# Mathematical Principles in Visual Computing: Rigid Transformations

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# Learning goals

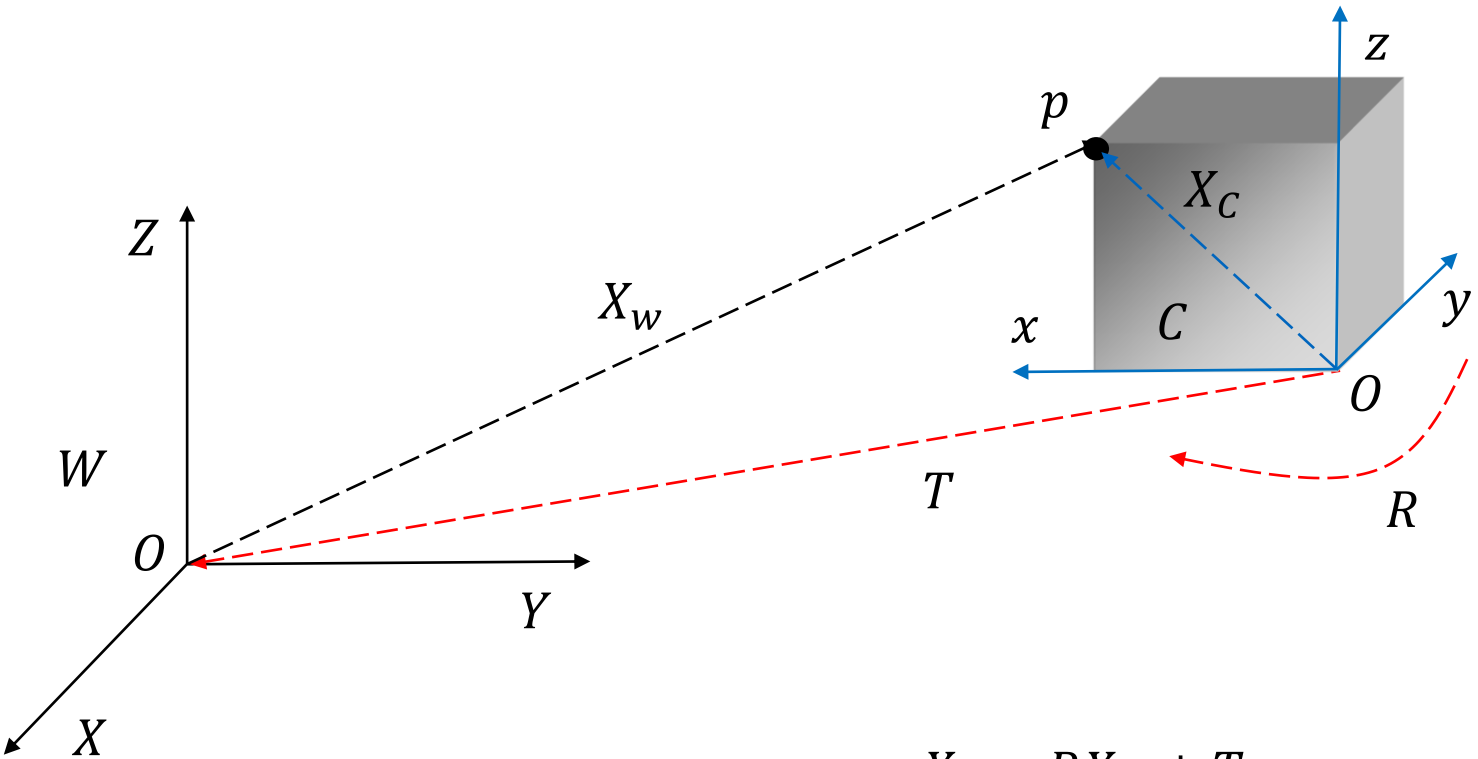
- Understand the problems of dealing with rotations
- Understand how to represent rotations
- Understand the terms SO(3) etc.
- Understand the use of the tangent space
- Understand Euler angles, Axis-Angle, and quaternions
- Understand how to interpolate, filter and optimize rotations

# Outline

- Rigid transformations
- Problems with rotation matrices
- Properties of rotation matrices
- Matrix groups SO(3), SE(3)
- Manifolds
- Tangent space
- Skew-symmetric matrices
- Exponential map
- Euler angles, angle-axis, quaternions
- Interpolation
- Filtering
- Optimization

# Motivation: 3D Viewer

# Rigid transformations




$x \in \mathbb{R}^n$

$\mathrm{T} \in \mathbb{R}^n$

$R \in \mathbb{R}^{n \times n}$

- Coordinates are related by:

$$X_c = RX_w + T$$

$$\begin{bmatrix} X_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ 1 \end{bmatrix}$$

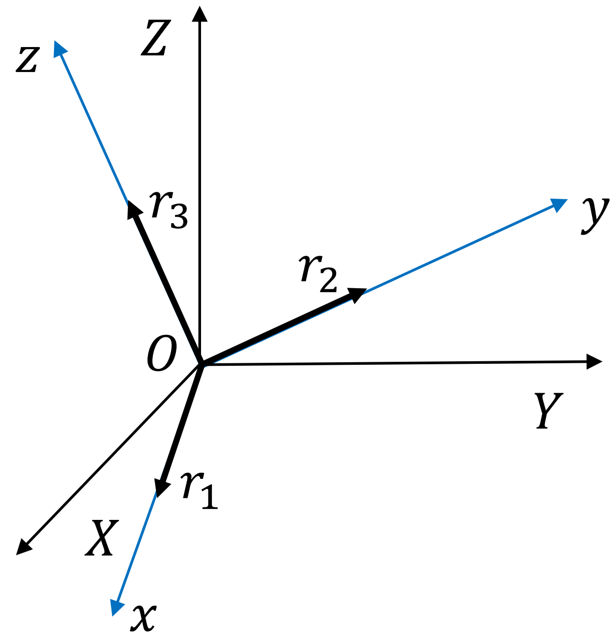
- Rigid transformation belong to the matrix group SE(3)
- What does this mean?

