Robot Vision:
Geometric Algorithms – Part 1

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Outline

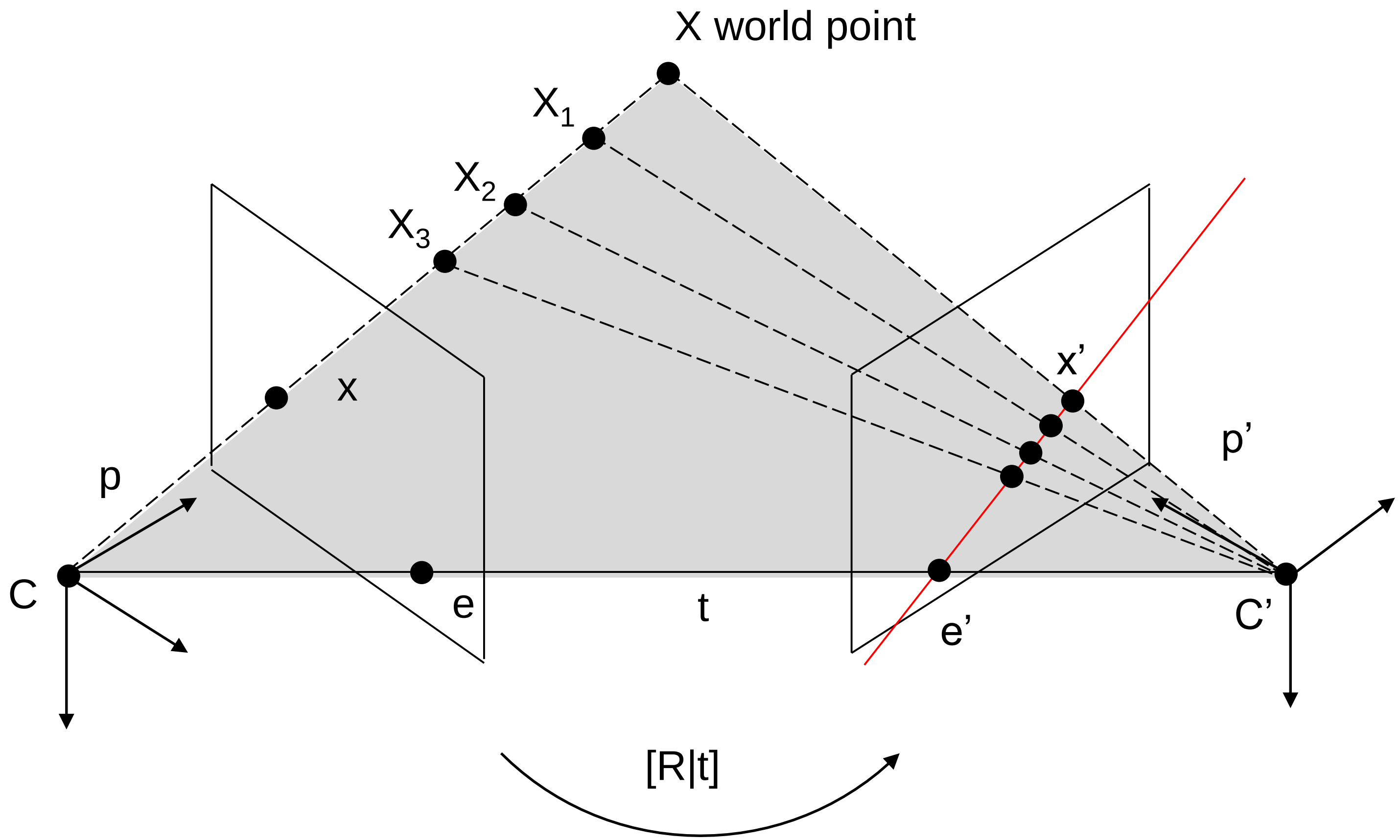
- Geometric Algorithms
  - Epipolar constraint derivation
  - Fundamental matrix
  - 8-point algorithm
  - Linear equation system solving

Learning goals

- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand the calculation of the fundamental matrix
- Understand linear equation system solving

Epipolar constraint

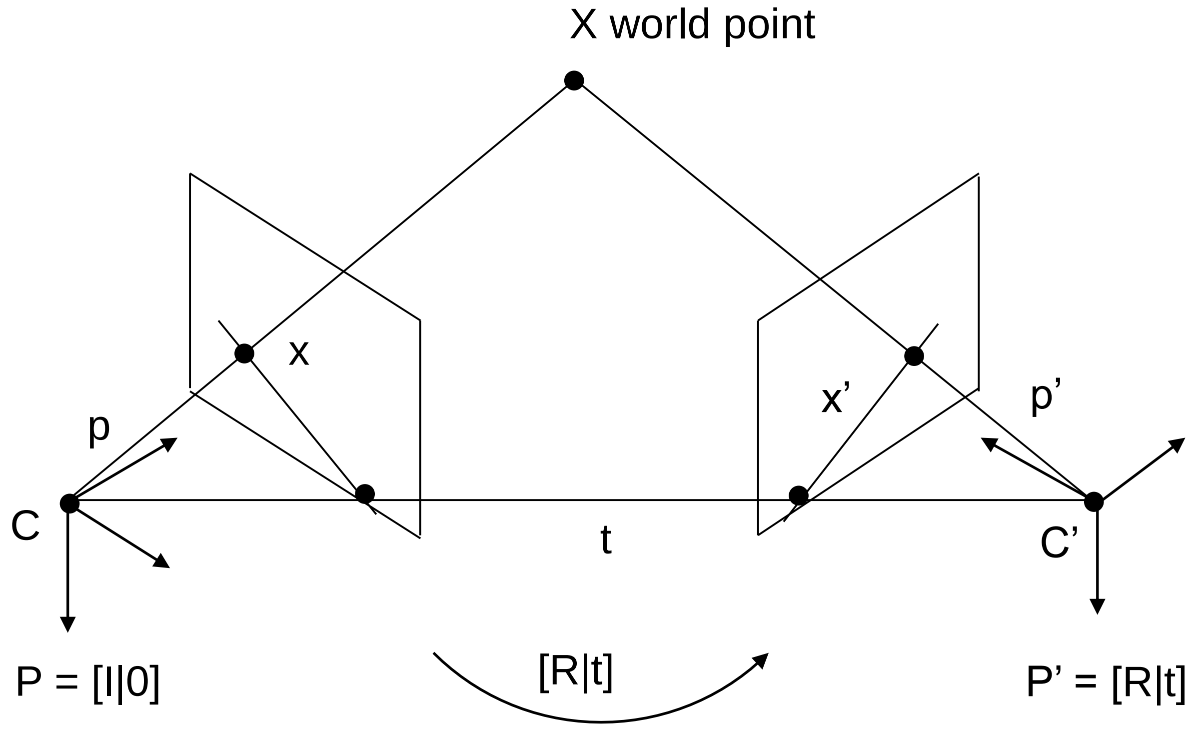
- The epipolar constraint is a mathematical relationship between the point correspondences of two images



