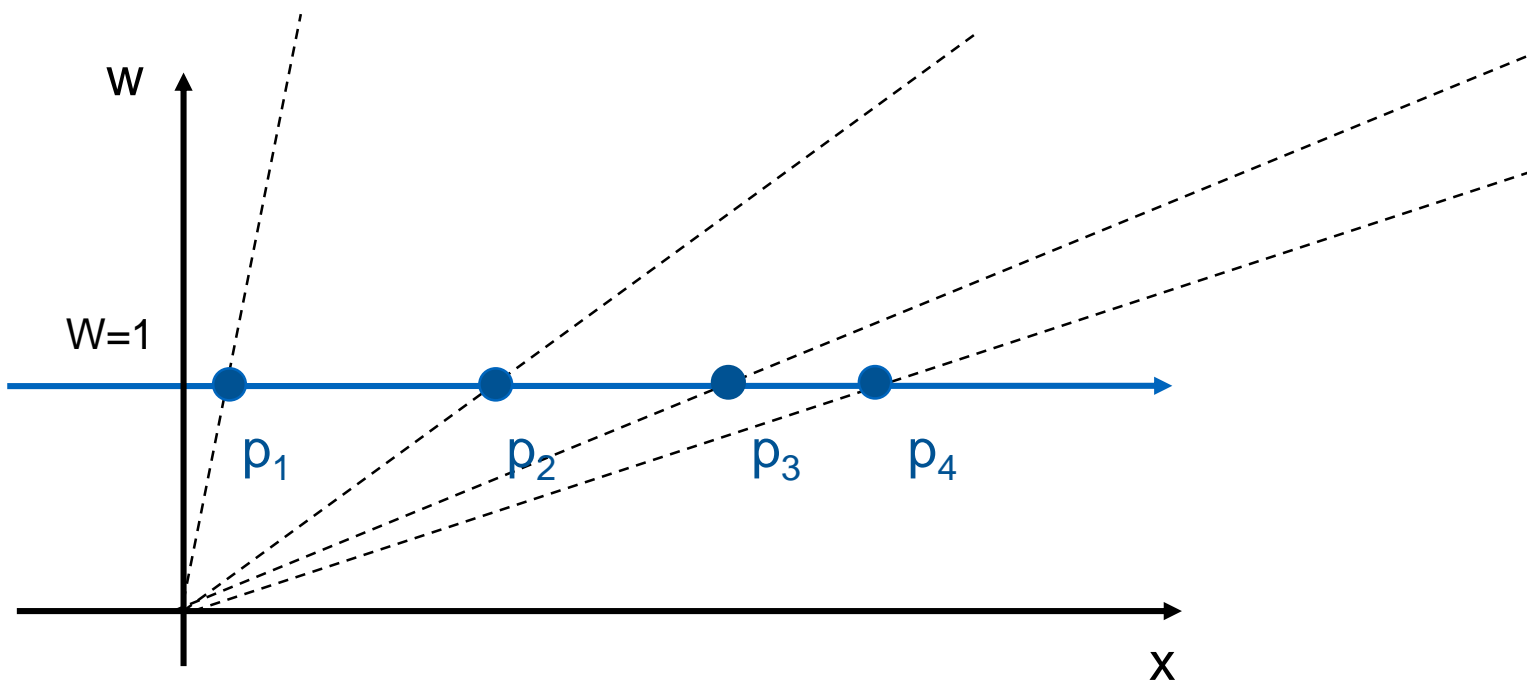
- Homogeneous coordinates
- Points, Lines
- Duality

1D Euclidean geometry



Euclidean coordinate:
 $p_1 = [x]$

1D projective geometry



homogeneous coordinate:
 $p_1 = [x, w] \approx [\hat{x}, 1]$

2D case - Homogeneous coordinates

- projective plane = Euclidean plane + a new line of points
- The projective space associated to \mathbb{R}^3 is called the projective plane P^2 .

