Mathematical Principles in Vision and Graphics: Solving Polynomial Systems

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Polynomial Systems in Computer Vision

Many Computer Vision problems can be solved by finding the roots of a polynomial system:

- camera pose estimation from point correspondences;
- camera relative motion estimation from point correspondences;
- image distortion calibration;
- point triangulation;
- ...

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Solving Polynomial Systems

- no general method;
- several mathematical tools exist. For a given problem, a tool can be more adapted than the others.

Gröbner Bases

- introduced in 1965 by Bruno Buchberger (now at the Johannes Kepler University in Linz) in his Ph.D. thesis (named after his advisor Wolfgang Gröbner) to study sets of polynomials

A Polynomial System

Let consider the following polynomial system:

$$\begin{array}{l} L_1 \\ L_2 \\ L_3 \end{array} \left\{ \begin{array}{rcl} 2x^2 + y^2 - 2z + 3z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2 y^2 + y^2 z^2 - 2 & = & 0 \end{array} \right.$$

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Replace L_1 by $L_1 - 2L_2$:

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Adding $y^2 L_2 - L_3$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \end{array} \left\{ \begin{array}{rcl} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \end{array} \right.$$

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Add $yL_1' - L_4$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \left\{ \begin{array}{rcl} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \\ 5z - 4z^2 + z^3 - 2 & = & 0 \end{array} \right.$$

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$5z - 4z^2 + z^3 - 2 = (z-1)^2(z-2)$

Each possible value for z gives a new polynomial system in x and y only.

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$$\mathbf{C} = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{n-1} \end{bmatrix}.$$

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    Its eigenvalues are the roots of $p(z)$ (because $p(z)$ is the characteristic polynomial $\det(z\mathbf{I} - \mathbf{C})$ of $\mathbf{C}$).
  - Sturm's bracketing method (slightly less stable but much faster).

Two Gröbner bases

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$$\left\{ y^2 - 4z + z^2 + 5, x^2 + z + z^2, 5z - 4z^2 + z^3 - 2 \right\}$$

is also a Gröbner basis.

A Gröbner basis is a set of polynomials $\{g_1, \ldots, g_t\}$, such that the system

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has the same solutions as the original one,

but with some specific properties that make the new system easier to solve than the original one, OR AT LEAST USEFUL to solve the original one.

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  After Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ 1 \end{bmatrix} = 0\,.$$

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- the remainder of polynomial divisions (used by Buchberger's algorithm).

Notations and Definitions

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The coefficients of polynomials and the variables take their values from a field.

Monomials

Definition. A **monomial** in $x_1, \ldots, x_n$ is a product of the form:

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots \cdot x_n^{\alpha_n},$$

where all the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers, sometimes noted $\mathbf{x}^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Examples: x, x^2, $x^2 y$, $x^2 y z^3$

Polynomials

Definition. A **polynomial** f in $x_1, \ldots, x_n$ with coefficients in a field k is a finite linear combination with coefficients in k of monomials. A polynomial is written in the form

$$f = \sum_{\alpha} a_\alpha \mathbf{x}^\alpha, \quad a_\alpha \in k$$

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- a_α the **coefficient** of the monomial $\mathbf{x}^\alpha$.
- If $a_\alpha \neq 0$, then we call $a_\alpha \mathbf{x}^\alpha$ a **term** of f.

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$k[x, y]$ is the set of polynomials in two variables: $x^2 - y \in k[x, y]$, $x^3 + 2xy + y^2 \in k[x, y]$.

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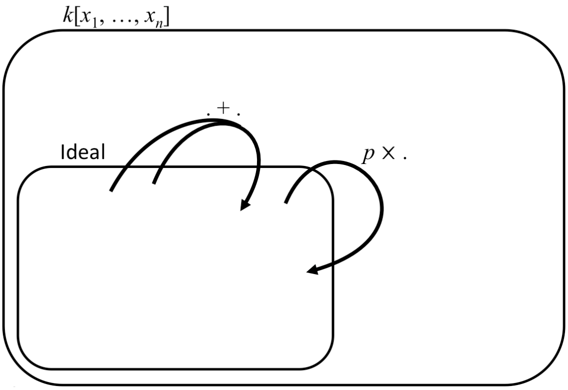