# Properties of rotation matrices

Rotation matrix:

$$R = [r_1, r_2, r_3] \in \mathbb{R}^{3\times 3}$$

$$R^T R = I, \det(R) = +1$$

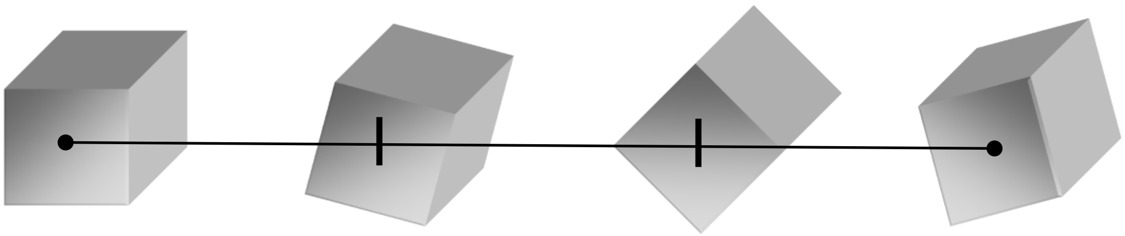



Coordinates are related by: $X_c = RX_w$

- Rotation matrices belong to the matrix group SO(3)
- What does this mean?

# Problems with rotation matrices

- Optimization of rotations (bundle adjustment)
  - Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Linear interpolation



- Filtering and averaging
  - E.g. averaging rotation from IMU or camera pose tracker for AR/VR glasses

# Matrix groups

- The set of all the nxn invertible matrices is a group w.r.t. the matrix multiplication:

$$GL(n) = (\{M \in \mathbb{R}^{n \times n} | \det(M) \neq 0\}, \times)$$

General linear group

- Reminder: A group is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element.
- The operation satisfies four conditions called the group axioms, namely closure, associativity, identity and invertibility.
- Closure means for all *a*, *b* in *G*, the result of the operation, *a* • *b*, is also in *G*.

# Matrix groups

- The set of all the nxn orthogonal matrices is a group w.r.t. the matrix multiplication:

$$O(n) = (\{A \in GL(n) | A^{-1} = A^T\}, \times)$$ Orthogonal group

$$A \in O(n) \Rightarrow \det(A) = \pm 1$$

# Matrix groups

- The set of all the nxn orthogonal matrices with determinant equal to +1 is a group w.r.t. the matrix multiplication:

$$SO(n) = (\{A \in O(n) |\, \det(A) = +1\},\times)$$

Special orthogonal group

- SO(3) … group of orthogonal 3x3 matrices with det=+1 …. "rotation matrices"

- $R_3 = R_1 * R_2$ … R3 is still an SO(3) element
- $R_3 = R_1 + R_2$ … R3 is NOT an SO(3) element. Not a rotation matrix anymore.

# Matrix groups

- The set of all the rigid transformations in $R^n$ is a group (not commutative) with the composition operation

$$\left(\left\{F: \mathbb{R}^n \rightarrow \mathbb{R}^n \mid \mathrm{F\ \ rigid}\right\}, \circ\right)$$

- The set is isomorphic to the special Euclidean group SE(n)

- The mathematical properties of a “rigid transformation” are specified by the special Euclidean group SE(n)

$$RT = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

# Special Euclidean group

- The special Euclidean group is constructed by the Cartesian product (a composition operation) from SO(n)xR$^n$.

$$SE(n) = (SO(n) \times \mathbb{R}^n, \times)$$

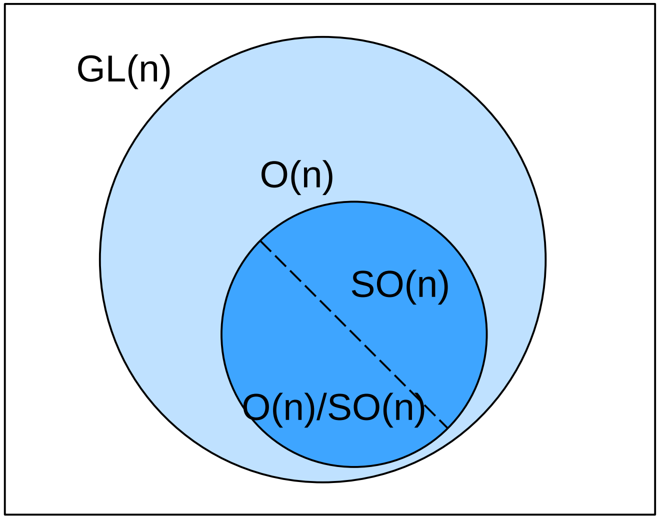
$$(M, t) \times (S, q) = (MS, Mq + t)$$

- The Cartesian product defines were the values of SO(n) and R$^n$ (rotation and translation) go to form the transformation matrix

$$RT = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

$$SE(3) = \begin{bmatrix} SO(3) & \mathbb{R}^3 \\ 0 & 1 \end{bmatrix}$$

# Matrix groups: Summary




$\mathbb{R}^{n \times n}$

Vector space of all the nxn matrices

$GL(n) = (\{M \in \mathbb{R}^{n \times n} | \det(M) \neq 0\}, \times)$ General linear group