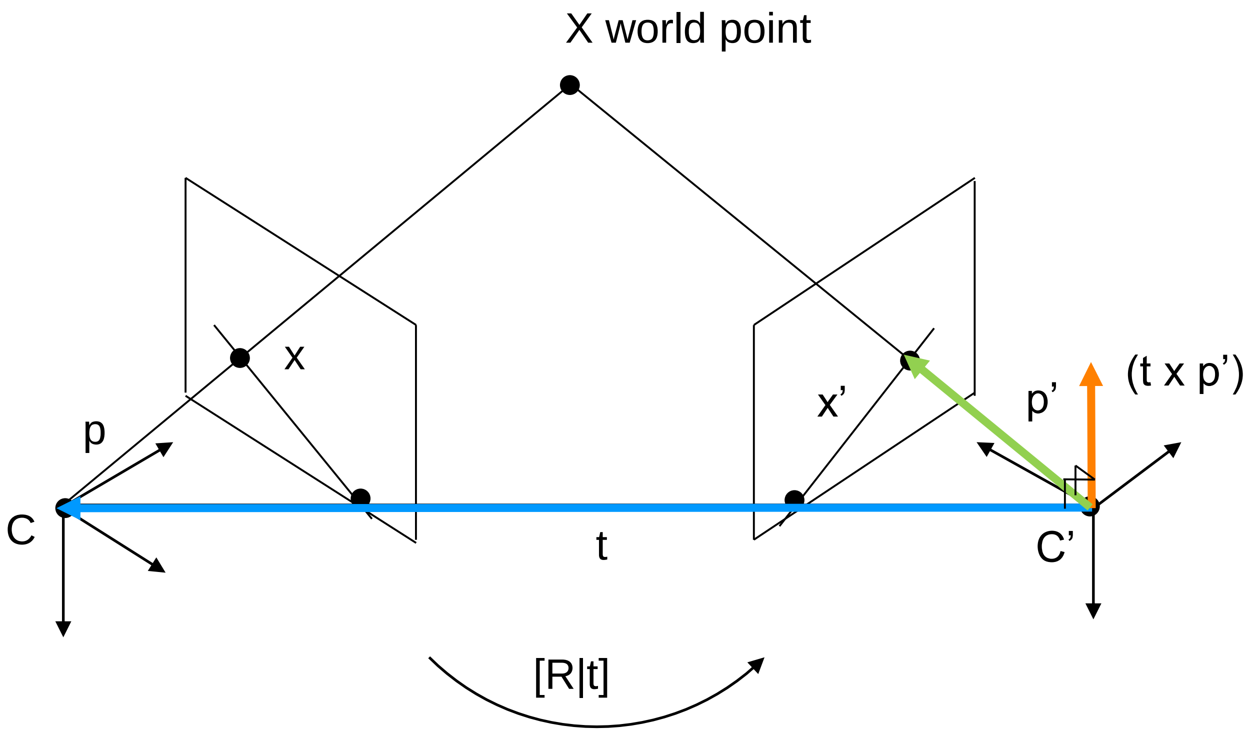
Epipolar constraint – derivation by coplanarity condition

- Vector p and t define a plane
- Vector p’ and t define also a plane
- Both planes must have the same normal
- What we seek is a relation between p and p’

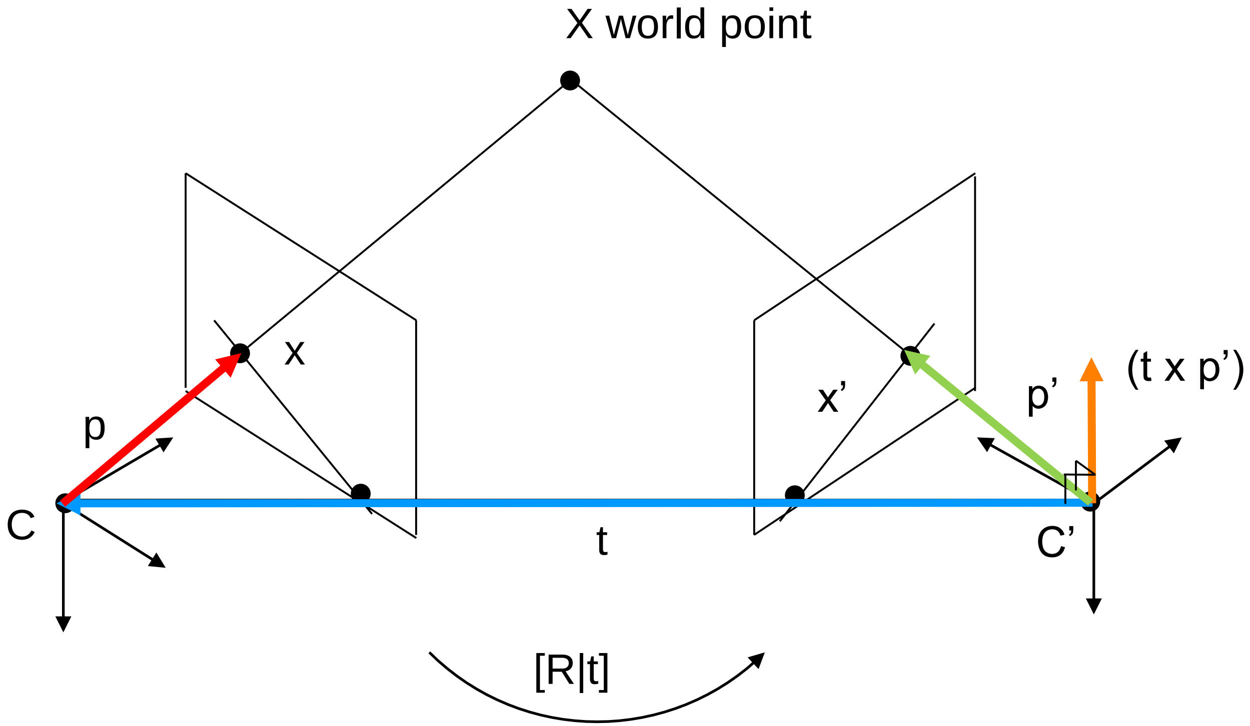
Epipolar constraint – derivation by coplanarity condition



