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# Robot Vision: Projective Geometry

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SS 2022

# Learning goals

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- Understand homogeneous coordinates
- Understand point, line, plane parameters and interpret them geometrically
- Understand point, line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the properties of parallel lines and planes in projective space
- Understand the concept of the line and plane at infinity

# Outline

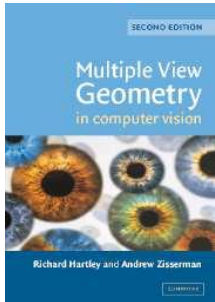
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- Differences between euclidean and projective geometry
- 2D projective geometry
  - 1D projective geometry
  - Homogeneous coordinates
  - Points, Lines
  - Duality
- 3D projective geometry
  - Points, Lines, Planes
  - Duality
  - Plane at infinity

# Literature

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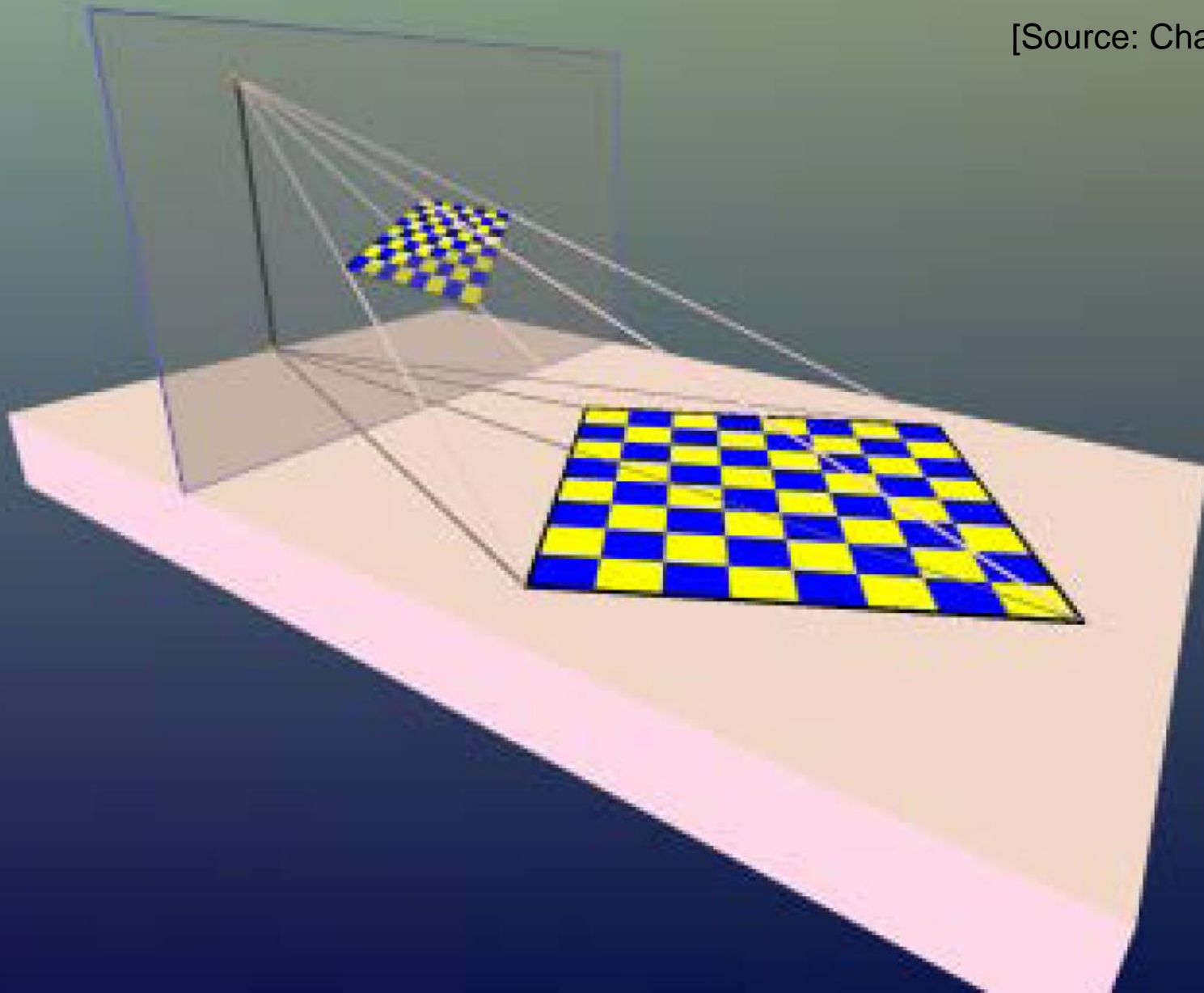
- Multiple View Geometry in Computer Vision. Richard Hartley and Andrew Zisserman. Cambridge University Press, March 2004.