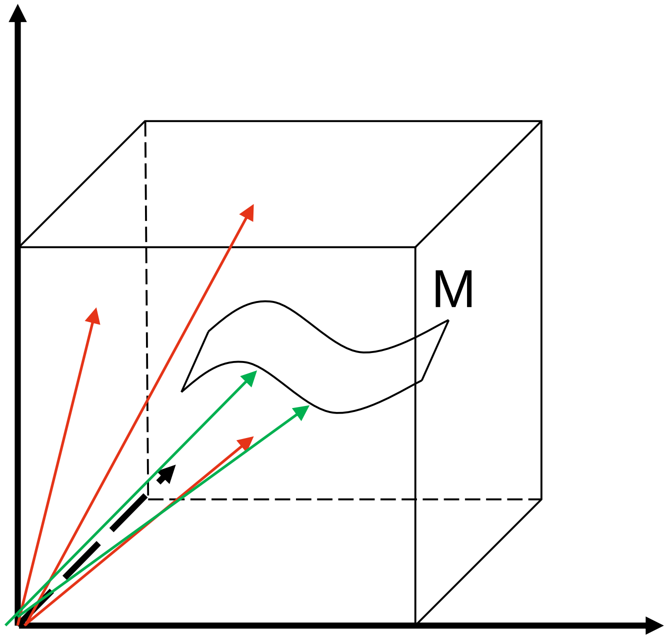
$O(n) = (\{A \in GL(n) | A^{-1} = A^T\}, \times)$ Orthogonal group

$SO(n) = (\{A \in O(n) | \det(A) = +1\}, \times)$ Special orthogonal group

$O(n)/SO(n) = (\{A \in O(n) | \det(A) = -1\})$ Set of orthogonal matrices which do not preserve orientation (not a group)

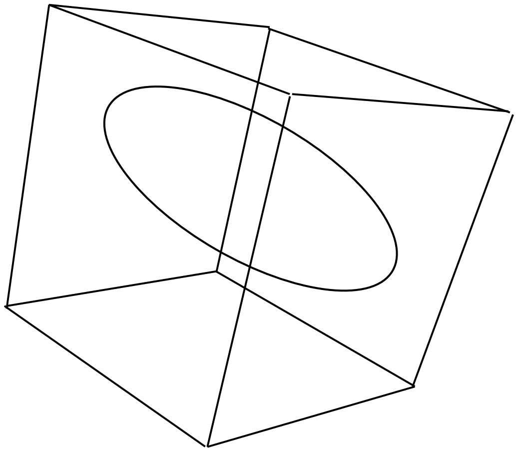
GL(n), O(n), SO(n) and SE(n) are all smooth manifolds
(e.g. surfaces, curves, solids immersed in some big vector space)

# Manifolds




- Non-mathematical definition:
  Manifolds are shapes like curves, surfaces, and volumes in Euclidean space (i.e. a vector space)
- A sphere is an example of a manifold of dimension 2.
- The matrix groups SO(3), SE(3) are manifolds.

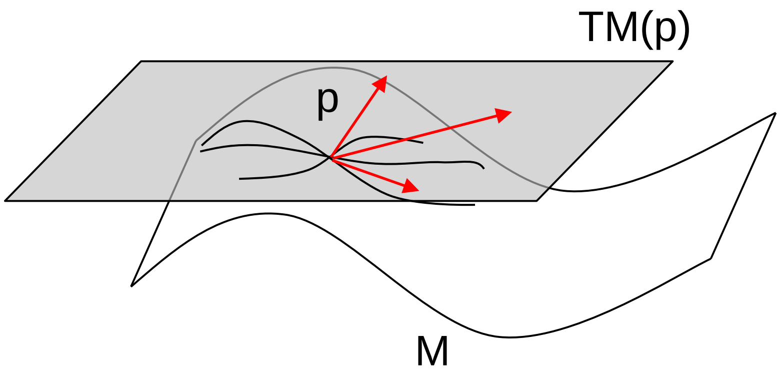
# Shapes of SO(2) and SO(3)



SO(2) … 1-manifold

SO(3) … 3-manifold (3-sphere)
A solid ball in $R^3$

# Tangent space of a manifold

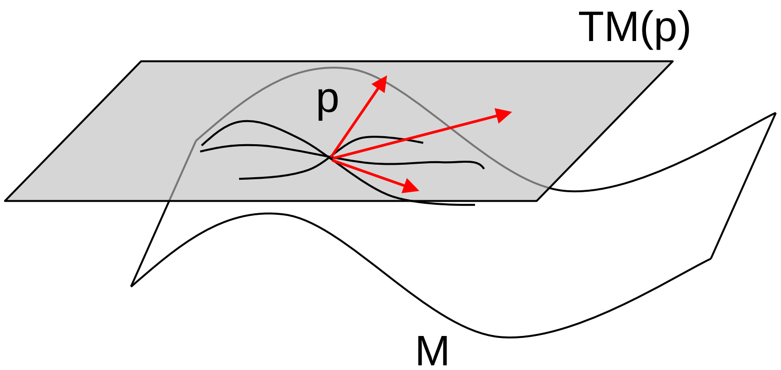



V = vector space

M … k-manifold

The tangent space of the manifold M in p (every point p on the manifold has a different tangent space) is isomorphic to a subspace of V.