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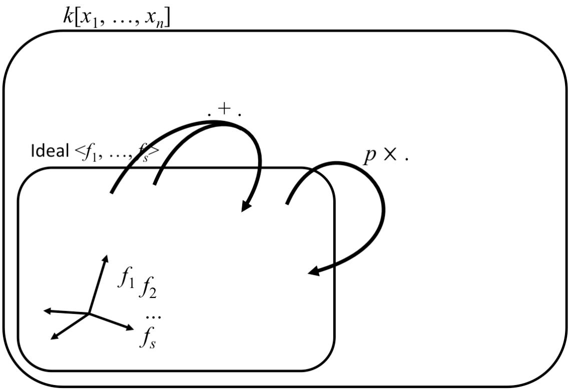

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- Is $x^2 \in I$? Yes, because $x^2 = x.(x - y^2) + y.xy$.
- Is $y \in I$? No, there is no $p_1, p_2 \in k[x_1, \ldots, x_n]$ such that $y = p_1.(x - y^2) + p_2.xy$.

Motivation for the Notion of Ideal

If $(a_1, \ldots, a_n) \in k^n$ is such that

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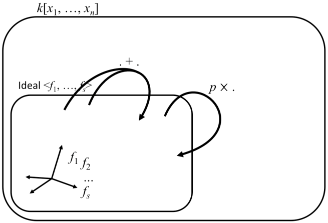
In other words, the ideal generated by a polynomial system is made of all the polynomials that can be added to the system without changing the solutions.



Affine Varieties - Sets of Solutions

Definition. The set

$$\mathbf{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0 \quad \forall\, 1 \leq i \leq s\}.$$

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- $\mathbf{V}(x^2+y^2-1)$ is the circle of radius 1 centered at the origin;
- $\mathbf{V}(x^2+y^2-1,x)$ is the set $\{(0,1),(0,-1)\}$.

Definition - Leading Term $\mathrm{LT}(f)$

Definition. Given a nonzero polynomial $f \in k[x]$, let

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We will write $\mathrm{LT}(f) = a_0 x^m$.

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To decide, we will define a *monomial order*.

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Monomial Order on $k[x]$

The only monomial order on $k[x]$ is the degree order, given by:

$$\ldots > x^{n+1} > x^n > \ldots > x^2 > x > 1.$$

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Let's first define an order on the variables: $x_1 > x_2 > \ldots > x_n$ (this is not a monomial order), and $x > y > z$.

Monomial Orders on $k[x_1, \ldots, x_n]$ - the Lexicographic Order $>_{lex}$

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$\rightarrow x^2yz^3 >_{lex} x^2z^4$ because $(2,1,3) - (2,0,4) = (0, \mathbf{1}, -1)$

Monomial Orders on $k[x_1, \dots, x_n]$ - the Graded Reverse Lexicographic Order $>_{grevlex}$

Let x^α and x^β be monomials in $k[x_1, \dots, x_n]$. $x^\alpha >_{grevlex} x^\beta$ if:

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Monomial Orders

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Computing Gröbner bases with $>_{lex}$ yields a polynomial system that can be easily solved.

Using the Monomial Orders

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r is called *a* remainder of f on division by F.

- Notation: $r = \overline{f}^F$;
- there exists an algorithm to compute the a_i's and r.

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Reordering F or changing the monomial order can produce different a_i and a different remainder r!

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(see `normalf` command in Maple or `PolynomialReduce` in Mathematica)

Using the Polynomial Division

Can we use the division to decide whether a given polynomial $f \in k[x_1, \dots, x_n]$ is a member of a given ideal $I = \langle f_1, \dots, f_s \rangle$, by computing the remainder on division?

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- On the other hand:
  there is no guarantee to find $\overline{f}^F = 0$ for every f in $I = \langle f_1, \ldots, f_s \rangle$, with $F = (f_1, \ldots, f_s)$.