- Mundy, J.L. and Zisserman, A., Geometric Invariance in Computer Vision, Appendix: Projective Geometry for Machine Vision, MIT Press, Cambridge, MA, 1992
- Available online: [www.cs.cmu.edu/~ph/869/papers/zisser-mundy.pdf](http://www.cs.cmu.edu/~ph/869/papers/zisser-mundy.pdf)

# Motivation – Image formation

[Source: Charles Gunn]



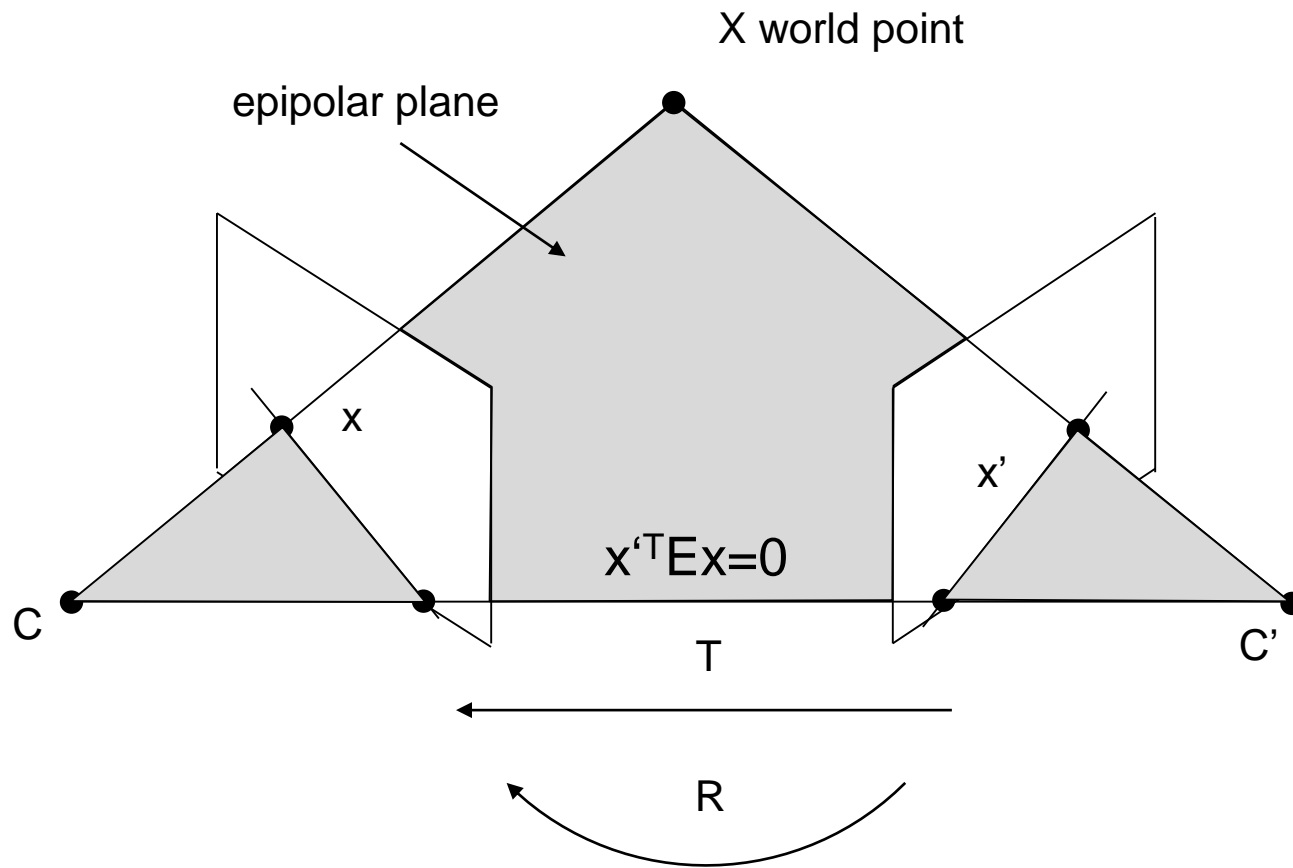


# Motivation – Parallel lines



[Source: Flickr]

# Motivation – Epipolar constraint



# Euclidean geometry vs. projective geometry

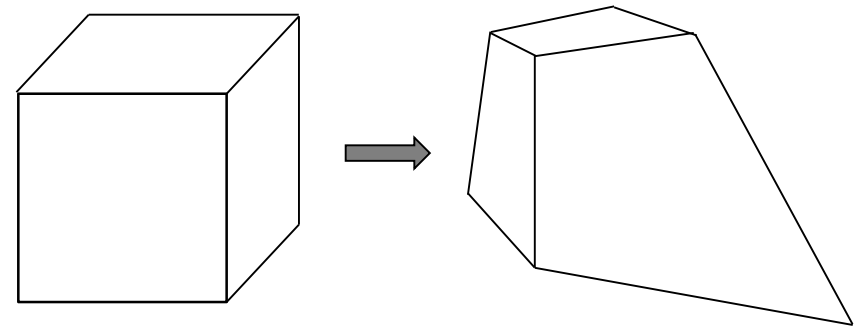
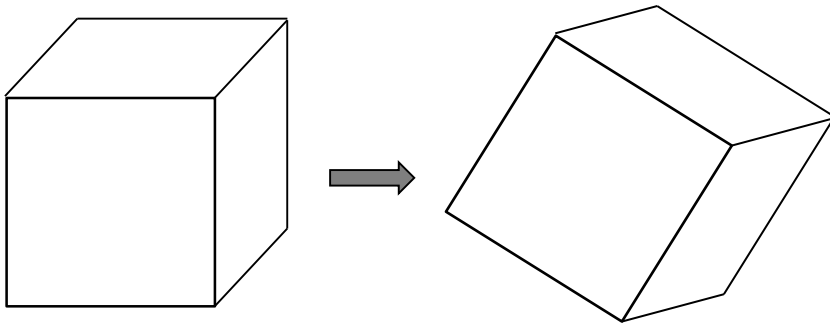
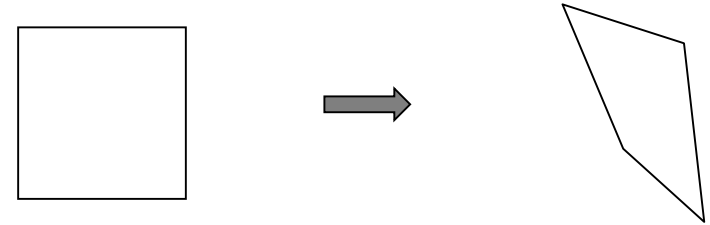
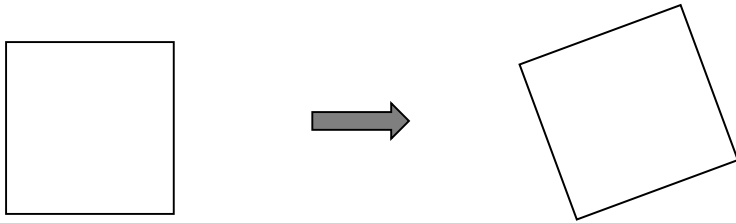
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Definitions:

- **Geometry** is the teaching of points, lines, planes and their relationships and properties (angles)
- Geometries are defined based on **invariances** (what is changing if you transform a configuration of points, lines etc.)
- Geometric transformations of Euclidean geometry **preserve** distances
- Geometric transformations of projective geometry do **NOT preserve** distances
  
- Projective geometry was developed to explain the perspective changes of three-dimensional objects when projected to a plane.



# Difference between Euclidean and projective transformation



Euclidean transformation

Projective transformation

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# Projective Geometry

## 2D Projective Geometry

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# 2D projective geometry

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- Homogeneous coordinates
- Points, Lines
- Duality

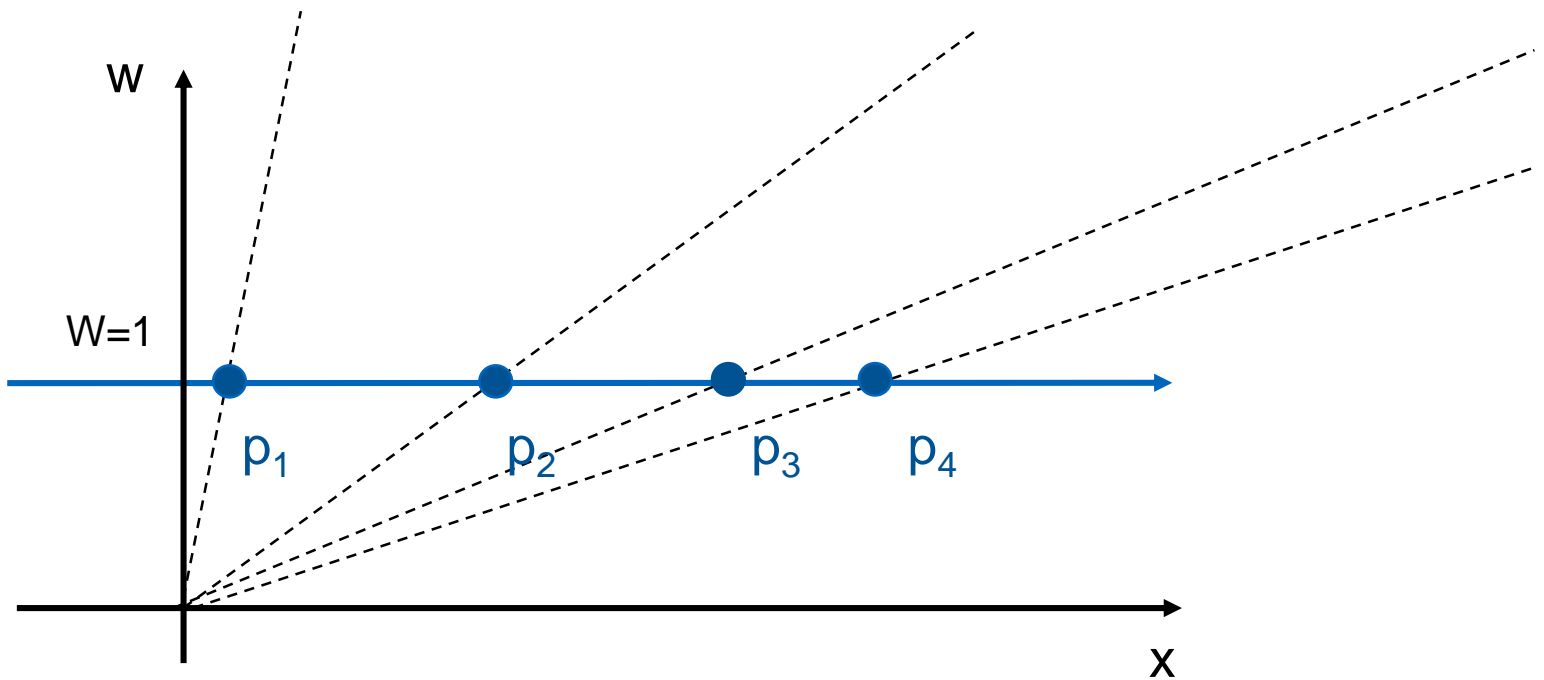
# 1D Euclidean geometry

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Euclidean coordinate:  
 $p_1 = [x]$

# 1D projective geometry



homogeneous coordinate:  
 $p_1 = [x, w] \approx [\dot{x}, 1]$



## 2D case - Homogeneous coordinates

- projective plane = Euclidean plane + a new line of points
- The projective space associated to  $\mathbb{R}^3$  is called the projective plane  $\mathbb{P}^2$ .

