
Mathematical Principles in Vision and Graphics: Projective Geometry – Part 2

Ass.Prof. Friedrich Fraundorfer

SS 2018

Outline

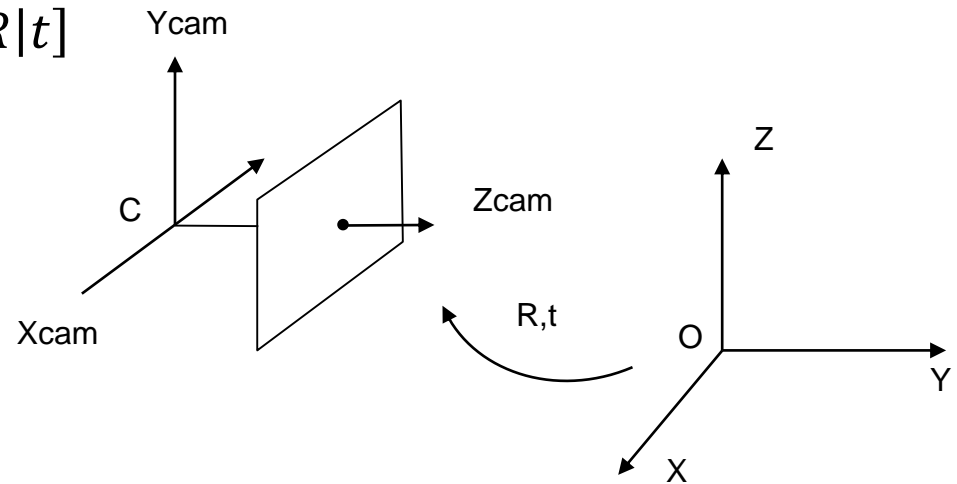
- Axioms of geometry
- Differences between Euclidean and projective geometry
- 2D projective geometry
 - Homogeneous coordinates
 - Points, Lines
 - Duality
- 3D projective geometry
 - Points, Lines, Planes
 - Duality
 - Plane at infinity
 - Image formation
 - **Parameterizing a camera matrix**
 - **Projection of points and lines**
- **Geometric relationships**
 - Epipolar constraint derivation
 - Stereo normal case
 - Triangulation
 - Camera pose estimation (algebraic, from points and lines)

Learning goals

- Understand the geometric interpretation of the camera matrix
- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation

Camera matrix (calibrated camera)

$$P = KR[I|-C] = K[R|-RC] = K[R|t]$$
$$t = -RC$$



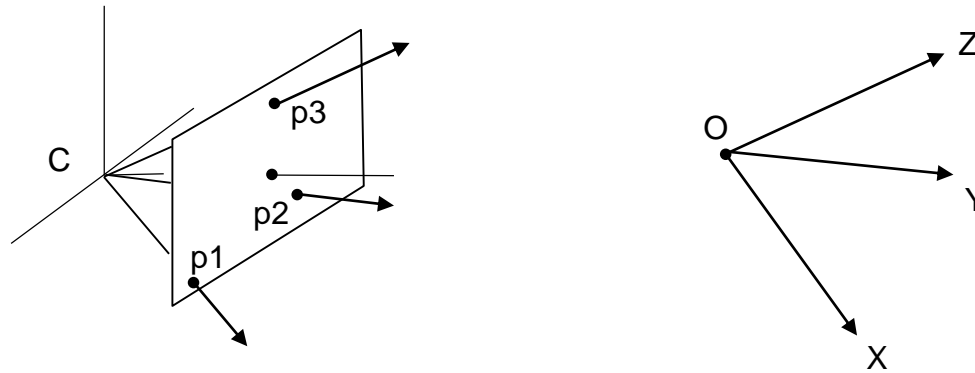
- Camera matrix is a coordinate transformation and then a projection
- C ... 3x1 coordinate of the camera center in world coordinate
- R ... 3x3 rotation matrix representing the orientation of the camera coordinate frame
- K ... 3x3 calibration matrix

Geometric interpretation of camera matrix entries

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = [p_1 \quad p_2 \quad p_3 \quad p_4] = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}$$

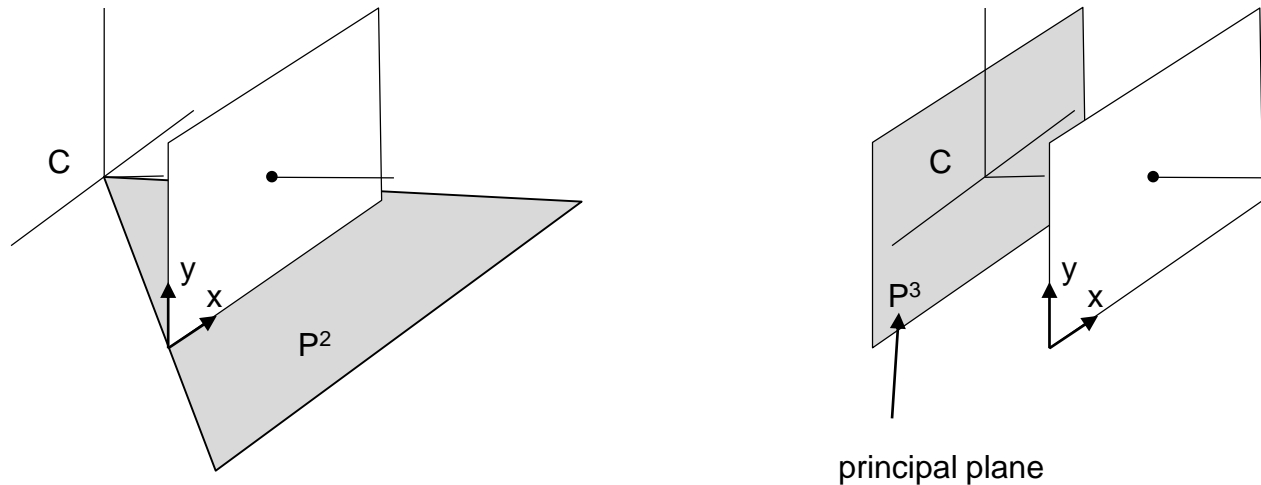
- Columns (3x1 vectors) are unit vectors of the coordinate system
- Rows transposed (1x4 vectors) are planes spanned by coordinate system axis