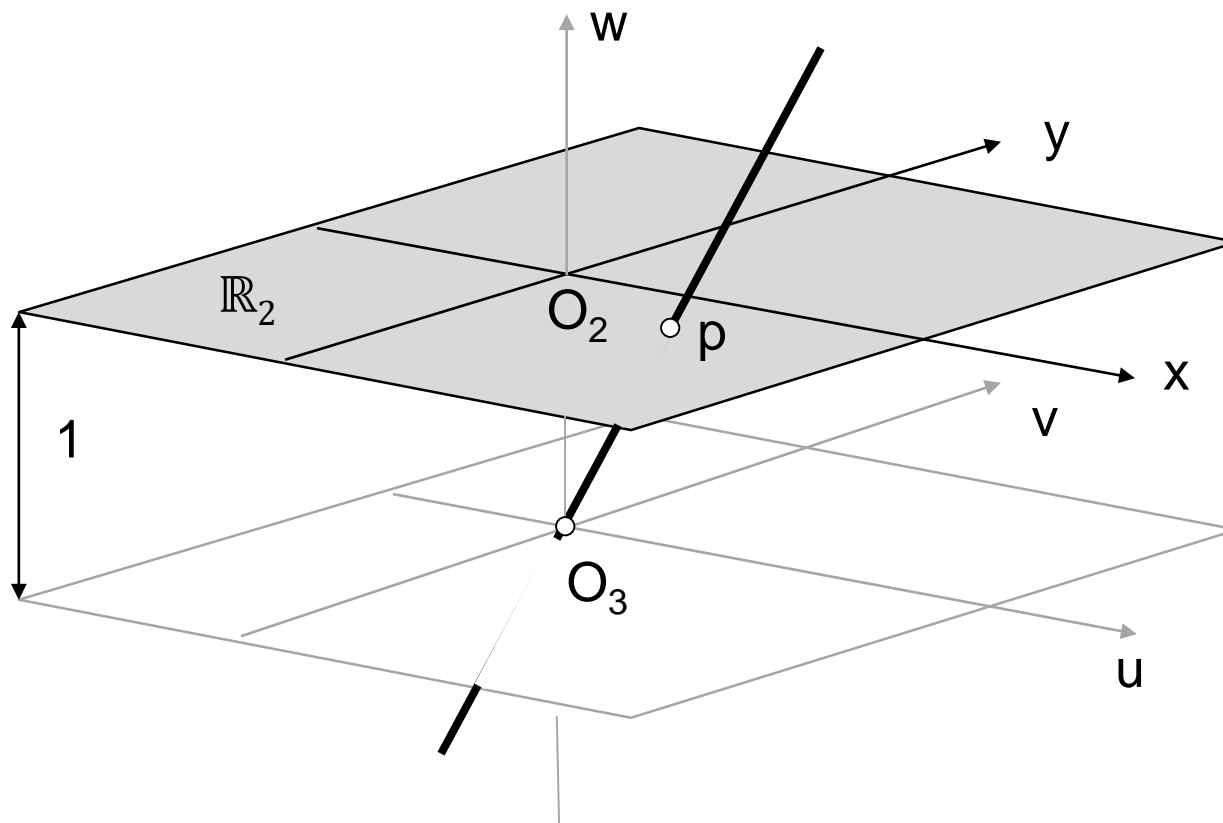
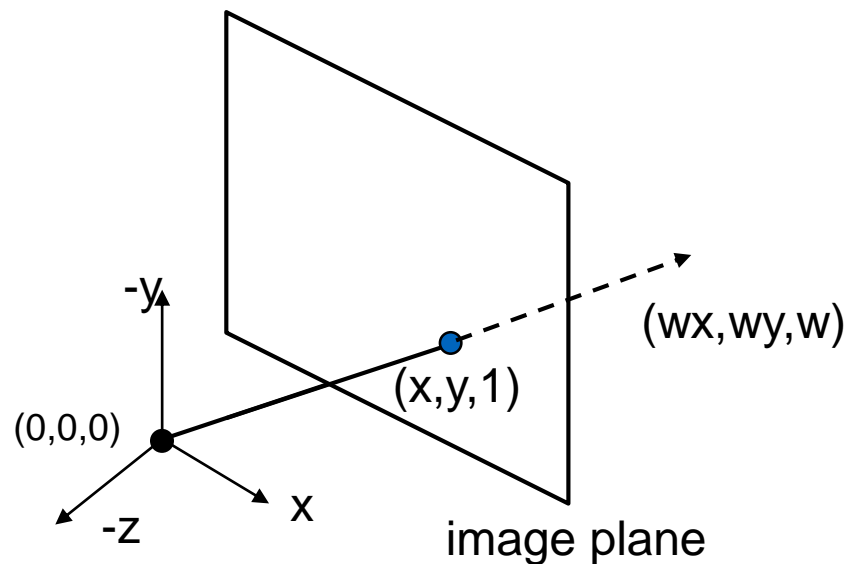