# Tangent space of a manifold



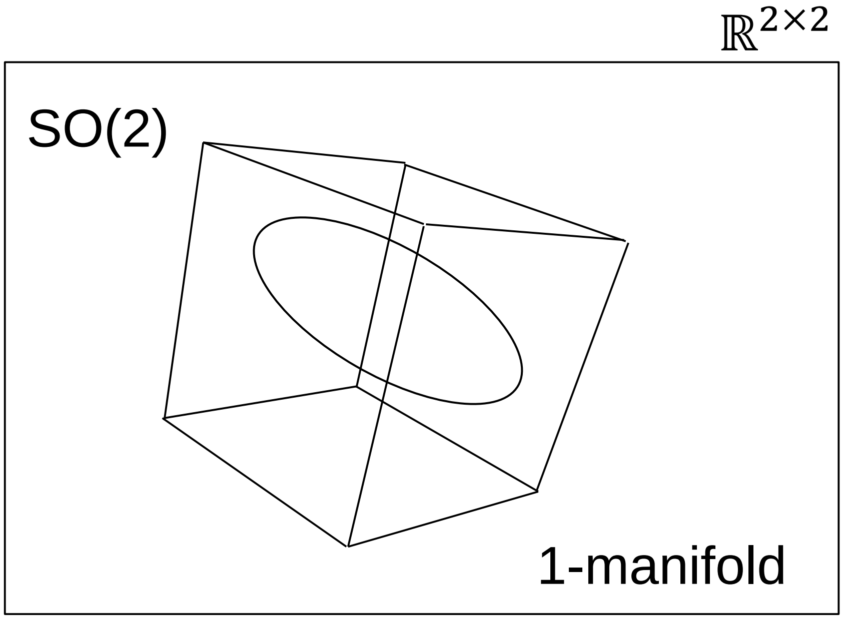

V = vector space with dimension n

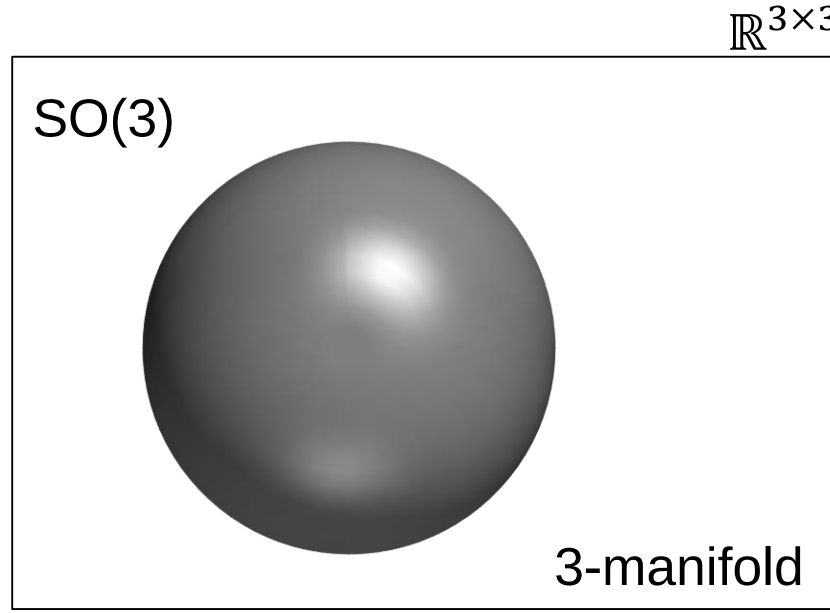
M … k-manifold

- TM(p) is a vector space (subspace of V) and has dimension k.

- 1-manifold (curves) -> 1 dim TM (lines)
- 2-manifold (surface) -> 2 dim TM (planes)
- 3-manifold (volumes, e.g. 3-sphere) -> 3 dim TM (full volumes)

# Tangent space of SO(2) and SO(3)




TSO(2) is a vector space
with dimension 1
subspace of $\mathbb{R}^{2\times 2}$

TSO(3) is a vector space
with dimension 3
subspace of $\mathbb{R}^{3\times 3}$

subspaces are defined by matrices

# Skew-symmetric matrices

- M is skew-symmetric iff $M^T$=-M

$$M^T = -M$$

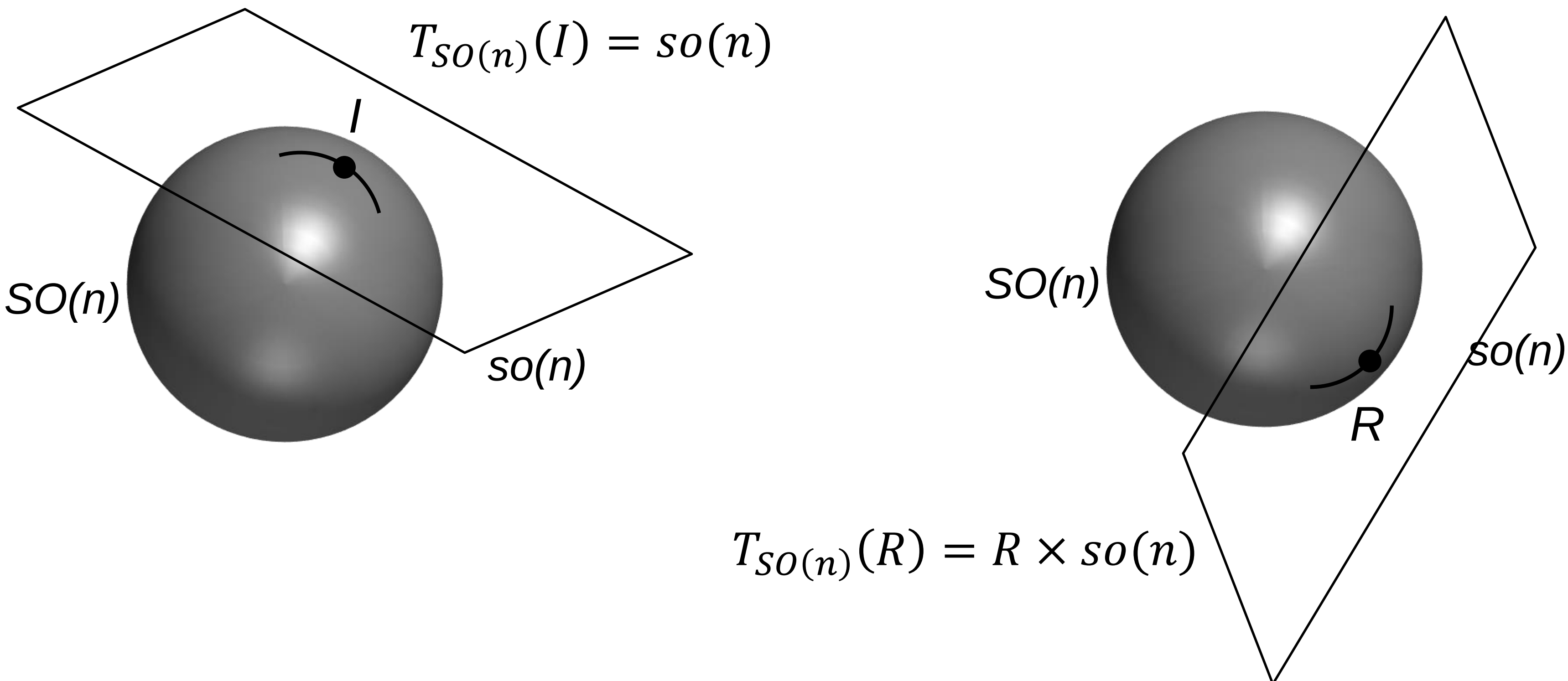
$$\begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$so(n) = (\{M \in \mathbb{R}^{n\times n} | M^T = -M\}, +, [])$$