Geometric interpretation of camera matrix entries



- p_1, p_2, p_3 are the images of the axis directions (vanishing points)
- x-axis ... $D=(1,0,0,0)$ is imaged as $p_1=PD$

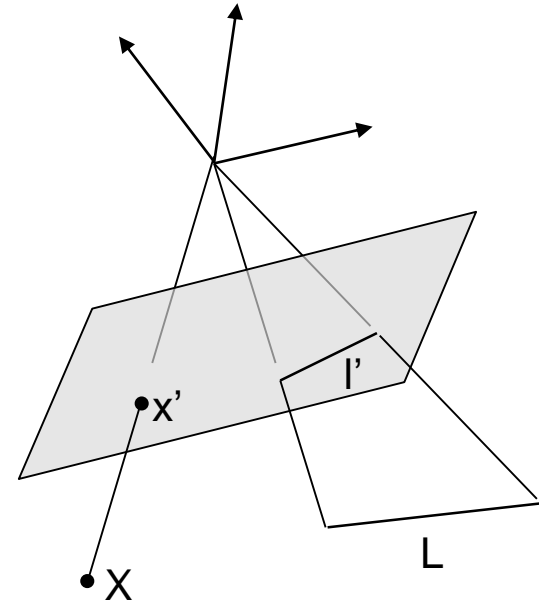
Geometric interpretation of camera matrix entries



- P^{3T} is the principal plane, the plane parallel to the image plane through the camera center
- $PX = (x, y, 0)$ all points on the principal plane have the following coordinates
- Plane condition, $P^{3T}X = 0$ (point X is on the plane if it fulfils the condition)
- Points on P^2 , $P^{2T}X = 0$, all points have coordinates $PX = (x, 0, z)$
- Plane P^2 is defined by the line $y=0$ and the camera center

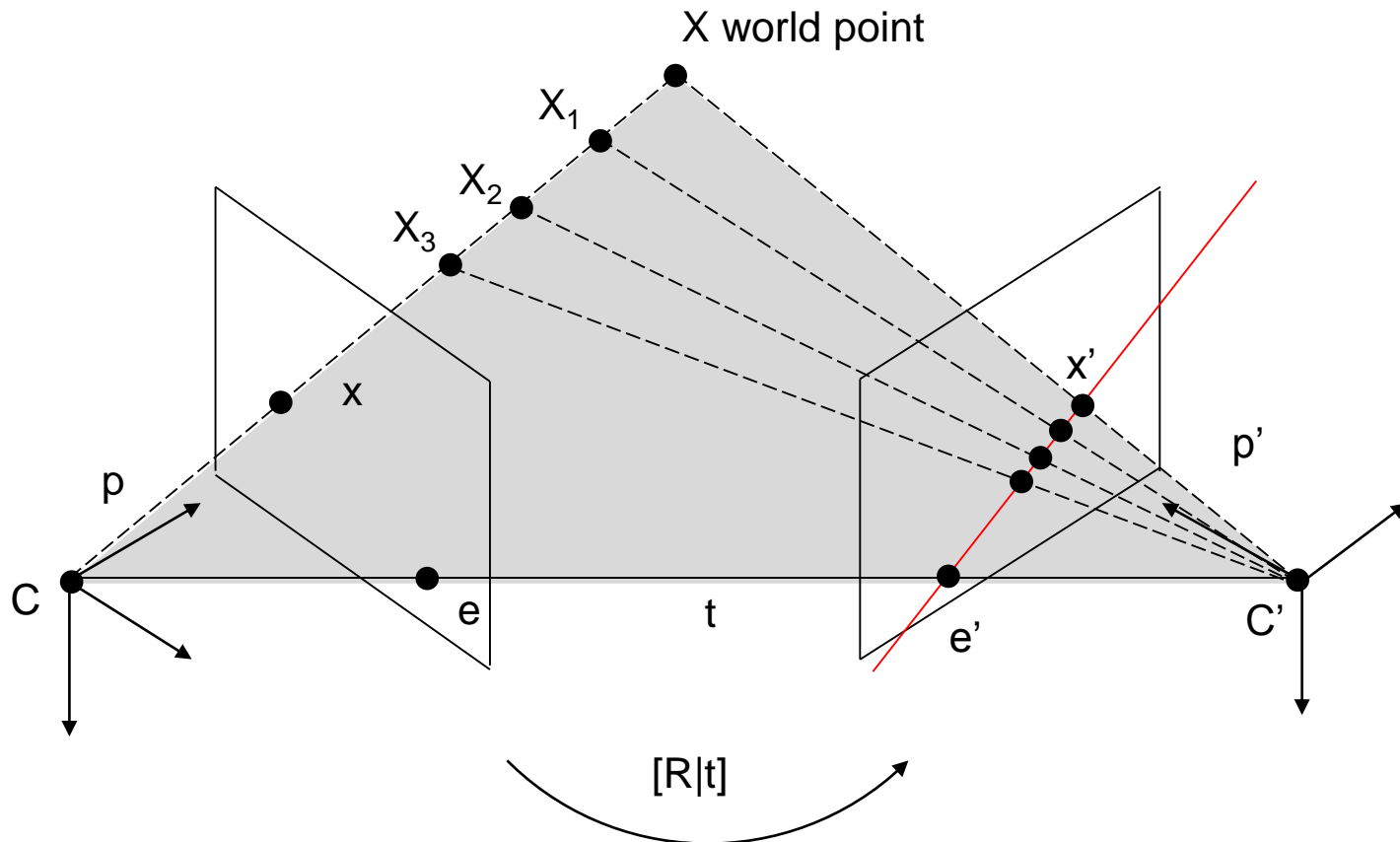
Point and line projection

- Point projection $x = PX$
- Line projection is more involved (line l is a 4x4 matrix)
- Therefore indirect projection:
 $l' = x' \times y' = PX \times PY$
 $L = \overline{XY}$