image coordinate:
 $p=[x,y]$
homogeneous coordinate:
 $p=[u,v,w] \approx [\hat{u}, \hat{v} 1]$

Points

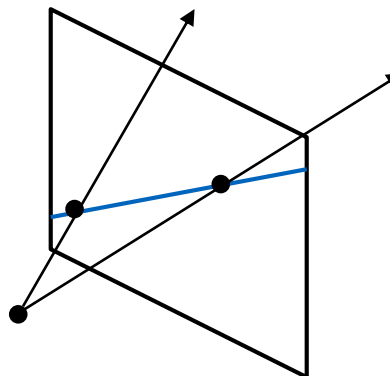
- A point in the image is a ray in projective space



- Each point (x,y) on the plane is represented by a ray (wx,wy,w)
 - all points on the ray are equivalent: $(x, y, 1) \cong (wx, wy, w)$

Lines

- A line in the image plane is defined by the equation $ax + by + cz = 0$ in projective space
- $[a,b,c]$ are the line parameters



- A point $[x,y,1]$ lies on the line if the equation $ax + by + cz = 0$ is satisfied
- This can be written in vector notation with a dot product:

$$0 = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\mathbf{l}^T \quad \mathbf{p}$

- A line is also represented as a homogeneous 3-vector \mathbf{l}

Calculations with lines and points

- Defining a line by two points

$$l = x \times y$$

- Intersection of two lines

$$x = l \times m$$

- Proof:

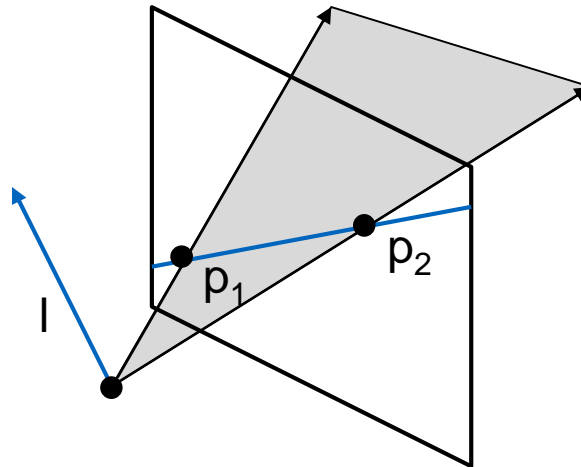
$$l = x \times y$$

$$x^T l = y^T l = 0$$

$$x^T (x \times y) = y^T (x \times y) = 0 \text{ (scalar triple product)}$$

Geometric interpretation of line parameters $[a,b,c]$

- A line \mathbf{l} is a homogeneous 3-vector, which is a ray in projective space
- It is \perp to every point (ray) \mathbf{p} on the line: $\mathbf{l}^T \mathbf{p} = 0$



What is the line \mathbf{l} spanned by rays \mathbf{p}_1 and \mathbf{p}_2 ?