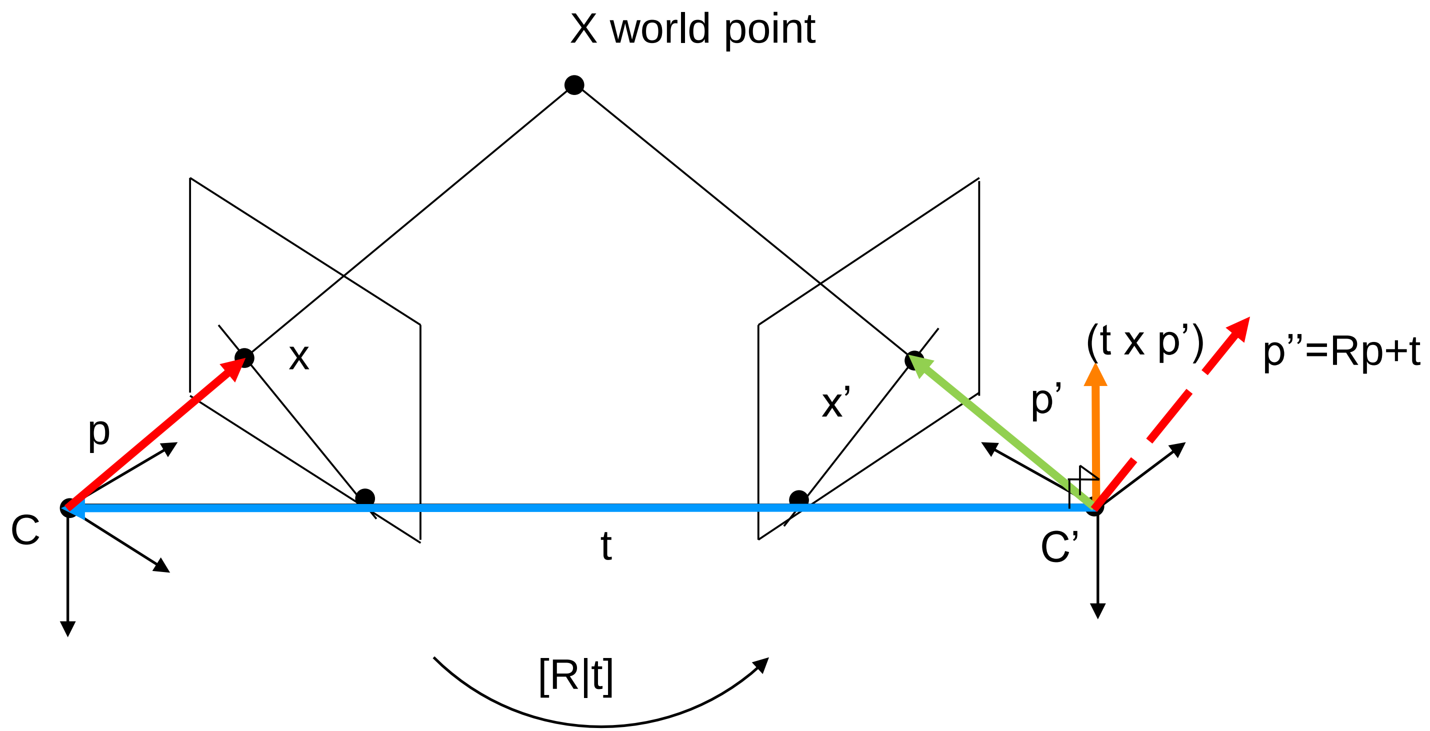
Epipolar constraint – derivation by coplanarity condition



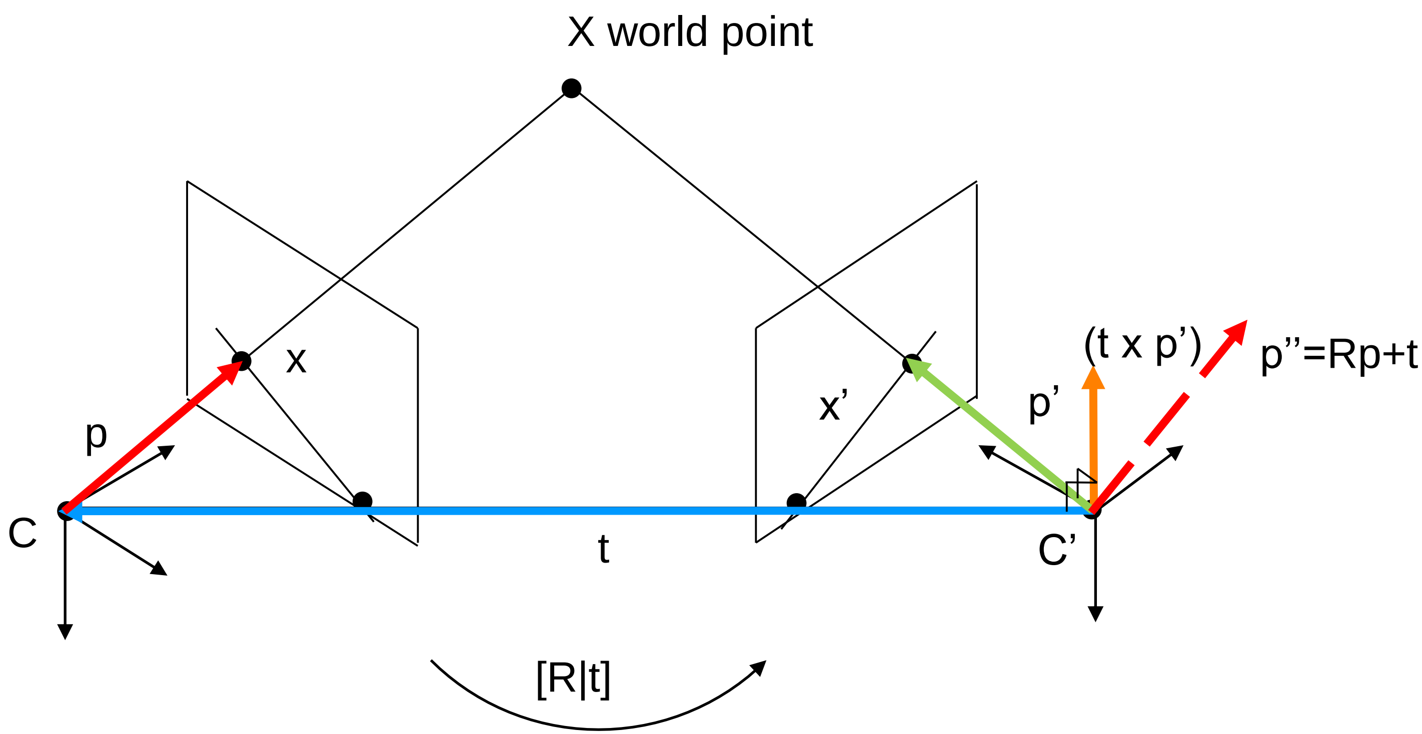
Epipolar constraint – derivation by coplanarity condition