Epipolar constraint

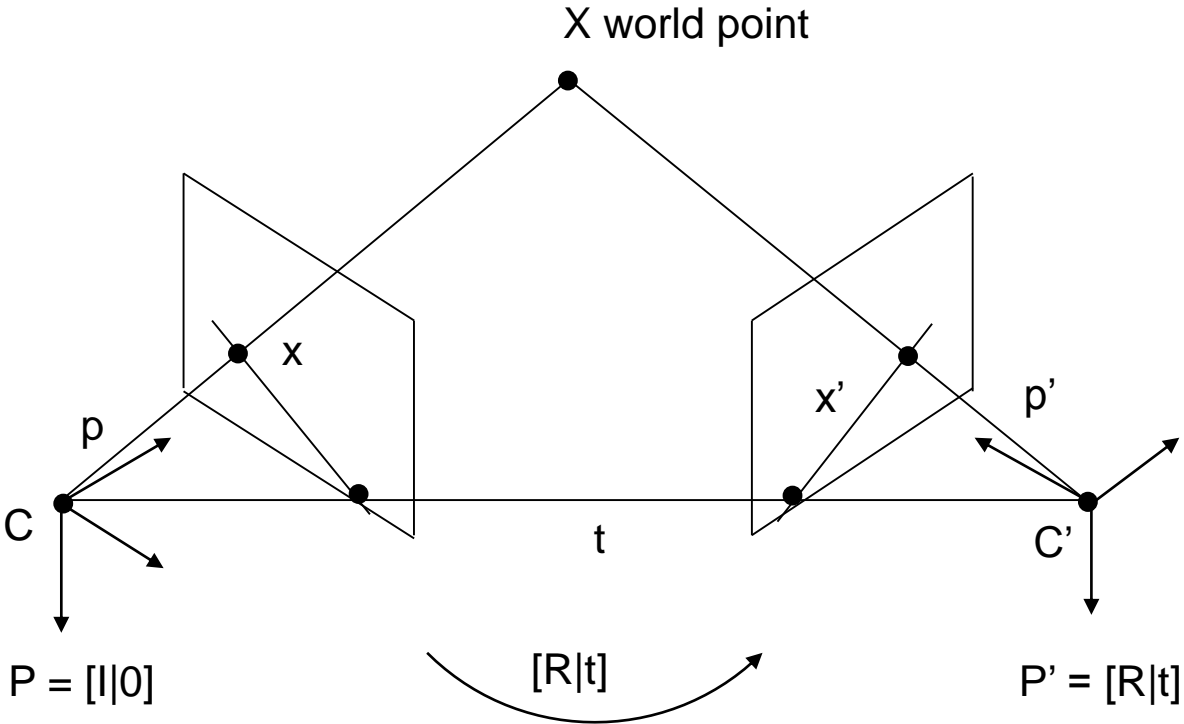
- An image point x lies on its corresponding epipolar line in the corresponding image.



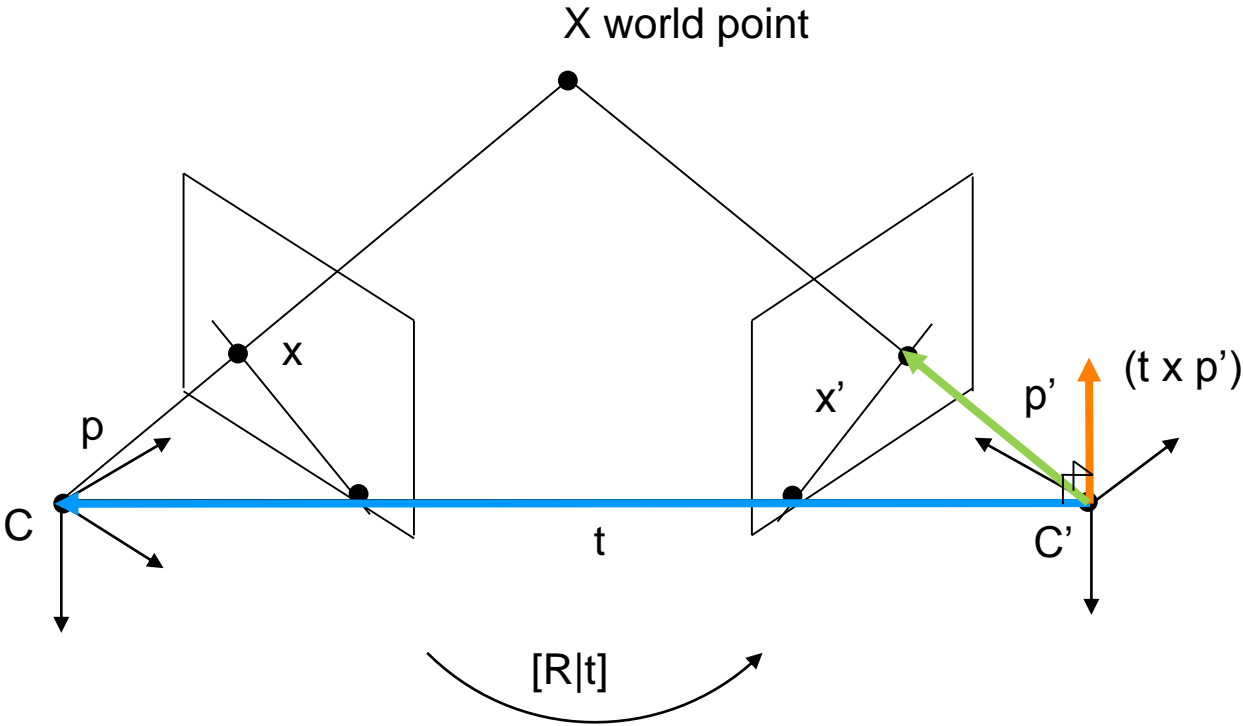
Epipolar constraint – derivation by coplanarity condition

- Vector p and t define a plane
- Vector p' and t define also a plane
- Both planes must have the same normal
- What we seek is a relation between p and p'