- \mathbf{l} is \perp to \mathbf{p}_1 and $\mathbf{p}_2 \Rightarrow \mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$
- \mathbf{l} is the plane normal

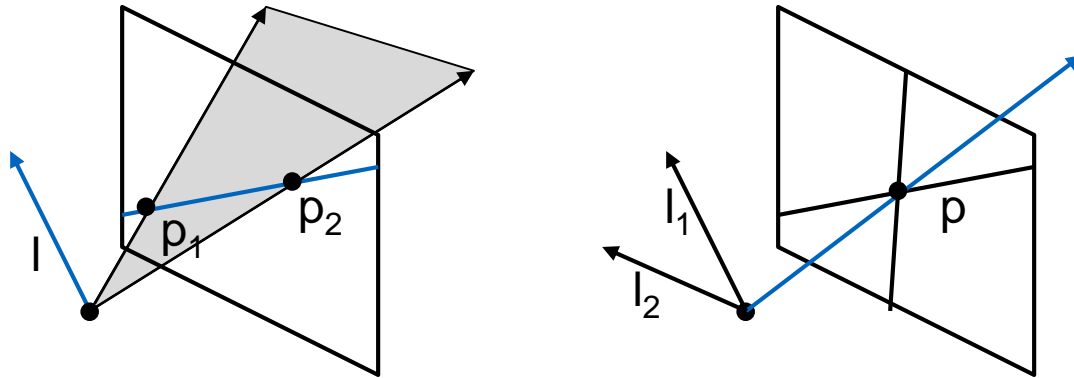
Point and line duality

Duality principle:

- To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

$$\begin{array}{ccc} x & \longleftrightarrow & l \\ x^T l = 0 & \longleftrightarrow & l^T x = 0 \\ x = l \times l' & \longleftrightarrow & l = x \times x' \end{array}$$

Point and line duality



What is the line l spanned by rays p_1 and p_2 ?

- l is \perp to p_1 and $p_2 \Rightarrow l = p_1 \times p_2$
- l is the plane normal

What is the intersection of two lines l_1 and l_2 ?

- p is \perp to l_1 and $l_2 \Rightarrow p = l_1 \times l_2$

Points and lines are dual in projective space

- given any formula, can switch the meanings of points and lines to get another formula

Intersection of parallel lines

- l and m are two parallel lines

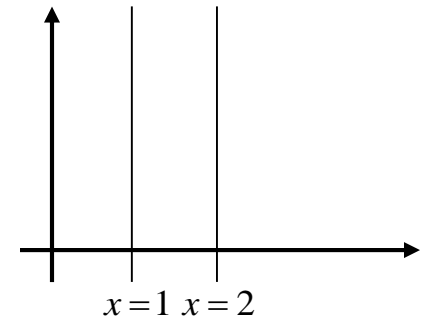
$l = (a, b, c)^T$ e. g. $(-1, 0, 1)^T$ (a line parallel to y-axis)

$m = (a, b, d)^T$ e. g. $(-1, 0, 2)^T$ (another line parallel to y-axis)

- Intersection of l and m

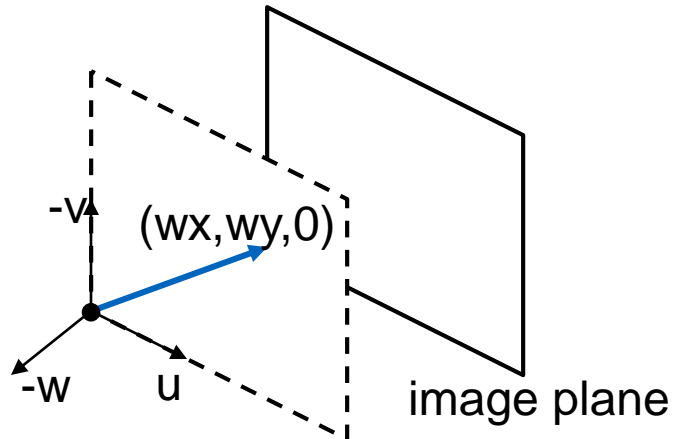
$$x = l \times m$$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = (d - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$



- A point $(x, y, 0)$ is called an ideal point, it does not lie in the image plane. But where does it lie then