Epipolar constraint – derivation by coplanarity condition



Epipolar constraint – derivation by coplanarity condition



$$t \times p' = t \times p''$$
$$t \times p' = t \times (Rp + t)$$
$$with\ p'' = Rp + t$$
$$t \times p' = t \times Rp + t \times t$$
$$p'^T (t \times p') = 0$$
$$p'^T (t \times Rp) = 0$$

$$p'^T \underbrace{[t]_x R}_{E} p = 0$$

$$\boxed{p'^T E p = 0}$$

E is called the Essential matrix

Fundamental matrix

- p,p' from the Essential matrix derivation are in normalized coordinates
- x,x' are image coordinates, x=Kp, x'=Kp'
- By replacing p,p' with x,x' one gets the Fundamental matrix

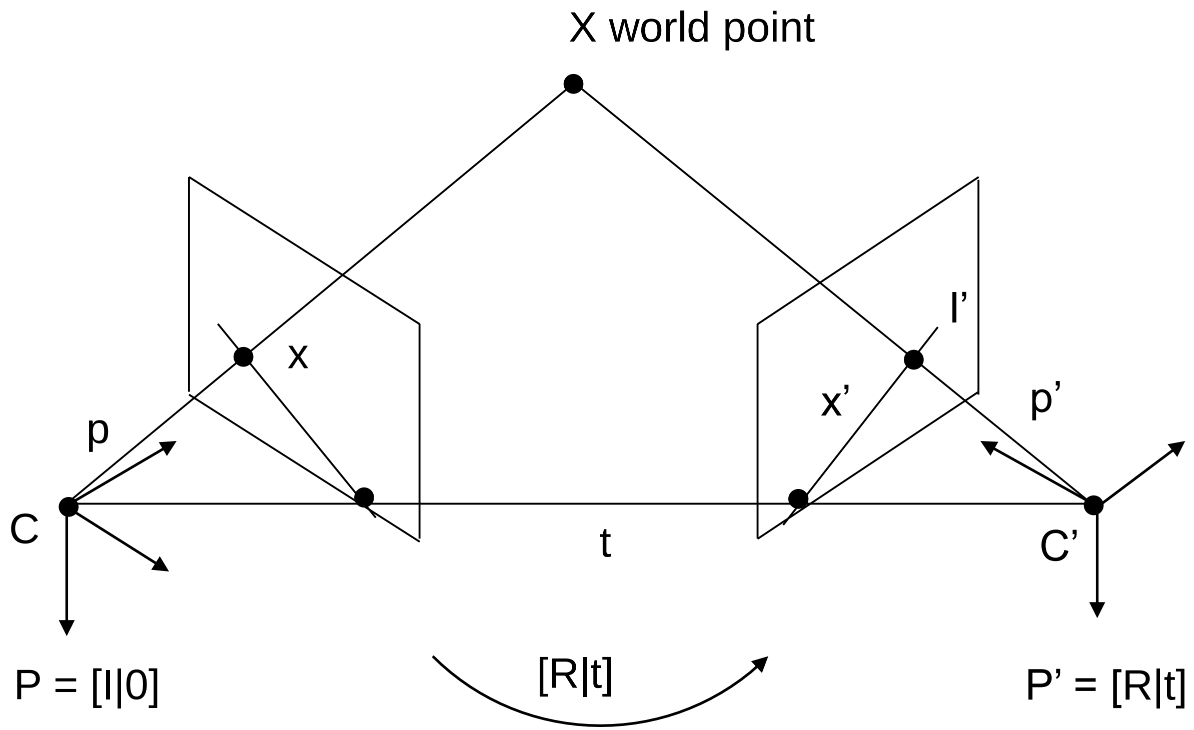
$$p = K^{-1}x$$

$$p' = K^{-1}x'$$

$$p'^T E p = 0$$
$$x'^T K^{-T} E K^{-1} x = 0$$
$$x'^T F x = 0$$
$$\mathrm{F} = K^{-T} E K^{-1}$$

Epipolar lines

- The corresponding line l’ to image coordinate x
- l’ is the line connecting the epipole e’ and the image coordinate x’
- Hypothesis: $l' = Fx$
- Point x’ must lie on l’, thus $x'^T l' = 0$
- Now $x'^T F x = 0$



Fundamental Matrix estimation – 8-point method

$$\boldsymbol{x}'^T F \boldsymbol{x} = 0$$

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

$$x'xf_1 + x'yf_2 + x'f_3 + y'xf_4 + y'yf_5 + y'f_6 + xf_7 + yf_8 + f_9 = 0$$

expanded epipolar constraint

$$(x'x, x'y, x', y'x, y'y, y', x, y, 1)\boldsymbol{f} = 0$$

expanded epipolar constraint as an inner product

- F … 3x3 fundamental matrix
- $f_1 \ldots f_9$ … individual entries of the fundamental matrix
- x,y,x',y' … image coordinates in pixel of a corresponding point