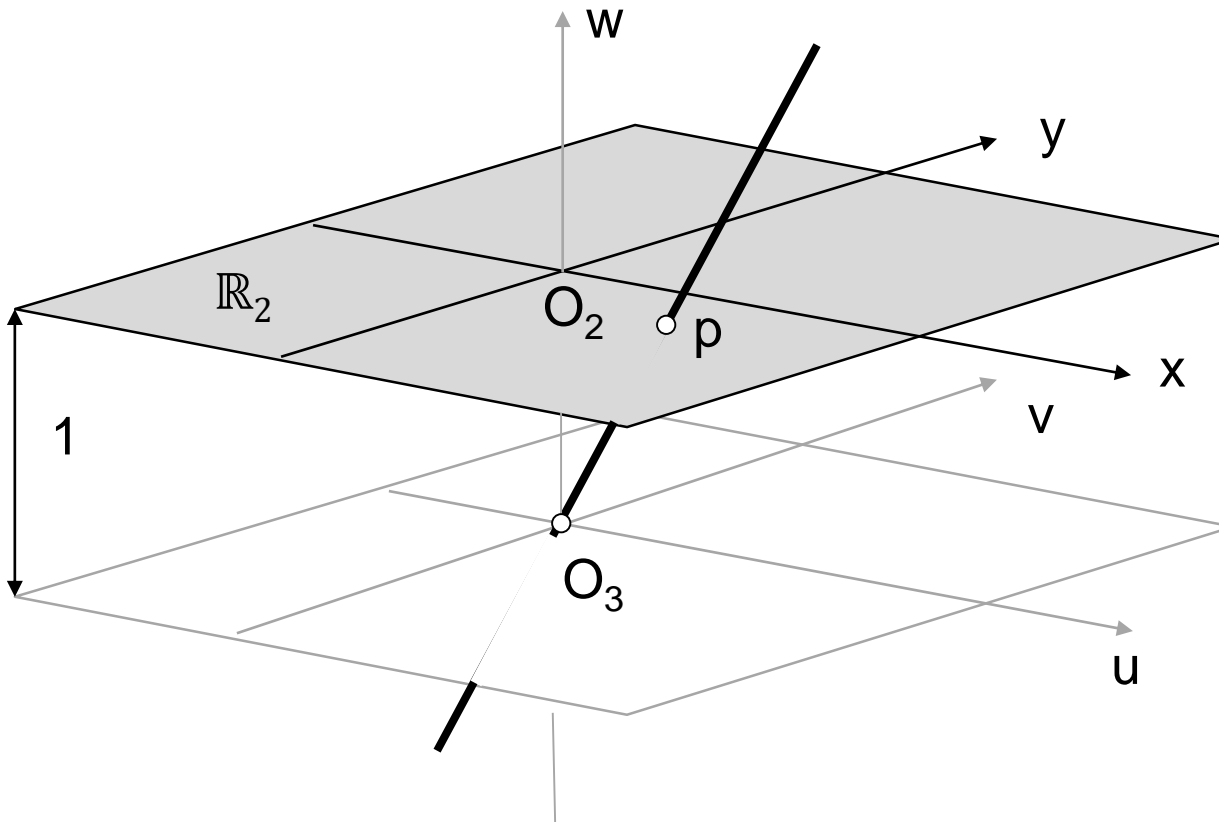
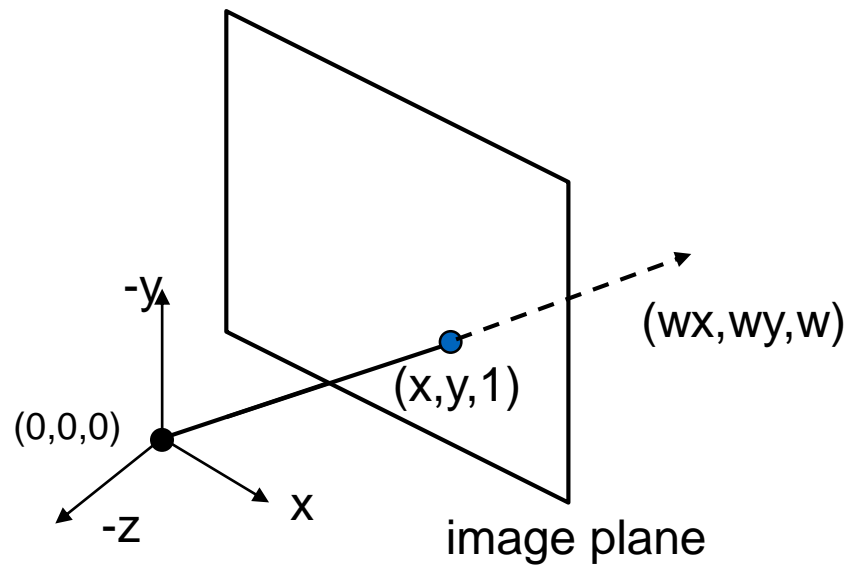


image coordinate:  
 $p=[x,y]$   
homogeneous coordinate:  
 $p=[u,v,w] \approx [u,v,1]$

# Points

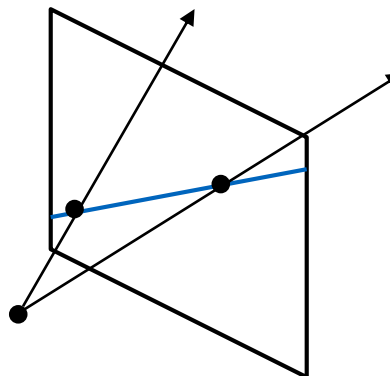
- A point in the image is a ray in projective space