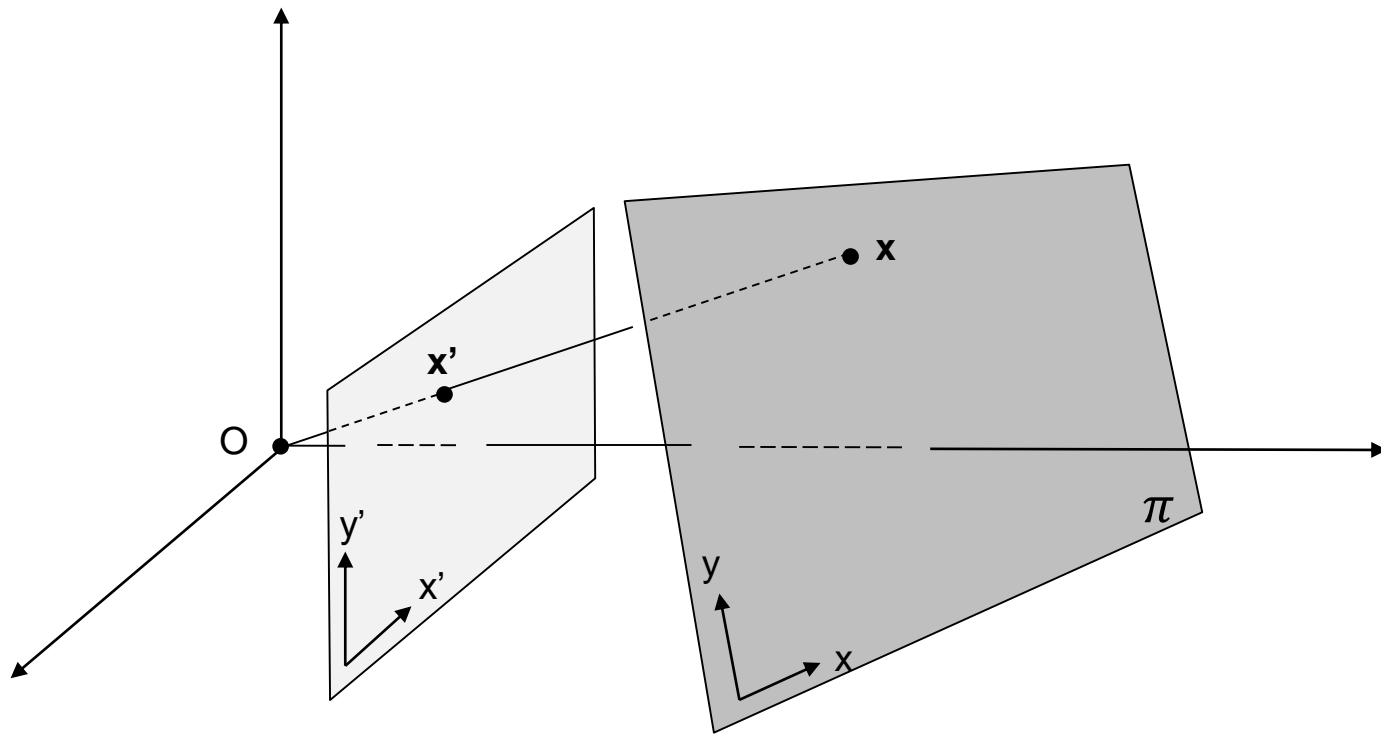
Ideal points and line at infinity



- Ideal point (“point at infinity”)
 - $p \equiv (x, y, 0)$ – parallel to image plane
 - It has infinite image coordinates
- All ideal points lie at the line at infinity
 - $l \equiv (0, 0, 1)$ – normal to the image plane
 - Why is it called a line at infinity?