Special orthogonal Lie algebra

- The term “algebra” means that so(n) is a vector space (more specific than group)
- We have addition in the vector space now (in SO(n) it was only multiplication)

# Skew-symmetric matrices

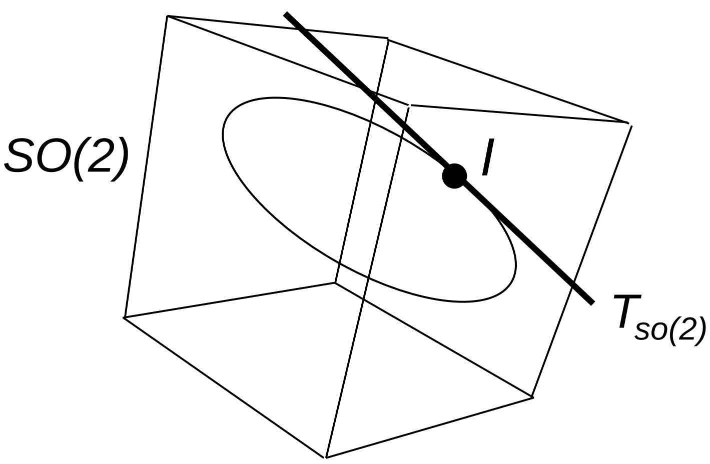


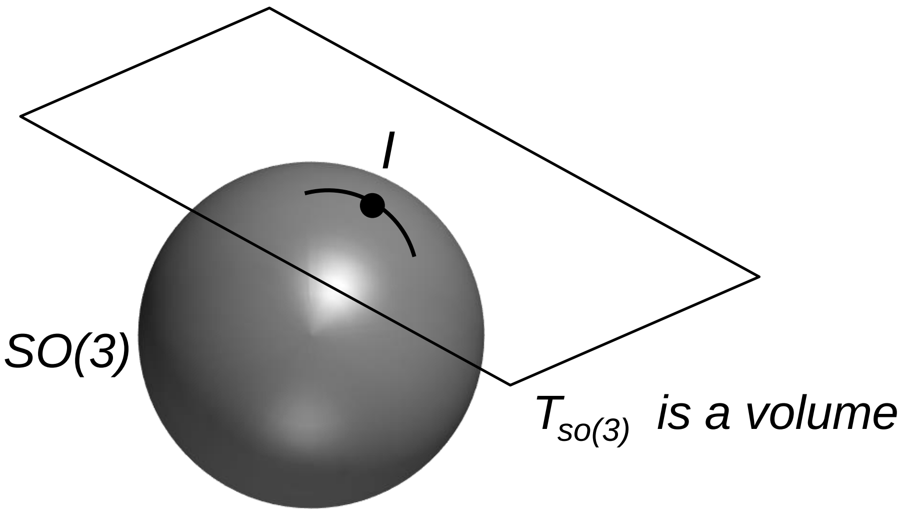

- The special orthogonal Lie algebra is the tangent space of SO(n) at identity.
- The tangent space of SO(n) in any other point R is a rotated version of so(n)
- $T_{SO(n)}(R)$ is not a skew-symmetric matrix anymore, but a rotated one

# so(2) and so(3)

$$\begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

- so(3) is a vector space of dimension 3
- so(2) is a vector space of dimension 1



# The hat operator

- The hat operator is used to form skew-symmetric matrices
- for so(3):

$$\hat{\cdot} : \mathbb{R}^3 \rightarrow so(3) \qquad (\widehat{x, y, z}) \rightarrow \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

- for so(2):

$$\hat{\cdot} : \mathbb{R} \quad \rightarrow so(2) \qquad \hat{x} \rightarrow \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$$

- Different notation
- $[t]_x = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$

# The hat operator

- The hat operator is used to define the cross-product in matrix form

$$a \times b = \hat{a} b \qquad \forall a, b \in \mathbb{R}^3$$

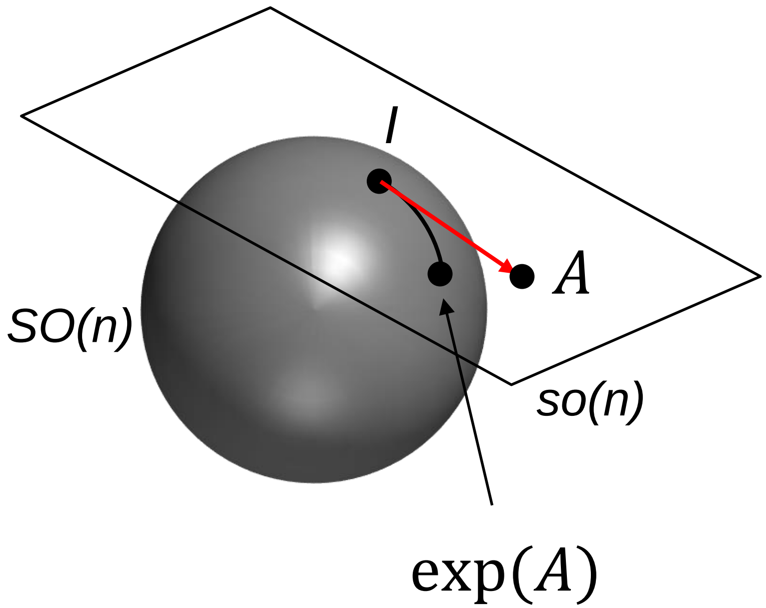
# Lie groups

- GL(n), O(n), SO(n) and SE(n) are all Lie groups (groups which are a smooth manifold where the operation is a differentiable function between manifolds)
- Also we have seen that the group of skew-symmetric matrices is called Lie algebra so(n) and is the tangent space of the special orthonormal group SO(n)
- But how to compute an element of the tangent space so(n) from SO(n) or vice versa?
- The exponential map!