- Each point  $(x,y)$  on the plane is represented by a ray  $(wx,wy,w)$ 
  - all points on the ray are equivalent:  $(x, y, 1) \cong (wx, wy, w)$

# Lines

- A line in the image plane is defined by the equation  $ax + by + cz = 0$  in projective space
- $[a,b,c]$  are the line parameters



- A point  $[x,y,1]$  lies on the line if the equation  $ax + by + cz = 0$  is satisfied
- This can be written in vector notation with a dot product:

$$0 = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\mathbf{l}^T \quad \mathbf{p}$

- A line is also represented as a homogeneous 3-vector  $\mathbf{l}$

# Calculations with lines and points

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- Defining a line by two points

$$l = x \times y$$

- Intersection of two lines

$$x = l \times m$$

- Proof:

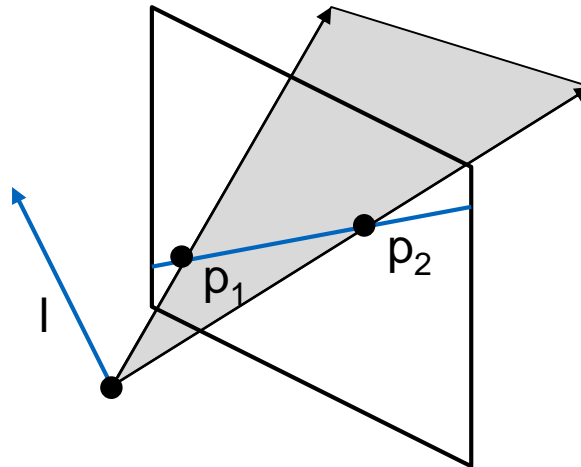
$$l = x \times y$$

$$x^T l = y^T l = 0$$

$$x^T (x \times y) = y^T (x \times y) = 0 \text{ (scalar triple product)}$$

# Geometric interpretation of line parameters $[a,b,c]$

- A line  $\mathbf{l}$  is a homogeneous 3-vector, which is a ray in projective space
- It is  $\perp$  to every point (ray)  $\mathbf{p}$  on the line:  $\mathbf{l} \mathbf{p} = 0$



What is the line  $\mathbf{l}$  spanned by rays  $\mathbf{p}_1$  and  $\mathbf{p}_2$  ?

- $\mathbf{l}$  is  $\perp$  to  $\mathbf{p}_1$  and  $\mathbf{p}_2 \Rightarrow \mathbf{l} = \mathbf{p}_1 \times \mathbf{p}_2$
- $\mathbf{l}$  is the plane normal



# Point and line duality

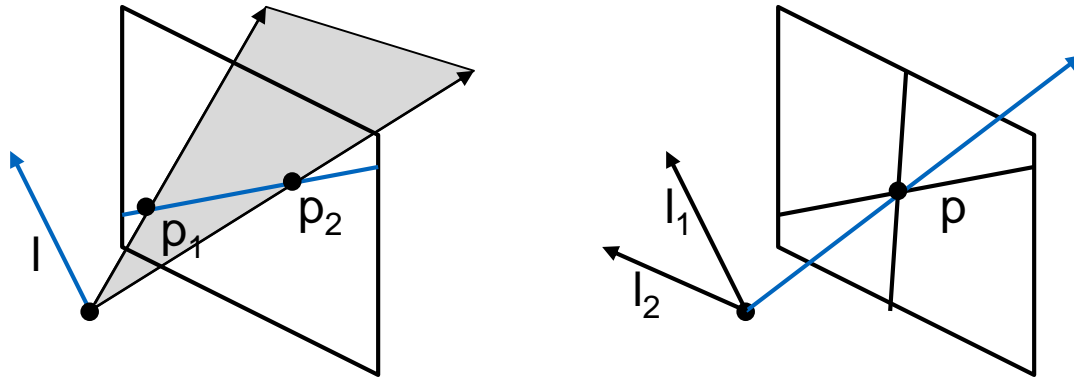
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Duality principle:

- To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

$$\begin{array}{ccc} \mathbf{x} & \longleftrightarrow & \mathbf{l} \\ \mathbf{x}^T \mathbf{l} = 0 & \longleftrightarrow & \mathbf{l}^T \mathbf{x} = 0 \\ \mathbf{x} = \mathbf{l} \times \mathbf{l}' & \longleftrightarrow & \mathbf{l} = \mathbf{x} \times \mathbf{x}' \end{array}$$

# Point and line duality



What is the line  $l$  spanned by rays  $p_1$  and  $p_2$  ?

- $l$  is  $\perp$  to  $p_1$  and  $p_2 \Rightarrow l = p_1 \times p_2$
- $l$  is the plane normal

What is the intersection of two lines  $l_1$  and  $l_2$  ?