Projective transformations

- Mapping between planes $x' = Hx$



Projective transformations

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H}\mathbf{x}$$

8DOF

projectivity=collineation=projective transformation=homography

To transform a point: $\mathbf{p}' = \mathbf{H}\mathbf{p}$

To transform a line: $\mathbf{l}\mathbf{p}=0 \rightarrow \mathbf{l}'\mathbf{p}'=0$

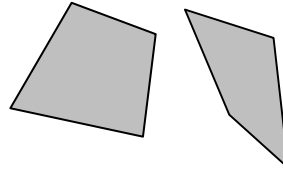
$$0 = \mathbf{l}\mathbf{p} = \mathbf{l}\mathbf{H}^{-1}\mathbf{H}\mathbf{p} = \mathbf{l}\mathbf{H}^{-1}\mathbf{p}' \Rightarrow \mathbf{l}' = \mathbf{l}\mathbf{H}^{-1}$$

lines are transformed by postmultiplication of \mathbf{H}^{-1}

Overview 2D transformations

Projective
8dof

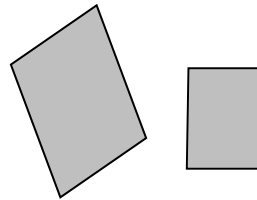
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity,
order of contact (intersection,
tangency, inflection, etc.),
cross ratio

Affine
6dof

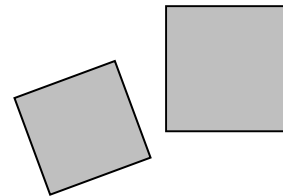
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas,
ratio of lengths on parallel
lines (e.g midpoints), linear
combinations of vectors
(centroids).

Similarity
4dof

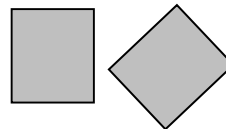
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.

Euclidean
3dof

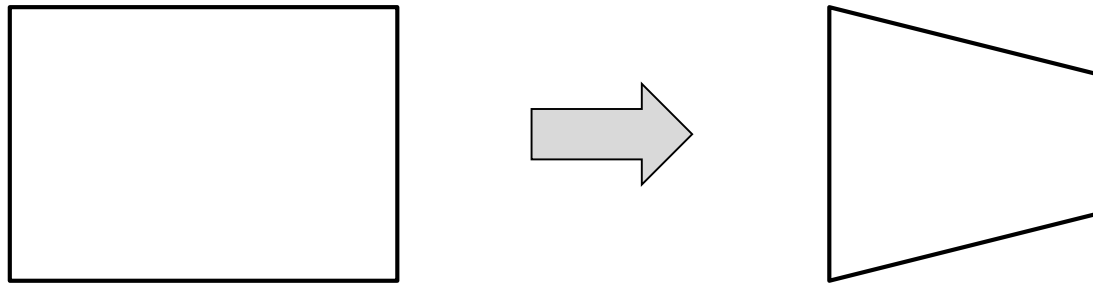
$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