Fundamental Matrix estimation – 8-point method

$$(x'x, x'y, x', y'x, y'y, y', x, y, 1)\boldsymbol{f} = 0$$

$$\begin{bmatrix} x_1{}'x_1 & x_1'y_1 & x_1{}' & y_1{}'x_1 & y_1{}'y_1 & y_1{}' & x_1 & y_1 & 1 \\ & & . & . & . & . & . & . & . \\ x_n{}'x_n & x_n'y_n & x_n{}' & y_n{}'x_n & y_n{}'y_n & y_n{}' & x_n & y_n & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{pmatrix} = A\boldsymbol{f} = 0$$

- Every point correspondence x_i'<->x_i gives one equation. An equation system can be created by stacking them
- 8 points needed for the equation system

Solving linear equation systems using SVD

- Homogeneous case: $Ax = 0$
- Inhomogeneous case: $Ax = b$
- A … mxn matrix

Ax=0

- Trivial solution … x = 0
- A non-zero solution is up to scale, i.e. if x is a solution then also kx (with k being any scalar) is a solution as well

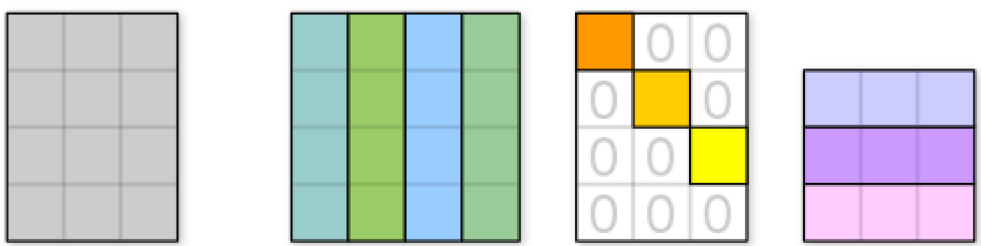
- Case m>=n: no exact solution, overdetermined equation system (e.g. 9x9)
- Case m=n-1: 1D null space (e.g. 8x9)
- Case m=n-2: 2D null space (e.g. 7x9)

- Solution: Compute null space or solution to overdetermined equation system using singular value decomposition (SVD)

Singular value decomposition (SVD)

$$M = USV^T$$

- SVD computes matrices U,S,V given M
- M … mxn matrix
- U … mxm unitary matrix (orthogonal matrix if M is real matrix)
- S … is a diagonal mxn matrix with non-negative real numbers on diagonal (singular values)
- V … nxn unitary matrix



$$\underset{m \times n}{M} = \underset{m \times m}{U} \quad \underset{m \times n}{S} \quad \underset{n \times n}{V^T}$$

[Image: Wikipedia (CC BY-SA 4.0)]

Ax=0, m=n-1

- e.g. 8 point correspondences for F, A … 8x9 matrix
- Solution x is the null space of A (1D null space)
- Null space can be computed using SVD
- The 1D null space is the column vector of V that belongs to the singular vector being 0.
- If x is a solution to Ax=0 then also kx is a solution with k being a scalar

Ax=0, m=n-1

- Example:

A = (8x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7
2	6	2	4	8	6	4	2	7

U = (8x8)

-0.24	0.26	-0.28	0.42	-0.34	-0.35	-0.53	-0.32
-0.34	0.60	0.36	-0.02	-0.12	-0.36	0.49	0.10
-0.45	0.08	-0.65	0.20	0.00	0.39	0.30	0.29
-0.38	-0.10	0.34	-0.31	-0.61	0.45	-0.24	0.05
-0.35	-0.24	0.34	0.36	0.34	-0.15	-0.32	0.58
-0.36	-0.24	0.27	0.32	0.29	0.23	0.22	-0.67
-0.34	-0.58	-0.22	-0.36	-0.13	-0.56	0.18	-0.06
-0.32	0.33	-0.12	-0.57	0.54	0.04	-0.37	-0.14

S = (8x9)

42.85	0	0	0	0	0	0	0	0
0	10.70	0	0	0	0	0	0	0
0	0	8.80	0	0	0	0	0	0
0	0	0	7.66	0	0	0	0	0
0	0	0	0	4.78	0	0	0	0
0	0	0	0	0	4.54	0	0	0
0	0	0	0	0	0	2.51	0	0
0	0	0	0	0	0	0	1.41	0

V = (9x9)

-0.25	-0.26	-0.25	-0.14	-0.31	0.29	0.78	0.00	0.00
-0.28	0.55	0.39	-0.34	-0.35	0.03	0.01	-0.44	-0.20
-0.37	-0.39	0.00	0.18	-0.37	-0.60	-0.14	0.04	-0.40
-0.32	-0.32	-0.11	-0.03	0.08	-0.06	-0.19	-0.62	0.59
-0.40	0.28	-0.09	-0.02	0.70	-0.38	0.34	0.00	-0.10
-0.36	-0.14	-0.27	-0.05	0.24	0.57	-0.37	-0.08	-0.51
-0.25	0.52	-0.61	0.27	-0.31	-0.02	-0.17	0.20	0.24
-0.38	0.01	0.55	0.63	0.01	0.29	0.06	0.19	0.18
-0.35	-0.10	0.16	-0.60	-0.01	-0.00	-0.21	0.58	0.30

x=null(A)	x = V(:,9)
x =	x =
0.0013	0.0013
-0.1953	-0.1953
-0.4040	-0.4040
0.5926	0.5926
-0.0992	-0.0992
-0.5054	-0.5054
0.2448	0.2448
0.1820	0.1820
0.2985	0.2985

Ax=?	A2x=?
ans =	ans =
-3.8858e-16	-7.7716e-16
0	0
4.4409e-16	8.8818e-16
-8.8818e-16	-1.7764e-15
8.8818e-16	1.7764e-15
-6.6613e-16	-1.3323e-15
1.3323e-15	2.6645e-15
-1.3323e-15	-2.6645e-15

Ax=0, m=n-2

- e.g. 7 point correspondences for F, A … 7x9 matrix
- Solution x is within the null space of A (2D null space)
- Null space can be computed using SVD
- The 2D null space is composed of the two column vector of V that belong to the singular vectors being 0.
- The solution x is a linear combination of the null space basis vectors, e.g. $x = x_1+w*x_2$