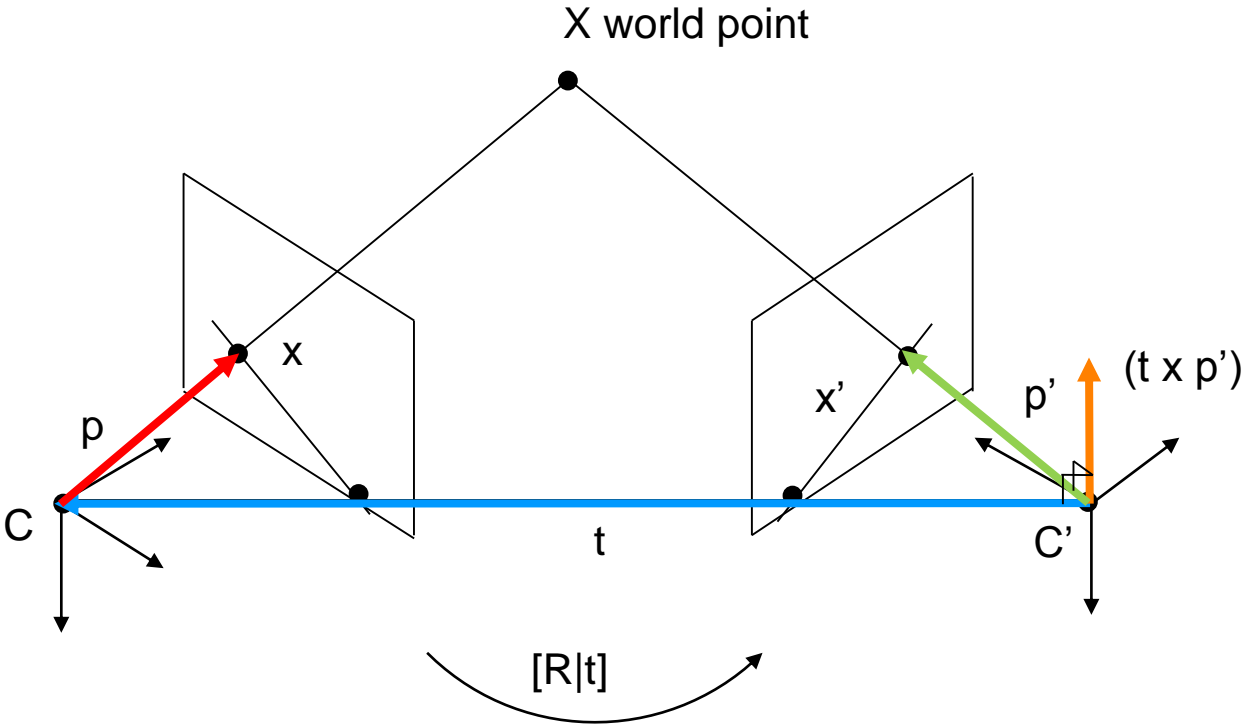
Epipolar constraint – derivation by coplanarity condition



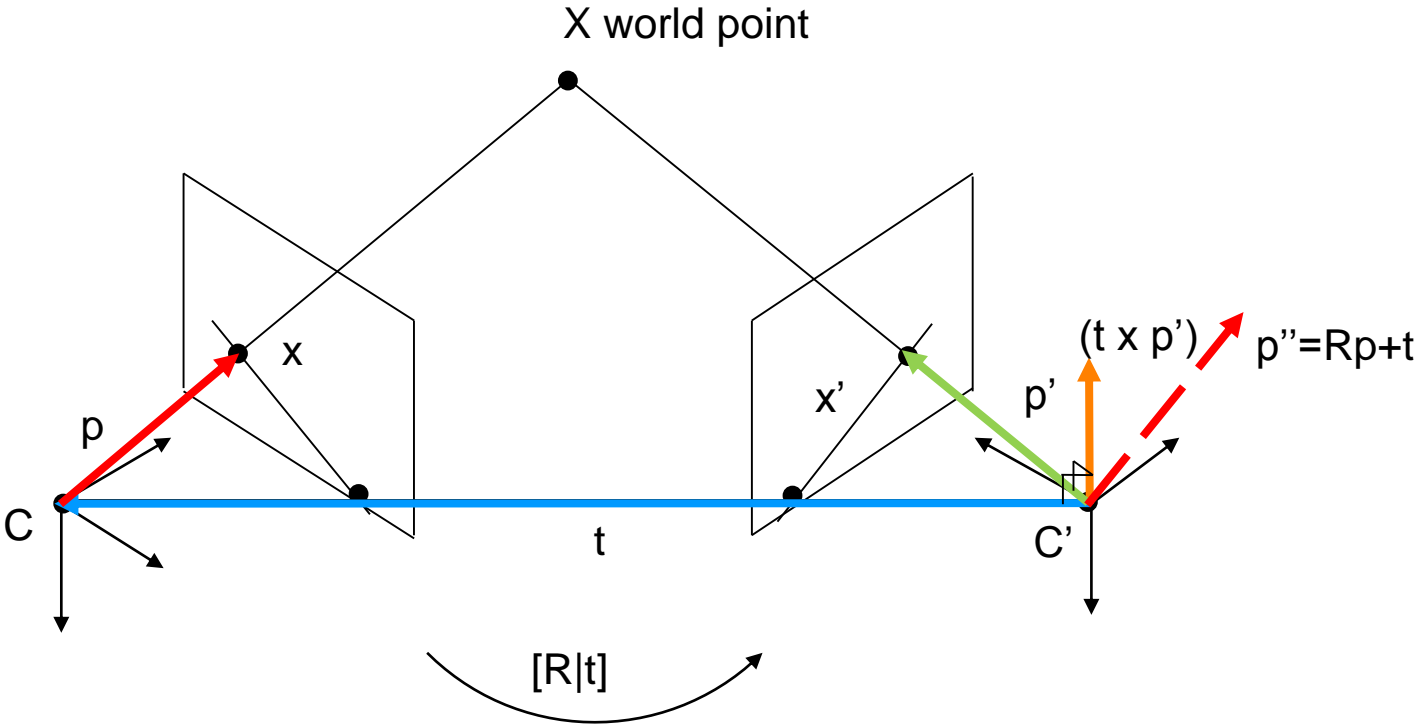
Epipolar constraint – derivation by coplanarity condition