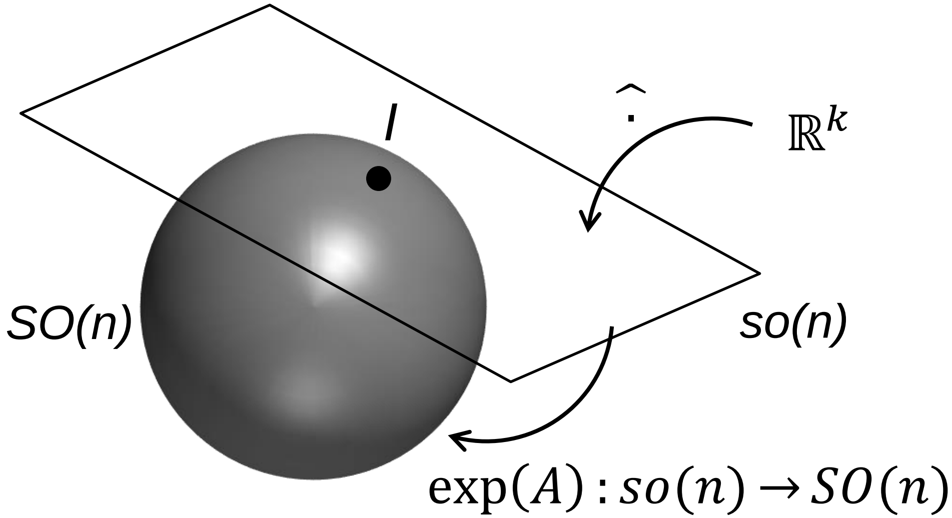
# Exponential map

- Given a Lie group G, with its related Lie algebra g=TG(I), there always exists a smooth map from Lie algebra g to the Lie group G called exponential map

$$exp: g \to G$$



# Exponential map




$\omega \in \mathbb{R}^k$

$\hat{\omega} \in so(n)$

$$\exp(\hat{\omega}) = \sum_{k=0}^{\infty} \frac{1}{k!} \hat{\omega}^k \in SO(n)$$

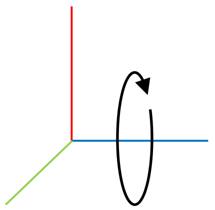
$\omega \in \mathbb{R}^k \to \exp(\hat{\omega})$

- Angle-axis representation for rotations:
  $\omega$… is the angle-axis representation ($\mathbb{R}^3$)
  $\exp(\omega)$ is the 3x3 rotation matrix (element of SO(3))

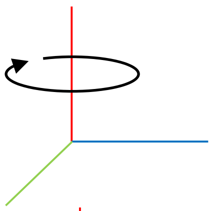
# Euler angles

- Euler's Theorem for rotations: Any element in SO(3) can be described as a sequence of three rotations around the canonical axes, where no successive rotations are about the same axis.

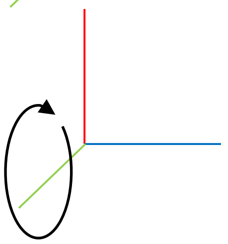
$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$



$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$



$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$For\ any\ R \in SO(3)\ there\ \exists \alpha, \beta, \gamma \mid R = R_x(\alpha) R_y(\beta)\ R_z(\gamma)$$

- $\alpha, \beta, \gamma$ are called Euler angles of R according to the XYZ representation (3 DOF/parameters)

# Euler angles

- Given M (element of SO(3)) there are 12 possible ways to represent it

$$M \in SO(3)\ there\ \exists \alpha, \beta, \gamma \mid M = R_x(\alpha) R_y(\beta)\ R_z(\gamma)$$

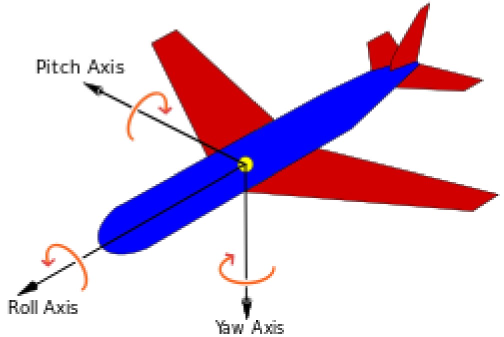
$$M \in SO(3)\ there\ \exists \alpha, \beta, \gamma \mid M = R_x(\alpha) R_z(\gamma) R_y(\beta)$$

$$M \in SO(3)\ there\ \exists \alpha, \beta, \gamma \mid M = R_x(\alpha) R_z(\gamma) R_x(\beta)$$

$$M \in SO(3)\ there\ \exists \alpha, \beta, \gamma \mid M = R_z(\alpha) R_x(\gamma) R_z(\beta)$$

…

- A common convention is ZYX corresponding to a rotation first around the x-axis (roll), then the y-axis (pitch) and finally around the z-axis (yaw)

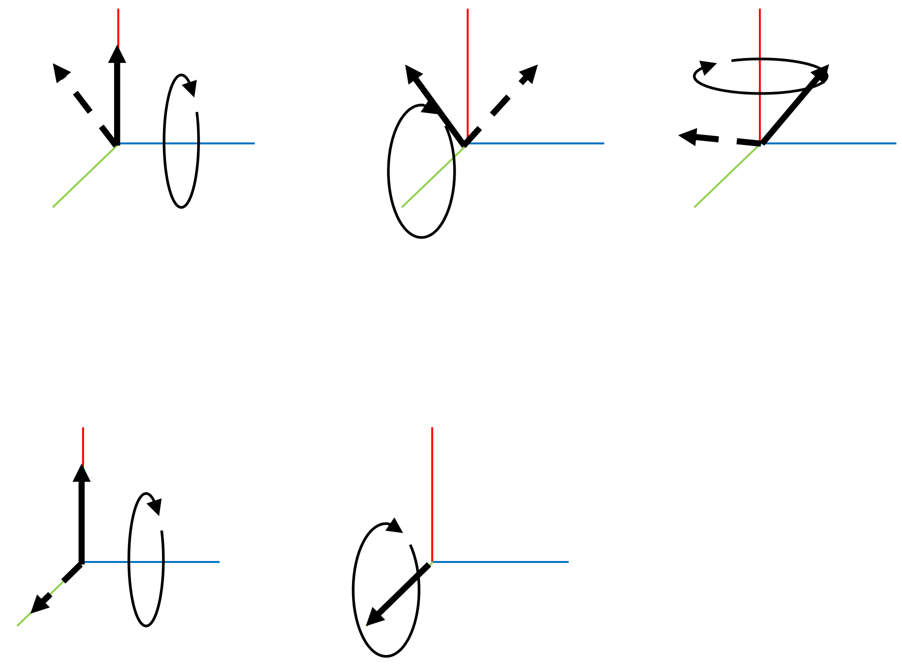





[https://en.wikipedia.org/wiki/Aircraft_principal_axes, CC BY-SA 3.0]

# Euler angles

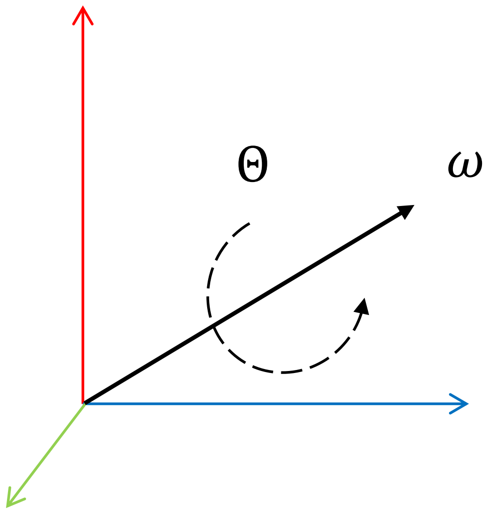
- The parameterization has singularities, called gimbal lock
- A gimbal lock happens when after a rotation around an axis, two axes align, resulting in a loss of one degree of freedom



cannot move (gimbal lock)

# Angle-Axis

- Euler's rotation theorem also states that any rotation can be expressed as a single rotation about some axis.




- The axis can be represented as a three-dimensional unit vector, and the angle by a scalar.
- 3 DOF/parameters
- Angle-axis defines a unique mapping and does not have gimbal lock

# Angle-Axis

- The operation to compute the rotation matrix SO(3) from the angle-axis parameters is by using the exponential map!

$$\omega \in \mathbb{R}^k \rightarrow \exp(\widehat{\omega})$$

- The exponential map can be computed in closed form using the Rodrigues formula

$$R = I + (sin\Theta)K + (1 - cos\Theta)K^2$$

$$K = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- There also exists the inverse

# Quaternions

- Quaternions are extensions of complex numbers, with 3 imaginary values instead of 1:

$$a+ib+jc+kd$$

- Like the imaginary component of complex numbers, squaring the components gives:

$$i^2=j^2=k^2=-1$$

- One way to express a quaternion is as a pair consisting of the real value and the 3D vector consisting of the imaginary components:

$$q=(a,w) \text{ with } w=(b,c,d)$$

- It is basically a 4-vector

# Quaternions

- If q=a+ib+jc+kd is a unit quaternion (||q||=1), then q corresponds to a rotation:

$$R(q) = \begin{bmatrix} 1-2c^2-2d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & 1-2b^2-2d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & 1-2b^2-2c^2 \end{bmatrix}$$

- Because q is a unit quaternion, we can write q as:

$$q = \left(\cos\left(\frac{a}{2}\right), \sin\left(\frac{a}{2}\right) w\right), \qquad \|w\| = 1$$

- It turns out the q corresponds to the rotation whose:
  - Axis of rotation is w and
  - Angle of the rotation is a

# Interpolation in SO(3)

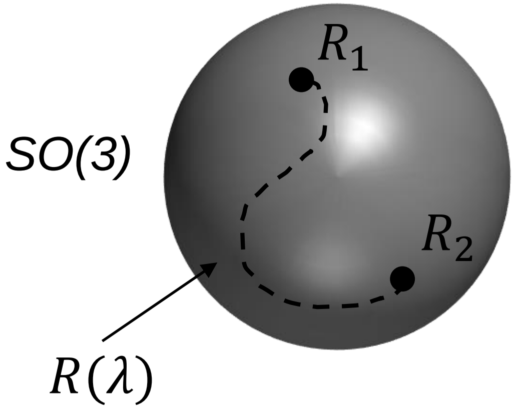
- Given two rotation matrices $R_1$,$R_2$ one would like to find a smooth path in SO(3) connecting these two matrices

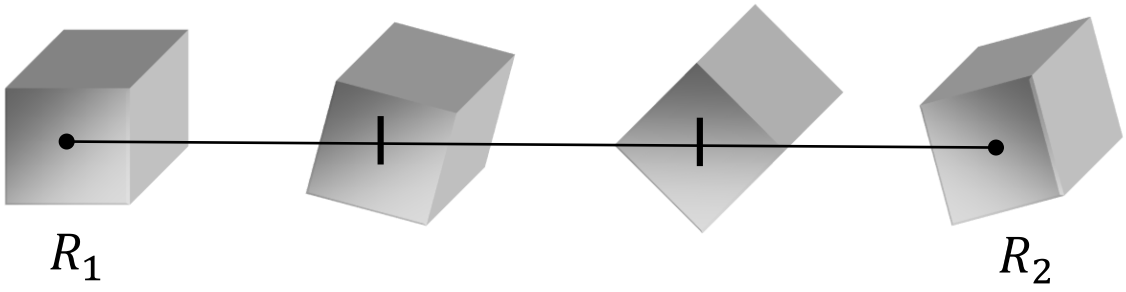
$$R(\lambda) \in SO(3), \lambda \in [0,1]$$

$$R(\lambda) \text{ smooth}$$

$$R(0) = R_1$$

$$R(1) = R_2$$



# Interpolation in SO(3)

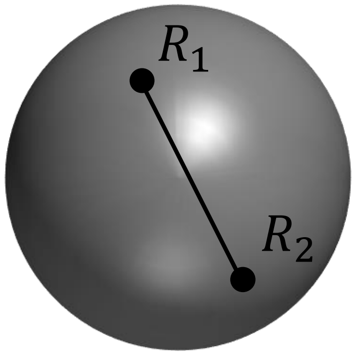
- Approach 1: Linearily interpolate $R_1$ and $R_2$ as matrices (naive approach)

$$R(\lambda) = \pi_{SO(3)}(\lambda R_1 + (1-\lambda)R_2)$$

Projection onto sphere
(not accurate)

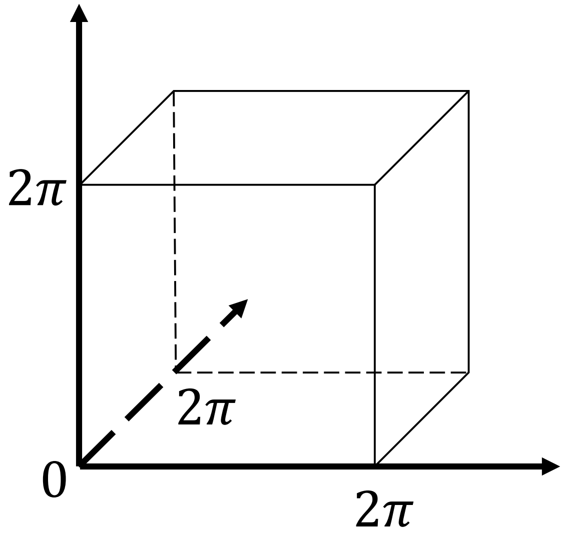
Not an element of SO(3),
not a rotation matrix at all

$$\pi_{SO(3)}(M) = arg \min_{R \in SO(3)} \|M - R\|_F^2$$



# Interpolation in SO(3)

- Approach 2: Linearily interpolate $R_1$ and $R_2$ using Euler angles

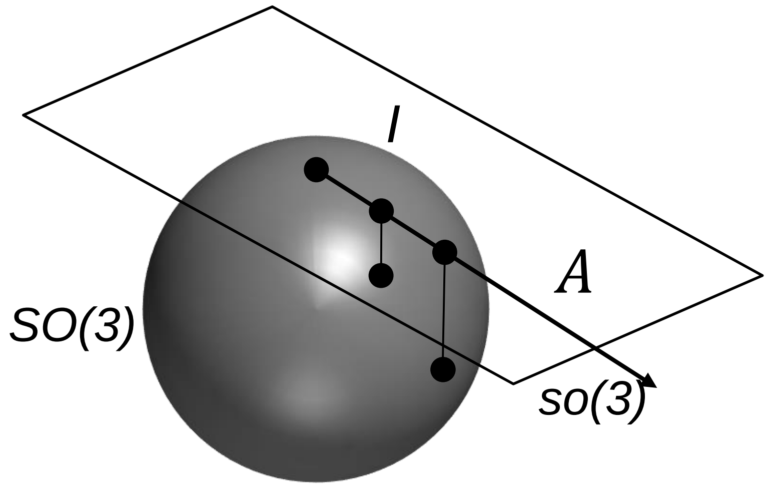



- Each axis is interpolated independently
- If $R_1$ and $R_2$ are too far apart -> not intuitive motion

# Interpolation in SO(3)

- Approach 3: Linearily interpolate $R_1$ and $R_2$ using angle-axis

$$\omega(\lambda) = \lambda\omega_1 + (1-\lambda)\omega_2$$




- Interpolation happens in tangent space (vector space) and is then projected using the exponental map onto the manifold

# Filtering in SO(3)

- Given n different noisy measurements for the rotation of an object

$$R_1, \ldots, R_n$$

- What is the filtered average of it?

# Filtering in SO(3)

- Possible approaches:

- Average the rotation matrices $R_i$ $\quad \frac{1}{n}\sum_{i=1}^{n} R_i$ (not rotation)
- Average the Euler angles of each $R_i$ $\quad \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i, \frac{1}{n}\sum_{i=1}^{n}\beta_i, \frac{1}{n}\sum_{i=1}^{n}\gamma_i\right)$
- Average the angle-axis of each $R_i$ $\quad \frac{1}{n}\sum_{i=1}^{n}\omega_i$
- Average the quaternions of each $R_i$ $\quad \frac{1}{n}\sum_{i=1}^{n} q_i$

(is rotation)

- All equally problematic and do not accurately respect the noise model

# Optimization in SO(3)

- Newton-Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Naive approach:
  - $X_n$ are elements of rotation matrix. Then the update step (addition) would not result in a rotation matrix.
- $X_n$ are Euler angles:
  - To evaluate $f(X_n)$ the rotation matrix has to be created from the Euler angles. Could lead to gimbal lock.
  - Derivatives of Euler angle construction has to be computed.
- $X_n$ are elements of the tangent space so(3)
  - Represents angle-axis notation
  - No gimbal lock
  - Minimal representation of 3 parameters

# Learning goals - Recap

- Understand the problems of dealing with rotations
- Understand how to represent rotations
- Understand the terms SO(3) etc.
- Understand the use of the tangent space
- Understand Euler angles, Axis-Angle, and quaternions
- Understand how to interpolate, filter and optimize rotations