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Division:

$$p = 0.(x^2 + 1) + 0.(xy) + \mathbf{y} \, .$$

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$$\forall f \in I \backslash \{0\} \;\; \exists\, g \in G \quad \text{such that } \mathrm{LT}(f) \text{ is divisible by } \mathrm{LT}(g)\,.$$

It can be shown that a Gröbner basis always exists for any ideal I and it is indeed a basis for I i.e. $I = \langle g_1, \dots, g_t \rangle$.

Link with the Division in $k[x_1, \ldots, x_n]$

If F is a Gröbner basis for I, then for any $g \in I$, and for any ordering of F, the remainder of the division of g by F is null.

Example

Let

$$I = \left\langle x^2 - y^2 + 1,\ xy - 1 \right\rangle .$$

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A Gröbner basis for I under the $>_{lex}$ order is:

$$\left\langle y^4 - y^2 - 1,\ x - y^3 + y \right\rangle .$$

We can check that $\mathrm{LT}(f) = x$ is divisible by $\mathrm{LT}(x - y^3 + y) = x$.

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The resulting polynomial $f = x + y - y^3$ is in I and its leading term x is not divisible neither by $\mathrm{LT}(x^2 - y^2 + 1) = x^2$ nor by $\mathrm{LT}(xy - 1) = xy$.

Remark (2)

Let's try the same operation on the Gröbner basis $\langle y^4 - y^2 - 1,\ x + y - y^3 \rangle$:

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$$\begin{aligned} & x(y^4 - y^2 - 1) + y^2(x - y^3 + y) \\ = \quad & -xy^2 + xy^4 - x + xy^2 + y^3 - y^4 \end{aligned}$$

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Let's try the same operation on the Gröbner basis $\langle y^4 - y^2 - 1,\ x + y - y^3 \rangle$:

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the leading term is xy^4, which is divisible by $\mathrm{LT}(x - y^3 + y) = x$.

Cool

If

- we use the monomial order $>_{lex}$ to compute a Gröbner basis and
- the solution set is finite,

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The system

$$\begin{cases} x^2 - y^2 + 1 & = & 0 \\ xy - 1 & = & 0 \end{cases}$$

has the same solutions as the system:

$$\begin{cases} y^4 - y^2 + -1 & = & 0 \\ x - y^3 + y & = & 0 \end{cases}$$

but the latter is much simpler to solve.

A More Ugly Example

A Gröbner basis for

$$\left\{ \begin{array}{ccc} x^2 - 2xz + 5 & = & 0 \\ xy^2 + yz + 1 & = & 0 \\ 3y^2 - 8xz & = & 0 \end{array} \right.$$

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$$\{-81 + 4320z - 86400z^2 + 766272z^3 - 2513488z^4 - 295680z^5 - 242496z^6 + 61440z^8, -2472389942760 + 1450790919y + 987224793696000z - 1312504296363936z^2 + 575639991700688z^3 + 711670127441280z^4 + 549519027506496z^5 - 103266809856000z^6 - 139421921341440z^7, 6503592729600 + 1450790919x - 257416379643438z + 3400639490020320z^2 - 1485707991955148 0z^3 - 1835782187164800z^4 - 1418473727285760z^5 + 263479449600000z^6 + 359882180198400z^7\}$$

Algorithms to Compute a Gröbner basis

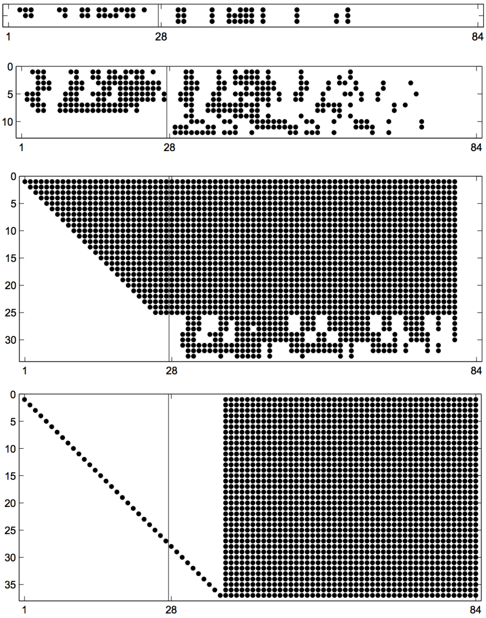
First algorithm to compute a Gröbner basis: the Buchberger algorithm.