- $p$  is  $\perp$  to  $l_1$  and  $l_2 \Rightarrow p = l_1 \times l_2$

Points and lines are dual in projective space

- given any formula, can switch the meanings of points and lines to get another formula

# Intersection of parallel lines

- $l$  and  $m$  are two parallel lines

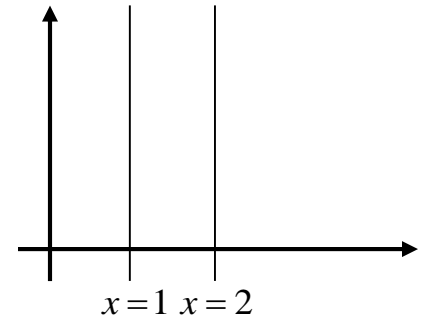
$l = (a, b, c)^T$  e. g.  $(-1, 0, 1)^T$  (a line parallel to y-axis)

$m = (a, b, d)^T$  e. g.  $(-1, 0, 2)^T$  (another line parallel to y-axis)

- Intersection of  $l$  and  $m$

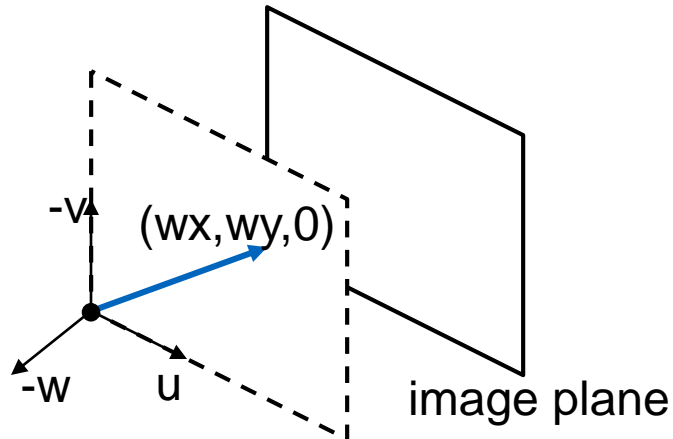
$$x = l \times m$$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = (d - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$



- A point  $(x, y, 0)$  is called an ideal point, it does not lie in the image plane. But where does it lie then

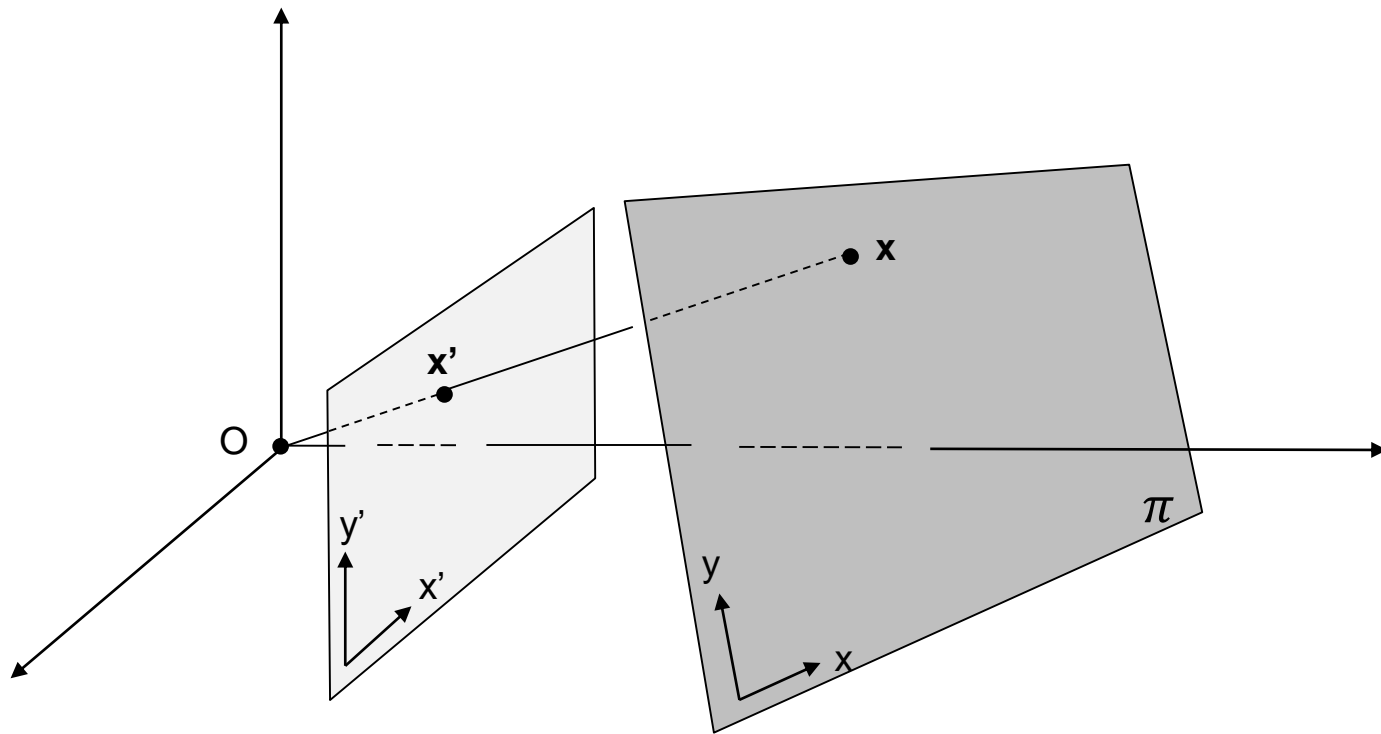
# Ideal points and line at infinity



- Ideal point (“point at infinity”)
  - $p \cong (x, y, 0)$  – parallel to image plane
  - It has infinite image coordinates
- All ideal points lie at the line at infinity
  - $l \cong (0, 0, 1)$  – normal to the image plane
  - Why is it called a line at infinity?