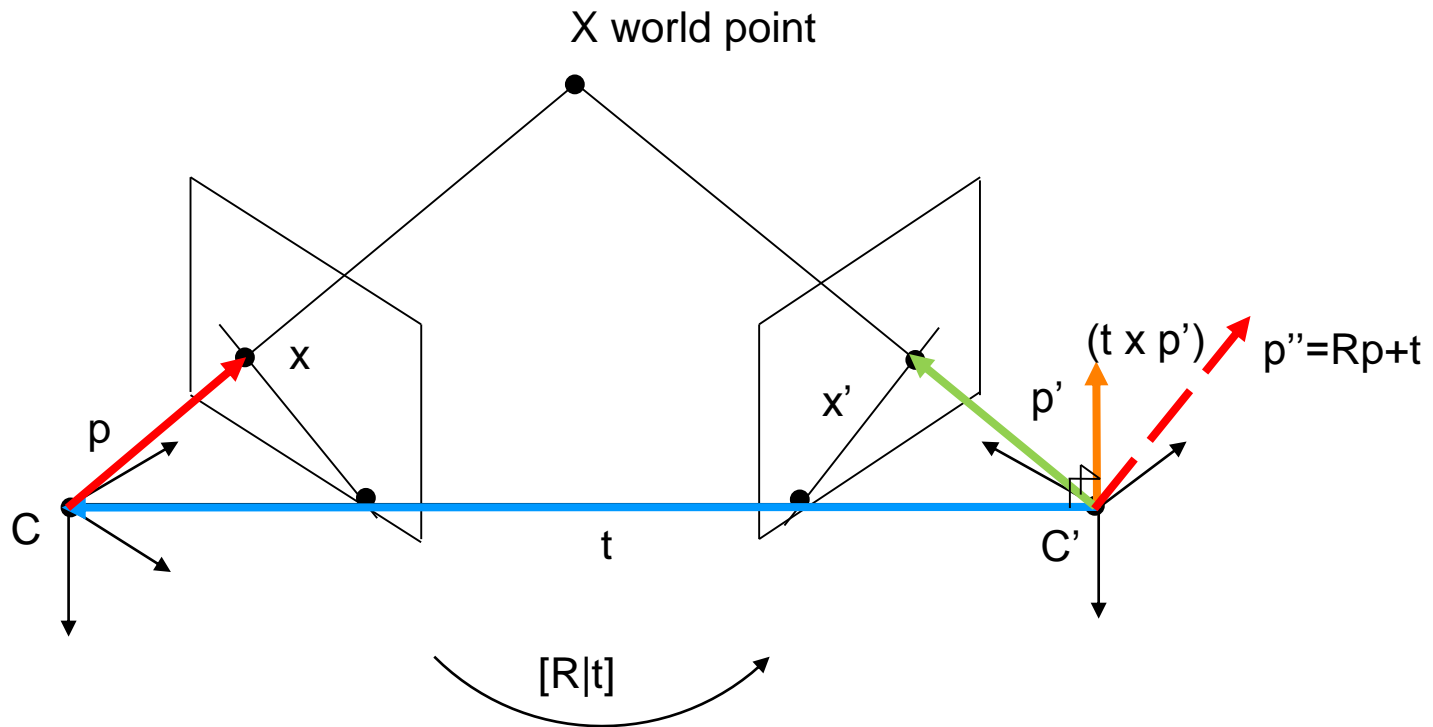
Epipolar constraint – derivation by coplanarity condition



Epipolar constraint – derivation by coplanarity condition



Epipolar constraint – derivation by coplanarity condition



$$\begin{aligned}
 t \times p' &= t \times p'' \\
 t \times p' &= t \times (Rp + t) \\
 \text{with } p'' &= Rp + t \\
 t \times p' &= t \times Rp + t \times t \\
 p'^T (t \times p') &= 0 \\
 p'^T (t \times Rp) &= 0
 \end{aligned}$$

$$p'^T \underbrace{[t]_x Rp}_{E} = 0$$

$$p'^T E p = 0$$

E is called the Essential matrix

Fundamental matrix

- p, p' from the Essential matrix derivation are in normalized coordinates
- x, x' are image coordinates, $x = Kp$, $x' = Kp'$
- By replacing p, p' with x, x' one gets the Fundamental matrix