More recent algorithms are more efficient ($F4$ and $F5$ algorithms by Faugère).

In [**?**]:

1. Start with $d \leftarrow 1$;
2. Multiply each equation of the current system by every possible monomial of degree d;
3. Simplify the system with Gauss-Jordan elimination;
4. If not a Gröbner basis, set $d \leftarrow d+1$, and iterate from 1.

Computation Steps for [?]



Damnit

Unfortunately, computation of Gröbner bases under the lexicographic ordering ($>_{lex}$) is often intractable for real problems.

Using the graded reverse lexicographical ordering ($>_{grevlex}$) usually yields more tractable computations.

Unfortunately, the resulting polynomial system is not necessarily easy to solve.

Fortunately, other properties of Gröbner bases can be used to find the solutions.

$>_{lex}$ versus $>_{grevlex}$: Example

Computing a Gröbner basis for

$$\left\{ \begin{array}{rcl} d_1^2 + A d_1 d_2 + d_2^2 - F^2 & = & 0 \\ d_1^2 + B d_1 d_3 + d_3^2 - F^2 & = & 0 \\ d_2^2 + C d_2 d_3 + d_3^2 - G^2 & = & 0 \\ d_2^2 + D d_2 d_4 + d_4^2 - F^2 & = & 0 \\ d_3^2 + E d_3 d_4 + d_4^2 - F^2 & = & 0 \end{array} \right.$$

under $>_{grevlex}$: less than a second (but 130 polynomials in a 96Kb text file).

under $>_{lex}$: more than a week

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$$a \sim b \text{ iff } a - b \in I \,.$$

For every element $a \in R$, we can define an *equivalence class*, or *coset* as:

$$[a] = \{ b \in R \,|\, a \sim b \} \,.$$

We have

$$a \sim b \Leftrightarrow [a] = [b] \Leftrightarrow a - b \in I$$

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etc.

The remainder of z divided by 7 is a standard representative of its class $[z]$.

$$[z] = [\text{remainder}(z/7)] \, .$$

Quotient Ring

If we define addition and multiplication between equivalence classes:

$$[a] + [b] = [a + b] \text{ and}$$

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Quotient Ring $k[x_1, \ldots, x_n]/I$

$k[x_1, \ldots, x_n]$ is a ring. If I is an ideal in $k[x_1, \ldots, x_n]$, $k[x_1, \ldots, x_n]/I$ is a quotient ring.

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If $G = (g_1, \ldots, g_t)$ is a Gröbner basis, and $f \in k[x_1, \ldots, x_n]$:

$$[f] = [\overline{f}^G]$$

with $\overline{f}^G$ the remainder of the division of f by G. (because $f = h_1.g_1 + \ldots + h_t.g_t + \overline{f}^G \Rightarrow f - \overline{f}^G \in I \Rightarrow [f] = [\overline{f}^G]$)

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and

$$k[x_1, \ldots, x_n]/I = \left\{ \left[\overline{f}^G\right] \middle| f \in k[x_1, \ldots, x_n] \right\}.$$

(does not depend on G, because the remainder does not depend on the Gröbner basis)

Finiteness Theorem

Theorem. Let's consider

- an ideal I in $\mathbb{C}[x_1, \ldots, x_n]$, and
- the corresponding quotient ring $A = \mathbb{C}[x_1, \ldots, x_n]/I$

$\dim(A)$ is finite $\Leftrightarrow$ $\mathbf{V}(I)$ is finite (i.e. the number of solutions is finite)

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Counter-example: $I =< x >\in \mathbb{C}[x, y]$, and
$A = \mathbb{C}[x, y]/I = \{[1], [x], [y], [y^2], [\sum_\alpha a_\alpha y^\alpha]\}$

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If the number of solutions is finite, $\dim(A)$ is finite, and m_f can be written as a matrix $\mathbf{M}_f$, which is called an *action matrix*.

wow

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Computing the Action Matrix

We need to identify a basis for the set of remainders.

The monomials in the basis of the set of remainders are the monomials that are not multiples of the leading terms of the polynomials in the Gröbner basis.