Ax=0, m=n-2

- Example:

A = (7x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7

U = (7x7)

-0.25	-0.34	0.36	-0.17	0.32	-0.72	-0.21
-0.36	-0.66	-0.31	0.26	0.35	0.38	-0.03
-0.48	-0.10	0.68	-0.01	-0.39	0.32	0.21
-0.40	0.07	-0.37	0.50	-0.50	-0.44	0.07
-0.37	0.15	-0.31	-0.54	0.17	-0.08	0.64
-0.39	0.15	-0.24	-0.47	-0.21	0.17	-0.69
-0.36	0.63	0.14	0.36	0.55	0.09	-0.11

S = (7x9)

40.64	0	0	0	0	0	0	0	0
0	10.28	0	0	0	0	0	0	0
0	0	8.76	0	0	0	0	0	0
0	0	0	6.41	0	0	0	0	0
0	0	0	0	4.54	0	0	0	0
0	0	0	0	0	3.29	0	0	0
0	0	0	0	0	0	1.53	0	0

V = (9x9)

-0.27	0.26	0.22	0.32	-0.30	0.35	-0.34	0.15	-0.60
-0.26	-0.52	-0.39	0.51	-0.06	-0.12	-0.40	-0.23	0.14
-0.40	0.30	0.01	-0.02	0.56	-0.46	-0.03	-0.34	-0.33
-0.32	0.34	0.08	-0.08	0.06	-0.13	-0.47	0.47	0.55
-0.38	-0.22	0.08	-0.21	0.44	0.72	0.03	-0.14	0.14
-0.35	0.19	0.24	-0.11	-0.55	-0.03	0.13	-0.60	0.31
-0.24	-0.53	0.68	0.07	-0.00	-0.29	0.18	0.26	-0.05
-0.41	-0.21	-0.44	-0.58	-0.30	-0.12	0.07	0.26	-0.28
-0.33	0.21	-0.25	0.48	0.01	0.06	0.67	0.28	0.13

$[x_1,x_2]$ =null(A)

ans =

0.15	-0.60
-0.23	0.14
-0.34	-0.33
0.47	0.55
-0.14	0.14
-0.60	0.31
0.26	-0.05
0.26	-0.28
0.28	0.13

$[x_1,x_2]$ = V(:,8:9)

ans =

0.15	-0.60
-0.23	0.14
-0.34	-0.33
0.47	0.55
-0.14	0.14
-0.60	0.31
0.26	-0.05
0.26	-0.28
0.28	0.13

$x = x_1+w*x_2$

Ax=?

w = 0.25

ans =
-3.3307e-16
-8.8818e-16
-1.3323e-15
-2.2204e-15
-8.8818e-16
4.4409e-16
-2.2204e-15
2.7638e-04

Ax=0, m>=n

- e.g. 9 point correspondences for F, A … 9x9 matrix
- Null space is of dimension 0 (if all equations are linearly independent)
- No exact solution, overdetermined equation system

Objective:
Find x that minimizes ||Ax|| subject to ||x||=1.

Algorithm:
x is the last column of V, where $A=UDV^T$ is the SVD of A

Ax=0, m>=n

- Example:

A = (9x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7
2	6	2	4	8	6	4	2	7
7	7	7	6	3	5	5	7	5

x = V(:,9)

x =

-0.03
-0.26
-0.40
0.52
-0.08
-0.50
0.26
0.20
0.37

U = (9x9)

-0.22	-0.26	-0.28	0.43	-0.08	-0.43	-0.55	0.28	-0.23
-0.32	-0.60	0.36	-0.03	0.06	-0.38	0.42	-0.26	-0.13
-0.42	-0.08	-0.65	0.20	0.03	0.38	0.20	-0.41	-0.03
-0.36	0.10	0.35	-0.29	-0.35	0.28	-0.46	-0.31	-0.38
-0.32	0.24	0.32	0.33	0.43	-0.06	-0.28	-0.28	0.53
-0.34	0.23	0.26	0.30	0.27	0.30	0.29	0.51	-0.42
-0.32	0.57	-0.23	-0.37	0.06	-0.58	0.16	-0.03	-0.13
-0.30	-0.34	-0.15	-0.60	0.39	0.17	-0.21	0.38	0.23
-0.37	0.03	0.07	0.06	-0.67	0.02	0.19	0.33	0.51

Ax=?

ans =

-6.3515e-02
-3.6645e-02
-9.3254e-03
-1.0446e-01
1.4559e-01
-1.1386e-01
-3.5147e-02
6.2170e-02
1.3899e-01

S = (9x9)

46.10	0	0	0	0	0	0	0	0
0	10.70	0	0	0	0	0	0	0
0	0	8.81	0	0	0	0	0	0
0	0	0	7.67	0	0	0	0	0
0	0	0	0	6.85	0	0	0	0
0	0	0	0	0	4.55	0	0	0
0	0	0	0	0	0	2.62	0	0
0	0	0	0	0	0	0	1.70	0
0	0	0	0	0	0	0	0	0.27

V = (9x9)

-0.28	0.26	-0.22	-0.11	-0.46	0.21	0.70	-0.20	-0.03
-0.30	-0.54	0.41	-0.31	-0.40	-0.05	0.01	0.35	-0.26
-0.38	0.39	0.01	0.19	-0.13	-0.67	-0.19	-0.08	-0.40
-0.32	0.32	-0.11	-0.03	-0.00	-0.03	-0.05	0.71	0.52
-0.37	-0.29	-0.13	-0.06	0.71	-0.20	0.45	0.03	-0.08
-0.35	0.14	-0.28	-0.06	0.15	0.61	-0.36	0.10	-0.50
-0.25	-0.52	-0.59	0.29	-0.26	-0.10	-0.23	-0.18	0.26
-0.39	-0.01	0.55	0.62	0.05	0.29	0.02	-0.20	0.20
-0.34	0.10	0.15	-0.61	0.08	-0.01	-0.29	-0.50	0.37

Ax=b

- Three cases:
  - m<n … more unknowns than equations. No unique solution, but a vector space of solutions.
  - m=n … a unique solution if A is invertible
  - m>n … more equations than unknowns. No exact solution exists.