# Projective transformations

- Mapping between planes  $x' = Hx$





# Projective transformations

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H}\mathbf{x}$$

8DOF

projectivity=collineation=projective transformation=homography

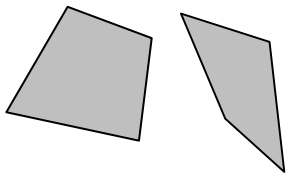
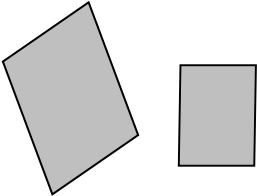
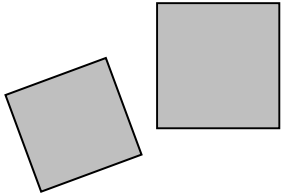
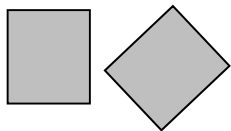
To transform a point:  $\mathbf{p}' = \mathbf{H}\mathbf{p}$

To transform a line:  $\mathbf{l}\mathbf{p}=0 \rightarrow \mathbf{l}'\mathbf{p}'=0$

$$0 = \mathbf{l}\mathbf{p} = \mathbf{l}\mathbf{H}^{-1}\mathbf{H}\mathbf{p} = \mathbf{l}\mathbf{H}^{-1}\mathbf{p}' \Rightarrow \mathbf{l}' = \mathbf{l}\mathbf{H}^{-1}$$

lines are transformed by postmultiplication of  $\mathbf{H}^{-1}$

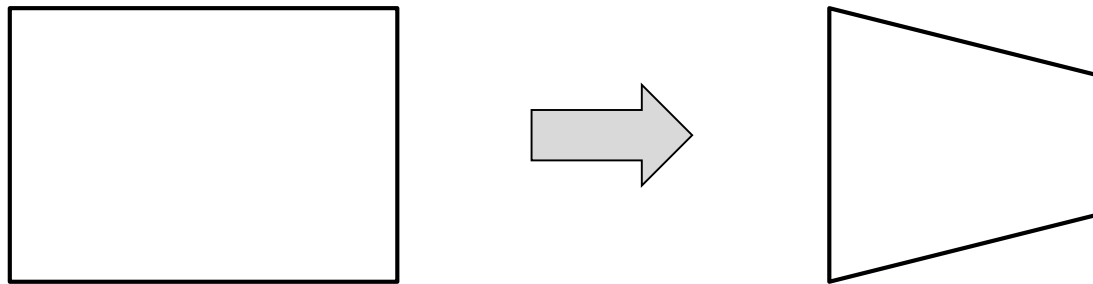
# Overview 2D transformations

Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		<p>Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio</p>
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<p>Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).</p>
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<p>Ratios of lengths, angles.</p>
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		<p>lengths, areas.</p>

# Effects of projective transformations

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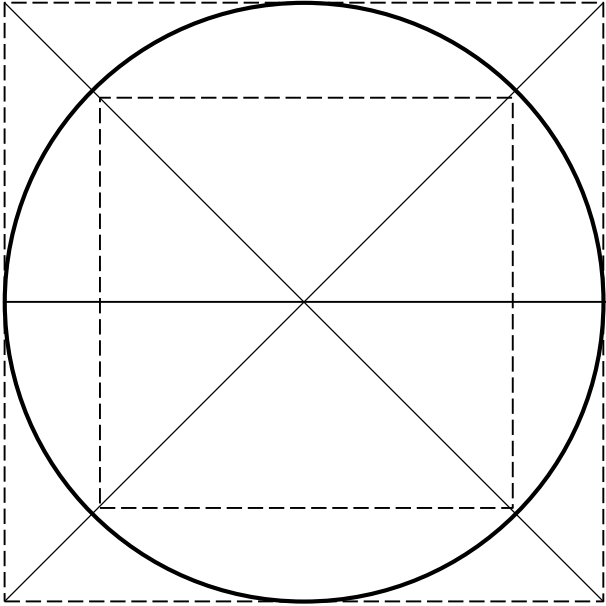
- Foreshortening effects can be imaged easily with primitive shapes



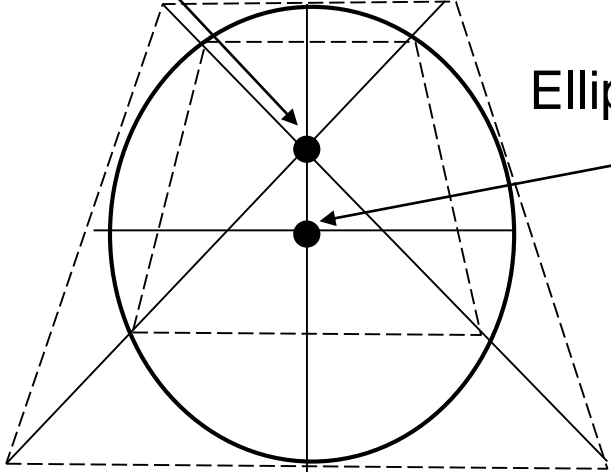
- But, how does an circle get transformed?

