

Mathematical Principles in Vision and Graphics:  
Sylvester Resultant  
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SS2018

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# Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown  $x$ :

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$$

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Sylvester matrix of  $f$  and  $g$ :

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$\det(\text{Syl}(f, g))$  is called the Sylvester Resultant of  $f$  and  $g$ .

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Let's consider  $y$  as a constant, and write these two polynomials as polynomials in  $x$ :

$$\begin{aligned} p_{1,y}(x) &= 6x^2 + (3y - y^2)x + (y + 1) \\ p_{2,y}(x) &= yx^2 + 5x + (4y - 1) \end{aligned}$$

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→ First solve for  $y$  (we will see later how it can be done). 