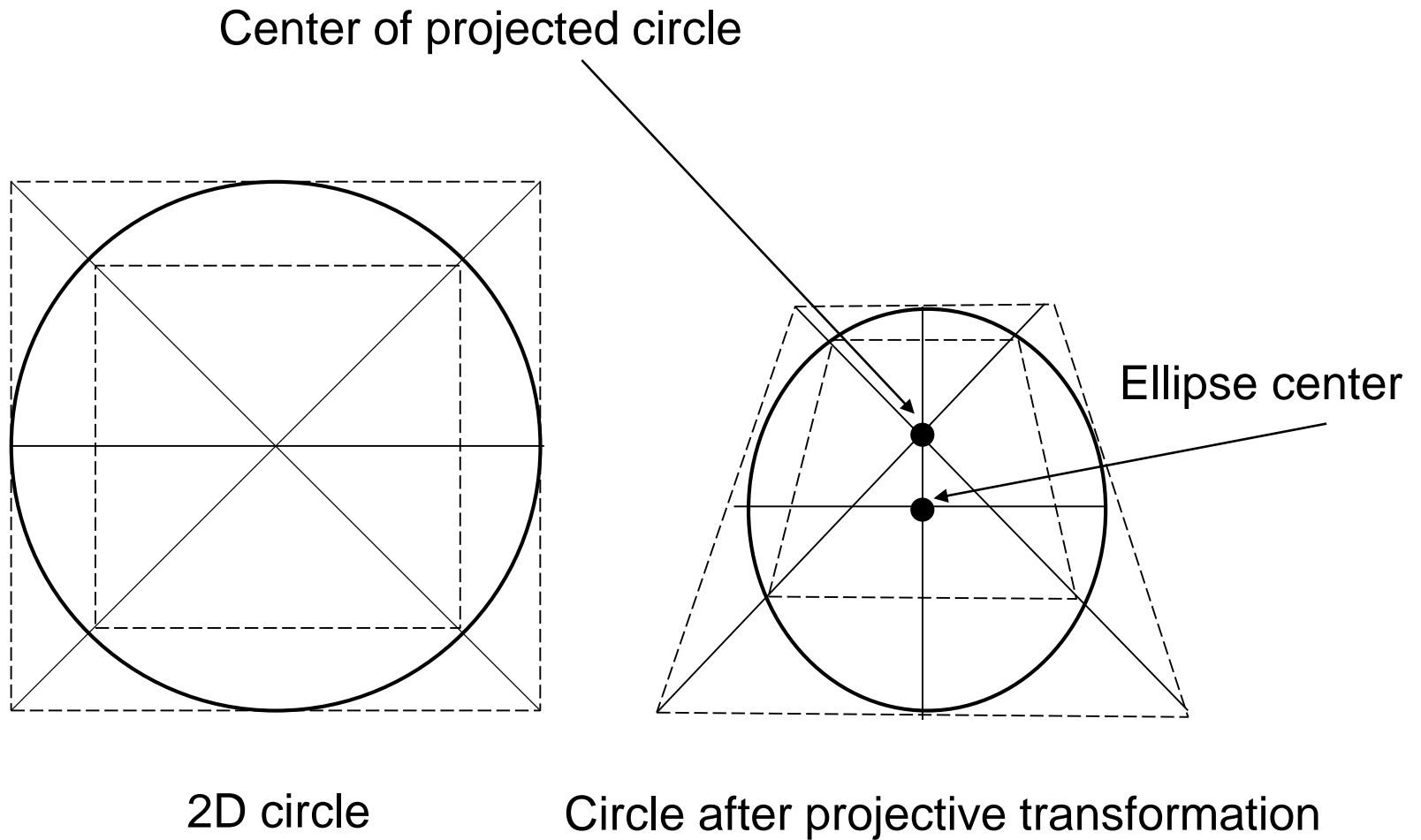
Effects of projective transformations

- Foreshortening effects can be imaged easily with primitive shapes



- But, how does a circle get transformed?

Effects of projective transformations



Projective Geometry

3D Projective Geometry

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3D projective geometry

- Points, Lines, Planes
- Duality
- Plane at infinity

3D projective geometry

- The concepts of 2D generalize naturally to 3D
 - The axioms of geometry can be applied to 3D as well
- 3D projective space = 3D Euclidean space + plane at infinity
 - Not so simple to visualize anymore (4D space)
- Entities are now points, lines and planes
 - Projective 3D points have four coordinates: $\mathbf{P} = (x,y,z,w)$
- Points, lines, and planes lead to more intersection and joining options than in the 2D case

Planes

- Plane equation

$$\Pi_1 X + \Pi_2 Y + \Pi_3 Z + \Pi_4 = 0$$
$$\Pi^T X = 0$$

- Expresses that point X is on plane Π

- Plane parameters

$$\Pi = [\Pi_1, \Pi_2, \Pi_3, \Pi_4]$$

- Plane parameters are normal vector + distance from origin

Join and incidence relations with planes

- A plane is defined uniquely by the join of three points, or the join of a line and point in general position
- Two distinct planes intersect in a unique line
- Three distinct planes intersect in a unique point

Three points define a plane

- X_1, X_2, X_3 are three distinct points, each has to fulfill the incidence equation. Equations can be stacked.