# Effects of projective transformations

Center of projected circle



2D circle



Ellipse center

Circle after projective transformation

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# Projective Geometry

## 3D Projective Geometry

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# 3D projective geometry

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- Points, Lines, Planes
- Duality
- Plane at infinity

# 3D projective geometry

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- The concepts of 2D generalize naturally to 3D
  - The axioms of geometry can be applied to 3D as well
- 3D projective space = 3D Euclidean space + plane at infinity
  - Not so simple to visualize anymore (4D space)
- Entities are now points, lines and planes
  - Projective 3D points have four coordinates:  $\mathbf{P} = (x,y,z,w)$
- Points, lines, and planes lead to more intersection and joining options than in the 2D case

# Planes

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- Plane equation

$$\begin{aligned}\Pi_1 X + \Pi_2 Y + \Pi_3 Z + \Pi_4 &= 0 \\ \Pi^T X &= 0\end{aligned}$$

- Expresses that point  $X$  is on plane  $\Pi$

- Plane parameters

$$\Pi = [\Pi_1, \Pi_2, \Pi_3, \Pi_4]$$

- Plane parameters are normal vector + distance from origin



## Join and incidence relations with planes

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- A plane is defined uniquely by the join of three points, or the join of a line and point in general position
- Two distinct planes intersect in a unique line
- Three distinct planes intersect in a unique point

# Three points define a plane

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- $X_1, X_2, X_3$  are three distinct points, each has to fulfill the incidence equation. Equations can be stacked.

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \Pi = 0 \quad (3 \times 4)(4 \times 1)$$

- Plane parameters are the solution vector to this linear equation system (e.g. SVD)
- Points and planes are dual

$$\begin{bmatrix} \Pi_1^T \\ \Pi_2^T \\ \Pi_3^T \end{bmatrix} X = 0$$

# Lines

- Lines are complicated
- Lines and points are not dual in 3D projective space
- Lines are represented by a 4x4 matrix, called Plücker matrix
- Computation of the line matrix from two points A,B

$$L = AB^T - BA^T \text{ (4x4) matrix}$$

- Matrix is skew-symmetric
- Example line of the x-axis

- $x1 = [0 \ 0 \ 0 \ 1]^T$
- $x2 = [1 \ 0 \ 0 \ 1]^T$
- $L = x1 * x2^T - x2 * x1^T$

$$L = \begin{matrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

# Lines

- Points and planes are dual, we can get new equations by substituting points with planes

$$L = AB^T - BA^T \text{ (} A, B \text{ are points)}$$
$$L^* = PQ^T - QP^T \text{ (} P, Q \text{ are planes)}$$

- The intersection of two planes  $P, Q$  is a line
- Lines are self dual, the same line  $L$  has a dual representation  $L^*$
- The matrix  $L$  can be directly computed from the entries of  $L^*$

$$\begin{aligned}l_{12} &= l_{34}^* \\l_{13} &= l_{42}^* \\l_{14} &= l_{23}^* \\l_{23} &= l_{14}^* \\l_{42} &= l_{13}^* \\l_{34} &= l_{12}^*\end{aligned}$$

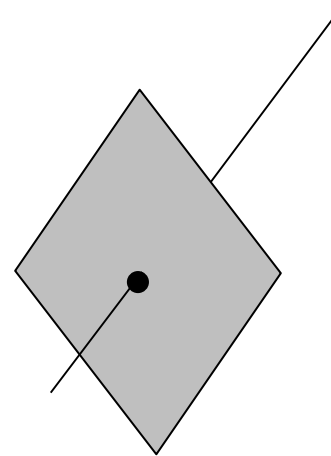
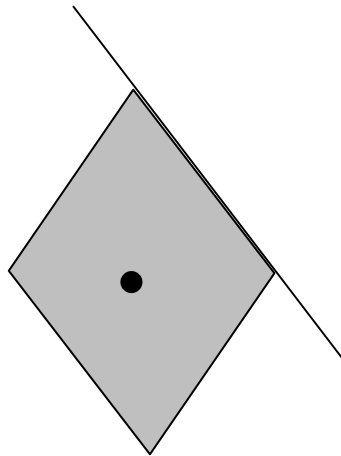
# Point, planes and lines

- A plane can be defined by the join of a point  $X$  and a line  $L$

$$\Pi = L * X$$

- A point can be defined by the intersection of a plane with a line  $L$

$$X = L \Pi$$

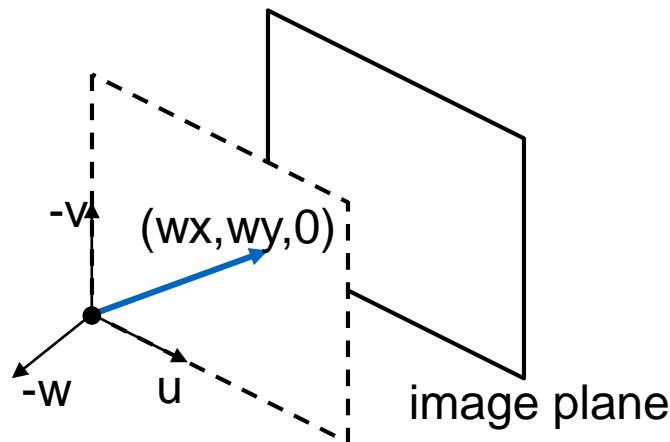


# Plane at infinity

- Parallel lines and parallel planes intersect at  $\Pi_\infty$
- Plane parameters of  $\Pi_\infty$

$$\Pi_\infty = (0,0,0,1)^T$$

- It is a plane that contains all the direction vectors  $D = (x_1, x_2, x_3, 0)^T$ , vectors that originate from the origin of 4D space
- Try to imagine an extension of the 2D case (see illustration below) to the 3D case...



## Recap - Learning goals

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- Understand homogeneous coordinates
- Understand points, line, plane parameters and interpret them geometrically
- Understand point, line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the properties of parallel lines and planes in projective space
- Understand the concept of the line and plane at infinity