$$p'^T E p = 0$$

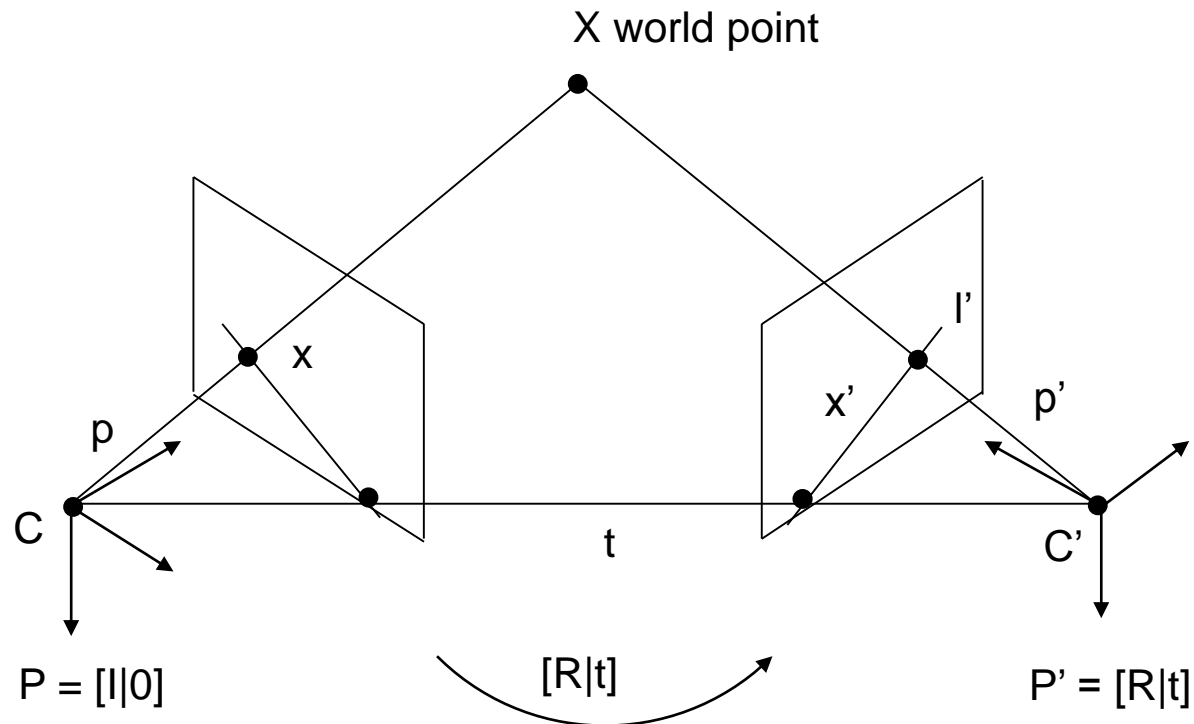
$$x'^T K^{-T} E K^{-1} x = 0$$

$$x'^T F x = 0$$

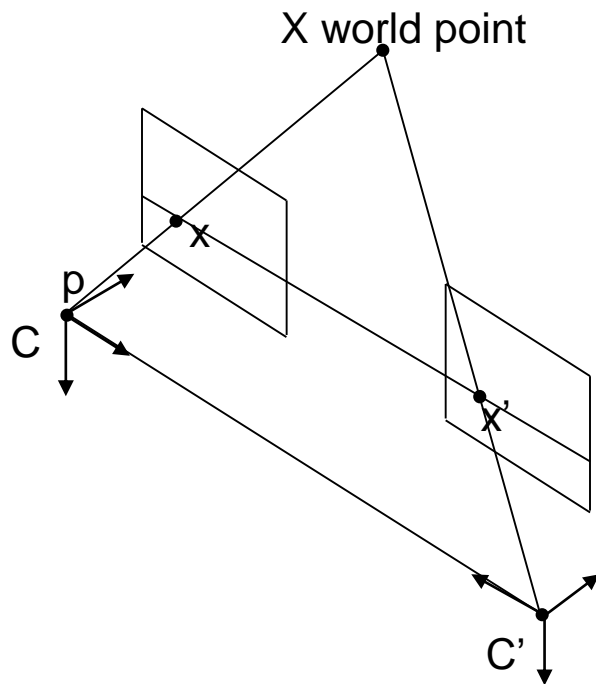
$$F = K^{-T} E K^{-1}$$

Epipolar lines

- The corresponding line l' to image coordinate x
- l' is the line connecting the epipole e' and the image coordinate x'
- Hypothesis: $l' = Fx$
- Point x' must lie on l' , thus $x'l' = 0$
- Now $x'Fx = 0$

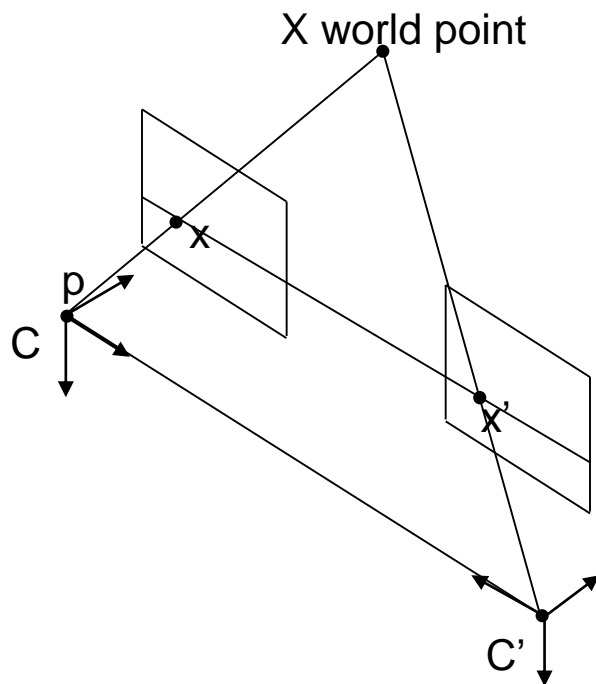


Stereo case



$$R = I_{3 \times 3} \quad T = [T_x \quad 0 \quad 0]^T$$

Stereo case



$$R = I_{3 \times 3} \quad T = [T_x \quad 0 \quad 0]^T$$

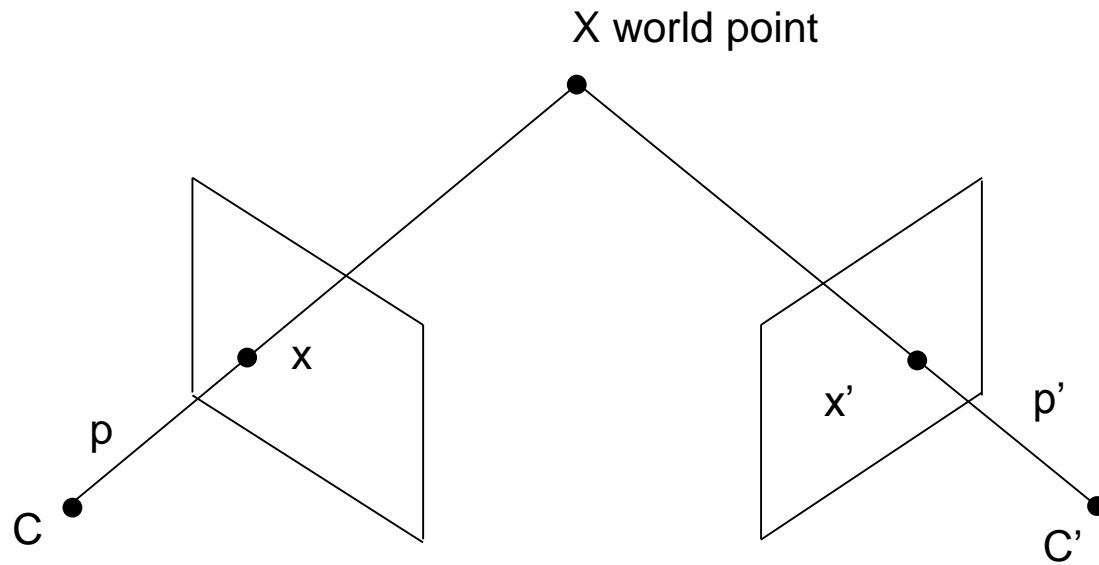
$$E = [T]_x R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix}$$

$$[x' \quad y' \quad 1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$[x' \quad y' \quad 1] \begin{bmatrix} 0 \\ -T_x \\ T_x y \end{bmatrix} = 0$$