Computing the Action Matrix (continued)

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Example: The Gröbner basis G under $>_{grevlex}$ for

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is $\{ \begin{array}{l} 3y^2 - 8xz,\ x^2 - 2xz + 5, \\ 160z^3 - 160xz + 415yz + 12x - 30y - 224z + 15, \\ 240yz^2 - 9xy + 1600xz + 18yz + 120z^2 - 120x + 240z, \\ 16xz^2 + 3yz - 40z + 3,\ 40xyz - 3xy + 6yz + 40z^2 \end{array} \}$

with leading terms $\{y^2, x^2, z^3, yz^2, xz^2, xyz\}$.

The monomials in B are the monomials that are not divisible by the leading terms: 1, x, y, z, xy, xz, yz, z^2

Computing the Action Matrix $\mathbf{M}_x$

The basis B is $\{[1], [x], [y], [z], [xy], [xz], [yz], [z^2]\}$

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etc.

(I used the Mathematica `PolynomialReduce` function to compute these remainders)

Computing the Action Matrix $\mathbf{M}_x$ (continued)

$$
\begin{aligned}
&[x.1] = [x] \\
&[x.x] = -5[1] + 2[xz] \\
&[x.y] = [xy] \\
&[x.z] = [xz] \\
&[x.(xy)] = -5[y] + \tfrac{3}{20}[xy] - \tfrac{2}{10}[yz] - 2[z^2]
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$$

and:

$$
\mathbf{M}_x = \begin{array}{c}
\begin{matrix} 1 & x & y & z & xy & xz & yz & z^2 \end{matrix} \\
\begin{bmatrix}
0 & -5 & 0 & 0 & 0 & & & \\
1 & 0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & -5 & & & \\
0 & 0 & 0 & 0 & 0 & : & : & : \\
0 & 0 & 1 & 0 & \frac{3}{20} & : & : & : \\
0 & 2 & 0 & 1 & 0 & & & \\
0 & 0 & 0 & 0 & -\frac{2}{10} & & & \\
0 & 0 & 0 & 0 & -2 & & &
\end{bmatrix}
\end{array}
\begin{matrix} \\ 1 \\ x \\ y \\ z \\ xy \\ xz \\ yz \\ z^2 \end{matrix}
$$

Computing the Possible Values for x

$$\mathbf{M}_x = \begin{bmatrix} 0 & -5 & 0 & 0 & 0 & -\frac{3}{8} & 0 & -\frac{3}{16} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{20} & 0 & \frac{3}{40} & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{10} & -\frac{3}{8} & -\frac{3}{20} & -\frac{3}{16} \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 \end{bmatrix}$$

Computing the Possible Values for x

$$\mathbf{M}_x = \begin{bmatrix} 0 & -5 & 0 & 0 & 0 & -\frac{3}{8} & 0 & -\frac{3}{16} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{20} & 0 & \frac{3}{40} & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{10} & -\frac{3}{8} & -\frac{3}{20} & -\frac{3}{16} \\ 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 \end{bmatrix}$$

The real eigenvalues for $\mathbf{M}_x$ are $-1.10137..$ and $0.9660..$, which are the possible values for x.

Computing the Complete Solution

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Possible strategies:

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Possible strategies:

1. do the same with $\mathbf{M}_y$ and $\mathbf{M}_z$, and check for every possible combination $(x, , y, z)$ if it is a valid solution.
2. for each possible value for x, plug it in the system and solve the resulting system (which is now only in y and z).

The first option is more stable numerically.

In Practice [Kukelova08]

Offline

input: polynomial system with random coefficients

↓

output: "elimination template" operations to apply to the system to obtain a Gröbner basis

Online

input: actual coefficients of the system

1. apply the elimination template to compute the coefficients of the Gröbner basis:

2. compute the action matrix(ces)

3. compute the solutions using the action matrix(ces)

The offline computations are done in $\mathbb{Z}/p\mathbb{Z}$ with p a large prime number. It speeds up the computations, and avoids numerical instability (can easily determine when a coefficient becomes null).

See Automatic solver software on
`http://cmp.felk.cvut.cz/minimal/`

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