Mathematical Principles in Visual Computing: Sylvester Resultant

Prof. Friedrich Fraundorfer

SS2024

Slides by Vincent Lepetit

April 23, 2024

Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown x:

$$\begin{aligned} f(x) &= a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 \\ g(x) &= b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 \end{aligned}$$

Sylvester matrix of f and g:

$$\mathrm{Syl}(f,g) = \begin{array}{l} \left[\begin{array}{ccccccc} a_m & & \ldots & & a_0 & & \\ & \ddots & & & & \ddots & \\ & & a_m & & \ldots & & a_0 \\ b_n & & \ldots & & b_0 & & \\ & \ddots & & & & \ddots & \\ & & \ddots & & & \ddots & \\ & & & b_n & & \ldots & b_0 \end{array}\right] \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c} a \\ a \\ a \end{array}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \end{array}}\right\} m \text{ rows} \end{array}$$

f and g have a common root if and only if $\det(\mathrm{Syl}(f,g)) = 0$.

$\det(\mathrm{Syl}(f,g))$ is called the Sylvester Resultant of f and g.

Using the Sylvester Resultant

to solve a polynomial system of two equations and TWO unknowns:

$$p_1(x,y) = 6x^2 + 3xy - xy^2 + y + 1 \qquad \left\{ \begin{array}{l} p_1(x,y) = 0 \\ p_2(x,y) = 0 \end{array} \right.$$
$$p_2(x,y) = x^2y + 5x + 4y - 1$$

Let's consider y as a constant, and write these two polynomials as polynomials in x:

$$p_{1,y}(x) = 6x^2 + (3y - y^2)x + (y+1)$$
$$p_{2,y}(x) = yx^2 + 5x + (4y - 1)$$

$$\mathrm{Syl}(p_{1,y}, p_{2,y}) = \begin{bmatrix} 6 & (3y - y^2) & (y+1) & 0 \\ 0 & 6 & (3y - y^2) & (y+1) \\ y & 5 & (4y-1) & 0 \\ 0 & y & 5 & (4y-1) \end{bmatrix}$$

Using the Sylvester Resultant

$$\begin{aligned} p_1(x,y) &= 6x^2+3xy-xy^2+y+1 \\ p_2(x,y) &= x^2y+5x+4y-1 \end{aligned} \qquad \begin{cases} p_1(x,y)=0 \\ p_2(x,y)=0 \end{cases}$$

$$\begin{aligned} p_{1,y}(x) &= 6x^2+(3y-y^2)x-xy^2+(y+1) \\ p_{2,y}(x) &= yx^2+5x+(4y-1) \end{aligned}$$

$p_{1,y}$ and $p_{2,y}$ should have roots in common, and $\det(\mathrm{Syl}(p_{1,y},p_{2,y}))=0$:

$$\det(\mathrm{Syl}(p_{1,y},p_{2,y})) = \begin{vmatrix} 6 & (3y-y^2) & (y+1) & 0 \\ 0 & 6 & (3y-y^2) & (y+1) \\ y & 5 & (4y-1) & 0 \\ 0 & y & 5 & (4y-1) \end{vmatrix} = 0$$

$\det(\mathrm{Syl}(p_{1,y},p_{2,y}))$ is a polynomial in y (only)!

$\det(\mathrm{Syl}(p_{1,y},p_{2,y})) = 4y^6-20y^5+153y^4-310y^3+781y^2-276y+36$

$\rightarrow$ First solve for y (we will see later how it can be done).