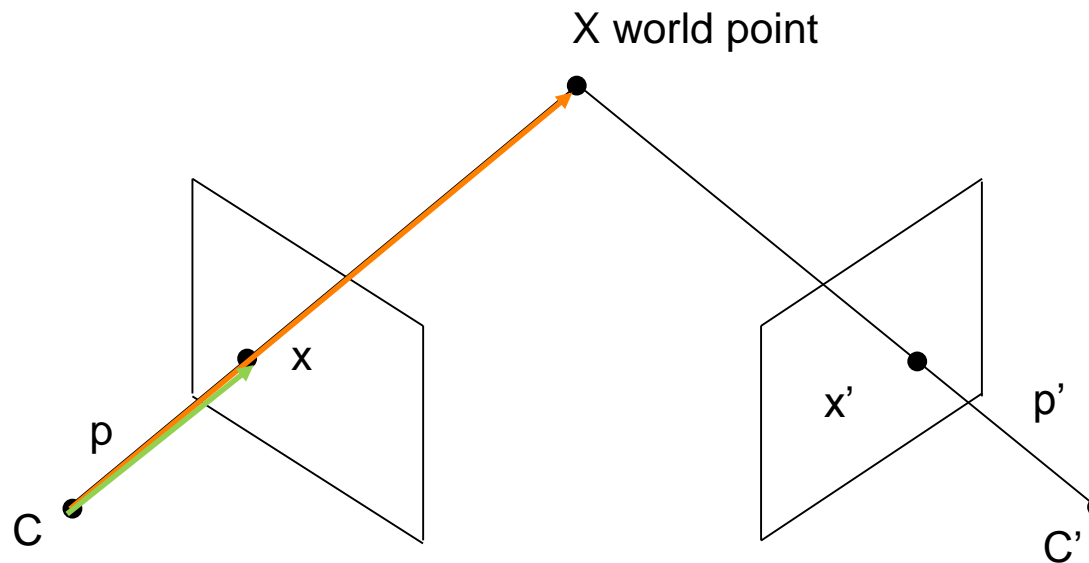
$$-y' T_x + T_x y = 0$$

Triangulation



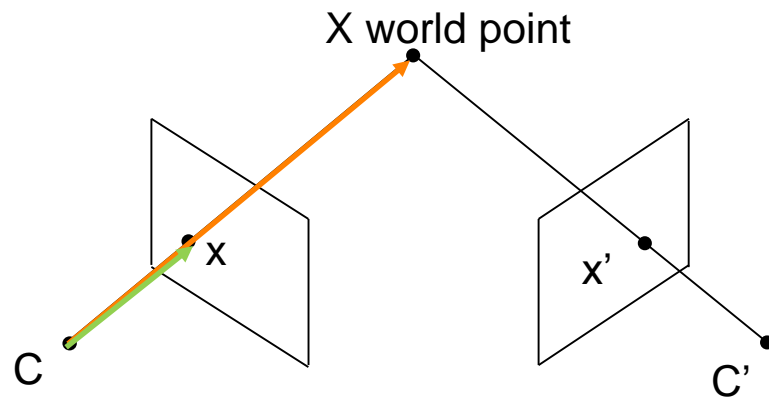
- Compute coordinates of world point X given the measurements x , x' and the camera projection matrices P and P'

Triangulation

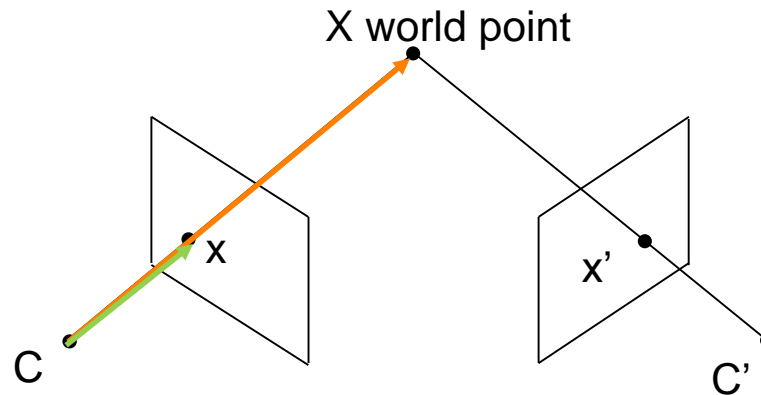


- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)
- $x \times (PX) = 0$ and $x' \times (P'X) = 0$
- Can be rewritten into equation system $AX = 0$ to solve for X