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \Pi = 0 \quad (3 \times 4)(4 \times 1)$$

- Plane parameters are the solution vector to this linear equation system (e.g. SVD)
- Points and planes are dual

$$\begin{bmatrix} \Pi_1^T \\ \Pi_2^T \\ \Pi_3^T \end{bmatrix} X = 0$$

Lines

- Lines are complicated
- Lines and points are not dual in 3D projective space
- Lines are represented by a 4x4 matrix, called Plücker matrix
- Computation of the line matrix from two points A,B

$$L = AB^T - BA^T \text{ (4x4) matrix}$$

- Matrix is skew-symmetric
- Example line of the x-axis
 - $x1 = [0 \ 0 \ 0 \ 1]^T$
 - $x2 = [1 \ 0 \ 0 \ 1]^T$
 - $L = x1 * x2^T - x2 * x1^T$

$$L = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Lines

- Points and planes are dual, we can get new equations by substituting points with planes

$$L = AB^T - BA^T \text{ (} A, B \text{ are points)}$$
$$L^* = PQ^T - QP^T \text{ (} P, Q \text{ are planes)}$$

- The intersection of two planes P, Q is a line
- Lines are self dual, the same line L has a dual representation L^*
- The matrix L can be directly computed from the entries of L^*

$$\begin{aligned}\ell_{12} &= \ell_{34}^* \\ \ell_{13} &= \ell_{42}^* \\ \ell_{14} &= \ell_{23}^* \\ \ell_{23} &= \ell_{14}^* \\ \ell_{42} &= \ell_{13}^* \\ \ell_{34} &= \ell_{12}^*\end{aligned}$$

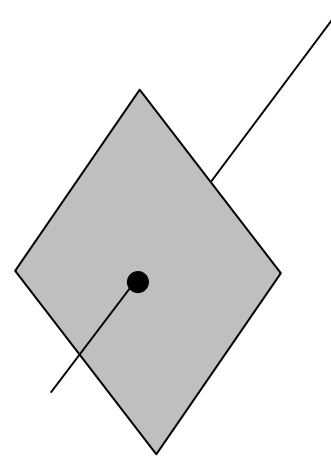
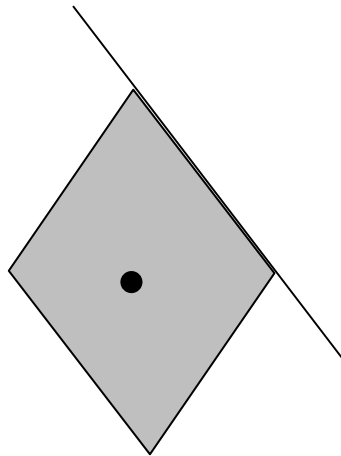
Point, planes and lines

- A plane can be defined by the join of a point X and a line L

$$\Pi = L * X$$

- A point can be defined by the intersection of a plane with a line L

$$X = L \Pi$$

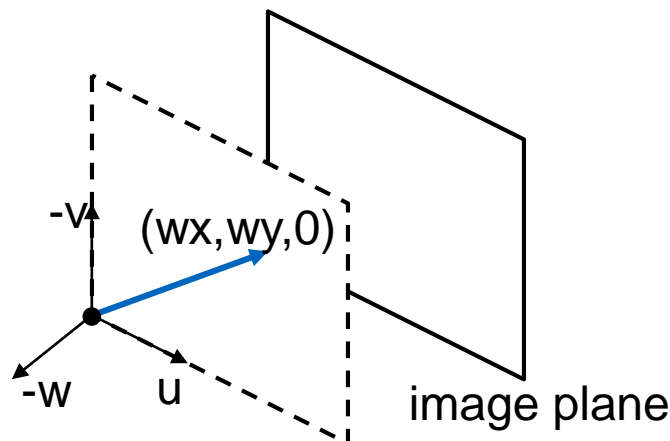


Plane at infinity

- Parallel lines and parallel planes intersect at Π_∞
- Plane parameters of Π_∞

$$\Pi_\infty = (0,0,0,1)^T$$

- It is a plane that contains all the direction vectors $D = (x_1, x_2, x_3, 0)^T$, vectors that originate from the origin of 4D space
- Try to imagine an extension of the 2D case (see illustration below) to the 3D case...



Recap - Learning goals

- Understand homogeneous coordinates
- Understand points, line, plane parameters and interpret them geometrically
- Understand point, line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the properties of parallel lines and planes in projective space
- Understand the concept of the line and plane at infinity