- A … mxn matrix

Ax=b, m=n

- The system will have a unique solution if A is invertible
- Simple case
- Compute inverse of A and then
- $x = A^{-1}b$

Ax=b, m=n

- Example:

A = (9x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7
2	6	2	4	8	6	4	2	7
7	7	7	6	3	5	5	7	5

b = (9x1)

2
6
3
1
7
3
4
7
4

x = inv(A)*b

```
x =
  -7.6360e-01
  -3.3165e+00
  -5.3983e+00
   7.7206e+00
  -4.0799e-01
  -7.1809e+00
   3.4356e+00
   2.6433e+00
   5.0943e+00
```

Ax-b = ?

```
ans =
  -1.0658e-14
  -1.0658e-14
  -1.4211e-14
  -1.4211e-14
  -1.7764e-14
  -2.1316e-14
  -1.0658e-14
  -3.5527e-15
  -1.7764e-14
```

Ax=b, m>n

- More equations than unknowns. No exact solution exists.
- Therefore we are seeking for a vector x that is closest to a solution of Ax=b.
- This means, we are seeking an x such that ||Ax-b|| is minimized
- Such an x is called the least-squares solution

Objective:
Find x that minimizes ||Ax-b||.

Algorithm:
1. Compute the SVD of $A=USV^T$
2. Set $b'=U^Tb$
3. Compute the vector y defined by $y_i=b_i'/s_i$, where s_i is the i-th diagonal entry of S
4. The solution is then x=Vy

Ax=b, m>n

- Example:

A = (10x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7
2	6	2	4	8	6	4	2	7
7	7	7	6	3	5	5	7	5
7	4	7	7	4	1	3	4	3

b = (10x1)

2
6
3
1
7
3
4
7
4
2

U = (10x10)

-0.21	-0.24	-0.30	-0.26	0.35	-0.15	-0.61	-0.42	-0.08	-0.22
-0.31	-0.59	0.31	-0.17	-0.09	-0.40	0.03	0.47	-0.21	-0.05
-0.40	-0.11	-0.63	0.12	0.26	0.31	0.25	0.41	0.14	-0.08
-0.34	0.05	0.36	-0.01	-0.34	0.50	-0.27	0.04	0.26	-0.50
-0.31	0.15	0.38	0.24	0.48	-0.15	-0.19	0.08	0.50	0.38
-0.33	0.17	0.30	0.14	0.40	0.06	0.43	-0.30	-0.50	-0.27
-0.31	0.54	-0.19	0.29	-0.28	-0.34	-0.34	0.24	-0.35	0.01
-0.28	-0.39	-0.12	0.53	-0.40	-0.09	0.13	-0.49	0.12	0.19
-0.36	0.05	0.02	-0.46	-0.17	0.37	-0.03	-0.11	-0.25	0.64
-0.28	0.28	-0.11	-0.48	-0.18	-0.43	0.38	-0.19	0.41	-0.17

S = (10x9)

47.98	0	0	0	0	0	0	0	0
0	11.04	0	0	0	0	0	0	0
0	0	8.84	0	0	0	0	0	0
0	0	0	7.85	0	0	0	0	0
0	0	0	0	7.64	0	0	0	0
0	0	0	0	0	5.43	0	0	0
0	0	0	0	0	0	3.49	0	0
0	0	0	0	0	0	0	1.89	0
0	0	0	0	0	0	0	0	0.73
0	0	0	0	0	0	0	0	0

V = (9x9)

-0.30	0.35	-0.26	-0.32	-0.28	0.19	0.55	0.45	-0.09
-0.30	-0.50	0.34	-0.35	-0.50	0.05	0.01	-0.27	-0.33
-0.39	0.42	0.00	-0.21	0.11	-0.34	-0.59	0.12	-0.37
-0.34	0.36	-0.11	-0.04	-0.05	-0.15	0.20	-0.74	0.35
-0.36	-0.32	-0.10	0.42	0.15	-0.62	0.36	0.14	-0.15
-0.33	0.01	-0.20	0.49	0.14	0.61	-0.05	-0.19	-0.42
-0.25	-0.46	-0.64	-0.32	0.16	0.10	-0.26	0.05	0.32
-0.38	-0.06	0.55	-0.21	0.60	0.22	0.14	0.14	0.24
-0.34	0.03	0.19	0.41	-0.49	0.08	-0.30	0.29	0.51

$b' = U^T b$

b_ = (10x1)

-12.13
-2.50
1.56
3.01
0.15
-3.49
-0.37
-0.43
0.42
3.93

$y_i = b_i'/s_i$

y = (9x1)

-0.25
-0.23
0.18
0.38
0.02
-0.64
-0.10
-0.23
0.57

$x=Vy$

x =

-0.51
-0.06
-0.03
0.41
0.56
-0.35
0.07
0.09
0.46

$Ax-b$

ans =

0.87
0.21
0.32
1.96
-1.51
1.05
-0.03
-0.73
-2.51
0.67

Ax=b, m>n

- Example:

A = (10x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7
2	6	2	4	8	6	4	2	7
7	7	7	6	3	5	5	7	5
7	7	7	6	3	5	5	7	5

b = (10x1)

2.00
6.00
3.00
1.00
7.00
3.00
4.00
7.00
4.00
4.10

U = (10x10)

-0.21	-0.26	-0.26	0.19	-0.40	-0.43	0.55	-0.28	-0.25	0
-0.30	-0.60	0.33	-0.17	0.00	-0.38	-0.40	0.29	-0.14	0.00
-0.39	-0.09	-0.66	0.09	-0.20	0.38	-0.17	0.43	-0.04	-0.00
-0.34	0.10	0.36	0.06	0.34	0.27	0.52	0.35	-0.41	-0.00
-0.30	0.23	0.24	-0.41	-0.43	-0.06	0.26	0.23	0.57	-0.00
-0.32	0.23	0.20	-0.26	-0.37	0.30	-0.31	-0.47	-0.45	0.00
-0.30	0.57	-0.27	-0.16	0.34	-0.58	-0.16	0.05	-0.14	-0.00
-0.27	-0.35	-0.23	-0.46	0.49	0.18	0.16	-0.43	0.24	-0.00
-0.35	0.04	0.14	0.48	0.08	0.01	-0.11	-0.19	0.27	-0.71
-0.35	0.04	0.14	0.48	0.08	0.01	-0.11	-0.19	0.27	0.71

S = (10x9)

49.21	0	0	0	0	0	0	0	0
0	10.71	0	0	0	0	0	0	0
0	0	8.85	0	0	0	0	0	0
0	0	0	8.11	0	0	0	0	0
0	0	0	0	7.65	0	0	0	0
0	0	0	0	0	4.55	0	0	0
0	0	0	0	0	0	2.64	0	0
0	0	0	0	0	0	0	1.75	0
0	0	0	0	0	0	0	0	0.29
0	0	0	0	0	0	0	0	0

V = (9x9)

-0.29	0.27	-0.13	0.45	0.23	0.21	-0.68	0.25	-0.03
-0.31	-0.53	0.47	0.23	0.40	-0.05	-0.01	-0.33	-0.26
-0.38	0.39	0.03	0.13	-0.16	-0.68	0.20	0.09	-0.40
-0.33	0.32	-0.11	0.02	0.03	-0.03	0.02	-0.72	0.51
-0.34	-0.31	-0.26	-0.66	-0.12	-0.19	-0.48	-0.04	-0.08
-0.34	0.13	-0.31	-0.13	0.02	0.61	0.35	-0.11	-0.50
-0.26	-0.51	-0.53	0.43	-0.20	-0.10	0.24	0.17	0.26
-0.39	-0.01	0.54	-0.01	-0.63	0.29	-0.01	0.20	0.20
-0.34	0.09	0.10	-0.30	0.56	-0.01	0.31	0.48	0.38

$b' = U^T b$

b_ = (10x1)

-12.72
-1.93
0.59
-3.96
0.23
-2.21
-0.80
-1.30
4.13
0.07

$y_i = b_i'/s_i$

y = (9x1)

-0.26
-0.18
0.07
-0.49
0.03
-0.49
-0.30
-0.74
14.08

$x=Vy$

x =

-0.76
-3.33
-5.44
7.78
-0.42
-7.23
3.46
2.66
5.12

$Ax-b$

ans =

0.00
-0.00
0.00
0.00
0.00
0
0.00
0
0.05
-0.05

Ax=b, m<n

- More unknowns than equations. No unique solution, but a vector space of solutions (which are exact).
- This is called a deficient-rank system.

Objective:
Find the general solution to Ax=b where A is an mxn matrix of rank r<n.

Algorithm:
1. Compute the SVD of $A=USV^T$, where the singular values are in descending order.
2. Set $b'=U^Tb$
3. Compute the vector y defined by $y_i=b_i'/s_i$, for i=1..r, and $y_i=0$ otherwise.
4. The solution x of minimum norm is then x=Vy
5. The general solution is $x = Vy+w_{r+1}v_{r+1}+\ldots+w_nv_n$, where $v_{r+1},\ldots,v_n$ are the last n-r columns of V

Ax=b, m<n

- Example:

A = (8x9)

1	3	5	3	4	3	7	4	1
2	9	4	2	8	2	5	7	5
7	3	6	6	8	9	9	6	5
5	7	6	5	3	6	2	7	8
2	2	7	5	6	5	1	9	5
4	3	6	6	6	6	1	9	4
6	1	9	7	5	5	1	2	7
2	6	2	4	8	6	4	2	7

b = (8x1)

2
6
3
1
7
3
4
7

U = (8x8)

-0.24	0.26	-0.28	0.42	-0.34	-0.35	-0.53	-0.32
-0.34	0.60	0.36	-0.02	-0.12	-0.36	0.49	0.10
-0.45	0.08	-0.65	0.20	0.00	0.39	0.30	0.29
-0.38	-0.10	0.34	-0.31	-0.61	0.45	-0.24	0.05
-0.35	-0.24	0.34	0.36	0.34	-0.15	-0.32	0.58
-0.36	-0.24	0.27	0.32	0.29	0.23	0.22	-0.67
-0.34	-0.58	-0.22	-0.36	-0.13	-0.56	0.18	-0.06
-0.32	0.33	-0.12	-0.57	0.54	0.04	-0.37	-0.14

S = (8x9)

42.85	0	0	0	0	0	0	0	0
0	10.70	0	0	0	0	0	0	0
0	0	8.80	0	0	0	0	0	0
0	0	0	7.66	0	0	0	0	0
0	0	0	0	4.78	0	0	0	0
0	0	0	0	0	4.54	0	0	0
0	0	0	0	0	0	2.51	0	0
0	0	0	0	0	0	0	1.41	0

V = (9x9)

-0.25	-0.26	-0.25	-0.14	-0.31	0.29	0.78	0.00	0.00
-0.28	0.55	0.39	-0.34	-0.35	0.03	0.01	-0.44	-0.20
-0.37	-0.39	0.00	0.18	-0.37	-0.60	-0.14	0.04	-0.40
-0.32	-0.32	-0.11	-0.03	0.08	-0.06	-0.19	-0.62	0.59
-0.40	0.28	-0.09	-0.02	0.70	-0.38	0.34	0.00	-0.10
-0.36	-0.14	-0.27	-0.05	0.24	0.57	-0.37	-0.08	-0.51
-0.25	0.52	-0.61	0.27	-0.31	-0.02	-0.17	0.20	0.24
-0.38	0.01	0.55	0.63	0.01	0.29	0.06	0.19	0.18
-0.35	-0.10	0.16	-0.60	-0.01	-0.00	-0.21	0.58	0.30

$b' = U^T b$

b_ = (8x1)

-11.42
1.79
1.50
-0.98
4.51
-3.51
-0.89
1.67

$y_i = b_i'/s_i$

y = (9x1)

-0.27
0.17
0.17
-0.13
0.94
-0.77
-0.35
1.18
0

$x=Vy$

x =

-0.78
-0.60
0.22
-0.53
0.97
-0.15
0.03
0.11
0.94

$Ax-b$

ans =

-0.00
0.00
-0.00
-0.00
-0.00
0.00
0.00
-0.00

$A(x+5*v)-b$

ans =

-0.00
-0.00
-0.00
-0.00
-0.00
0.00
0.00
-0.00

Recap - Learning goals

- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand the calculation of the fundamental matrix
- Understand linear equation system solving