Triangulation



Triangulation



$$x \times (PX) = 0 \text{ and } x' \times (P'X) = 0$$

$$x(P_3^T X) - (P_1^T X) = 0$$

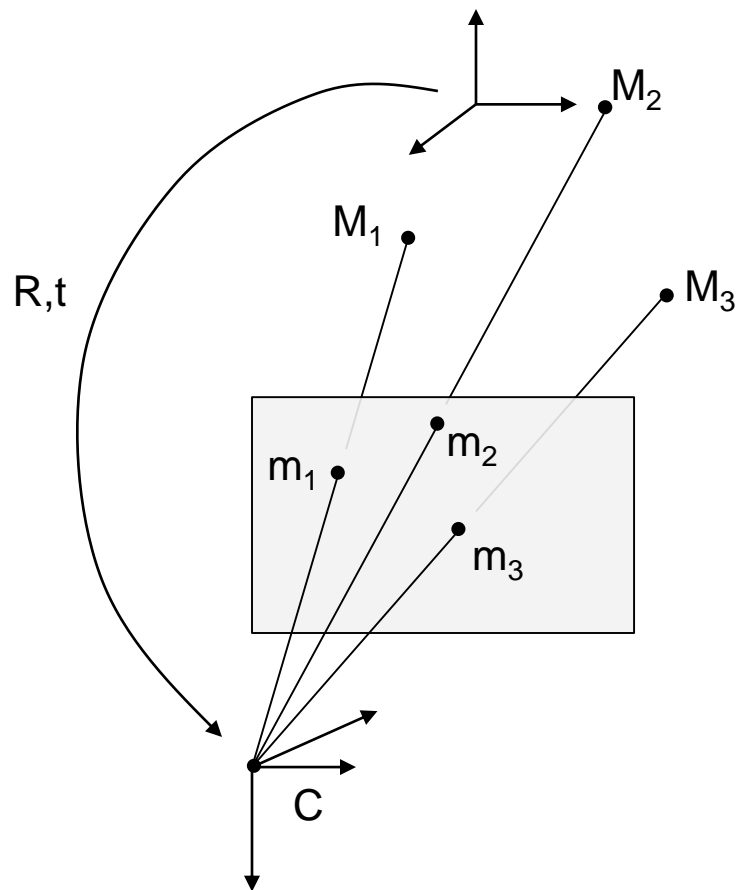
$$y(P_3^T X) - (P_2^T X) = 0$$

$$x(P_2^T X) - y(P_1^T X) = 0$$

$$P = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}$$

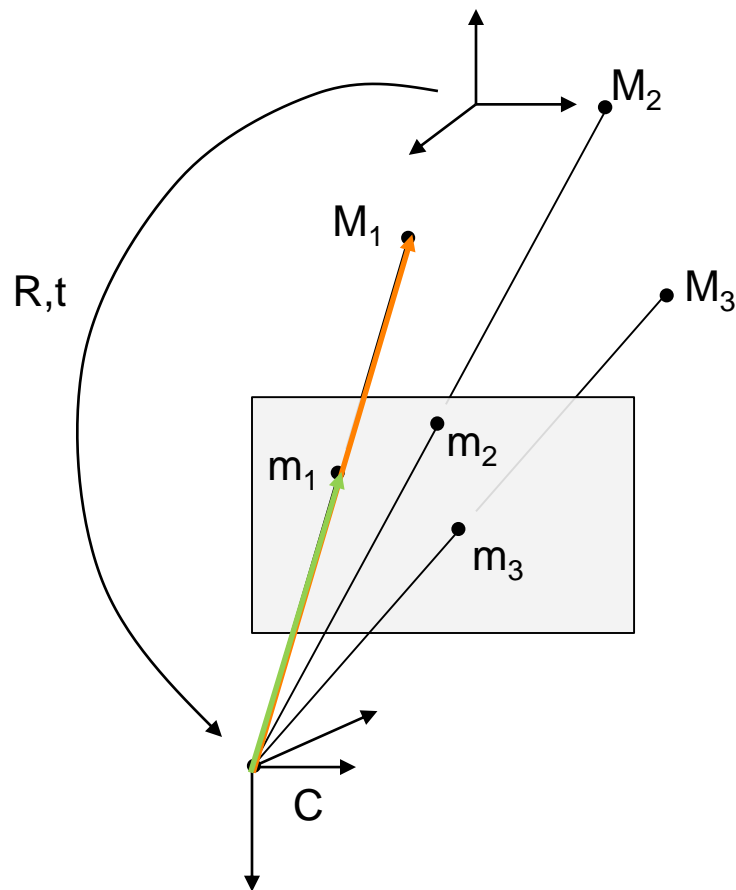
$$\begin{bmatrix} xP_3^T - P_1^T \\ yP_3^T - P_2^T \\ x'P_3^T - P_1^T \\ y'P_3^T - P_2^T \end{bmatrix} X = 0$$

Camera pose estimation



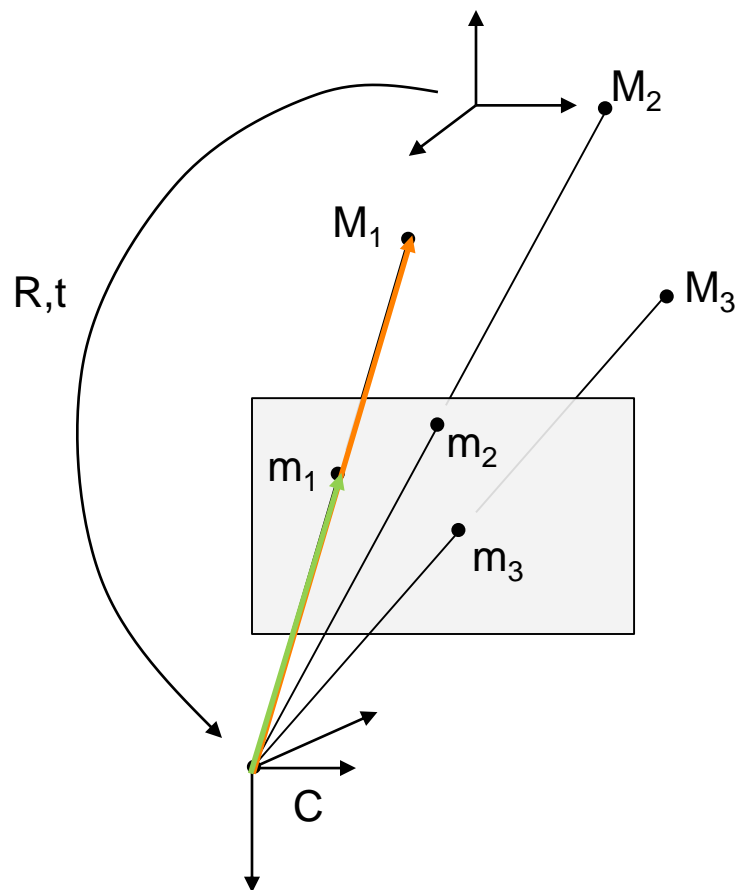
- perspective-n-point problem
- Goal is to estimate camera matrix P such that $m_1 = PM_1$
- $m_1, M_1, m_2, M_2, m_3, M_3$ are known
- Algebraic linear solution requires 6 3D-2D point correspondences, minimal nonlinear solution requires only 3

Camera pose estimation



- Derivation similar to Triangulation, but now entries of P are the unknowns instead of X
- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)

Camera pose estimation



- Derivation similar to Triangulation, but now entries of P are the unknowns instead of X
- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)

$$x \times (PX) = 0 \text{ for all pairs } x \leftrightarrow X$$

$$y(P_3^T X) - w(P_2^T X) = 0$$

$$x(P_3^T X) - w(P_1^T X) = 0$$

$$x(P_2^T X) - y(P_1^T X) = 0$$

$$\begin{bmatrix} 0 & -wX^T & yX^T \\ -wX^T & 0 & xX^T \\ -yX^T & xX^T & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$

Recap - Learning goals

- Understand the geometric interpretation of the camera matrix
- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation