Mathematical Principles in Visual Computing: Projective Geometry – Part 2

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Learning goals

- Understand the concept of vanishing points and vanishing lines
- Understand the calculation of vanishing points
- Understand the relation between vanishing points and camera orientation and calibration

Outline

- Vanishing points and lines
- Applications of vanishing points

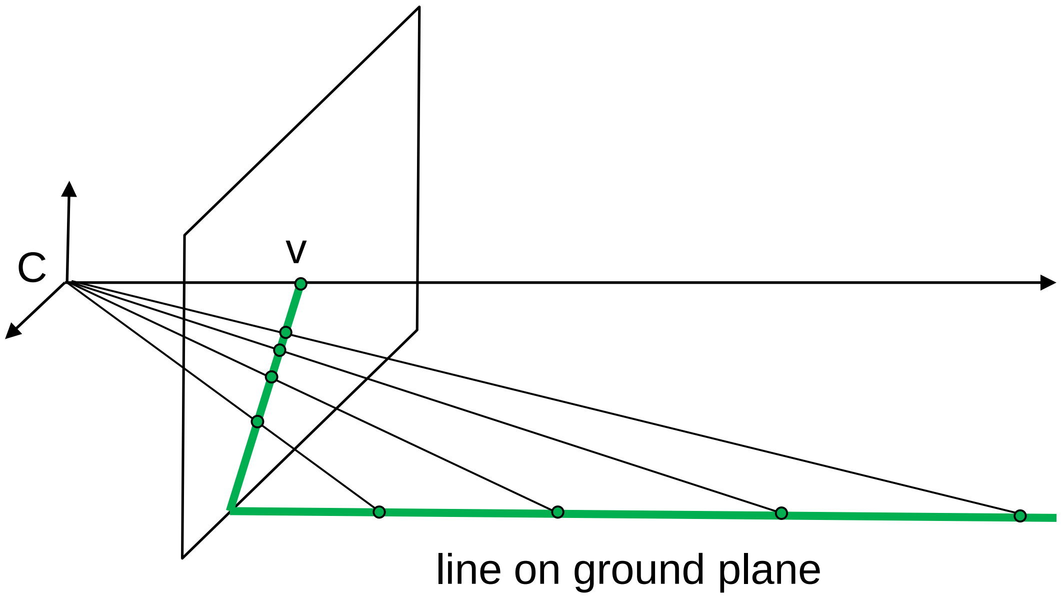
Vanishing points

Vanishing points

- Ideal points in the projective 3-space are located at infinity, and have homogeneous coordinates of the form (x, y, z, 0).
- These points are also called points at infinity.
- The image of an ideal point under a projective mapping is called a vanishing point.
- Recall that two parallel lines in the projective 3- space meet at an ideal point.
- Thus the images of two or more parallel world lines converge at a vanishing point.

[Source: Flickr]

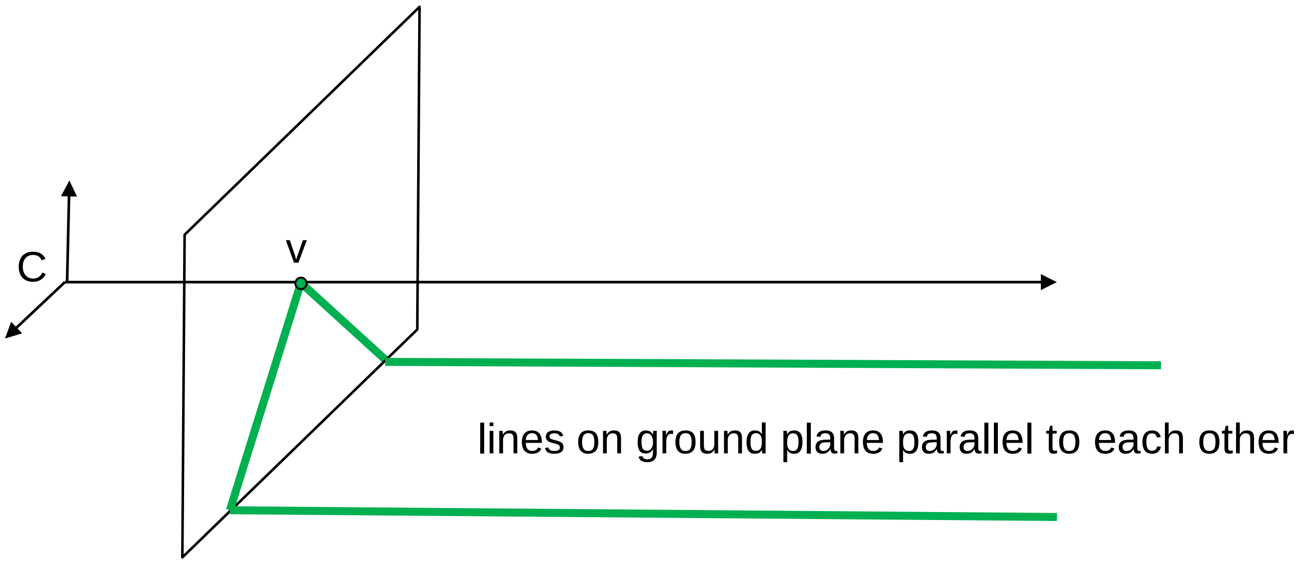
Vanishing points



The vanishing point v is the projection of a point at infinity.

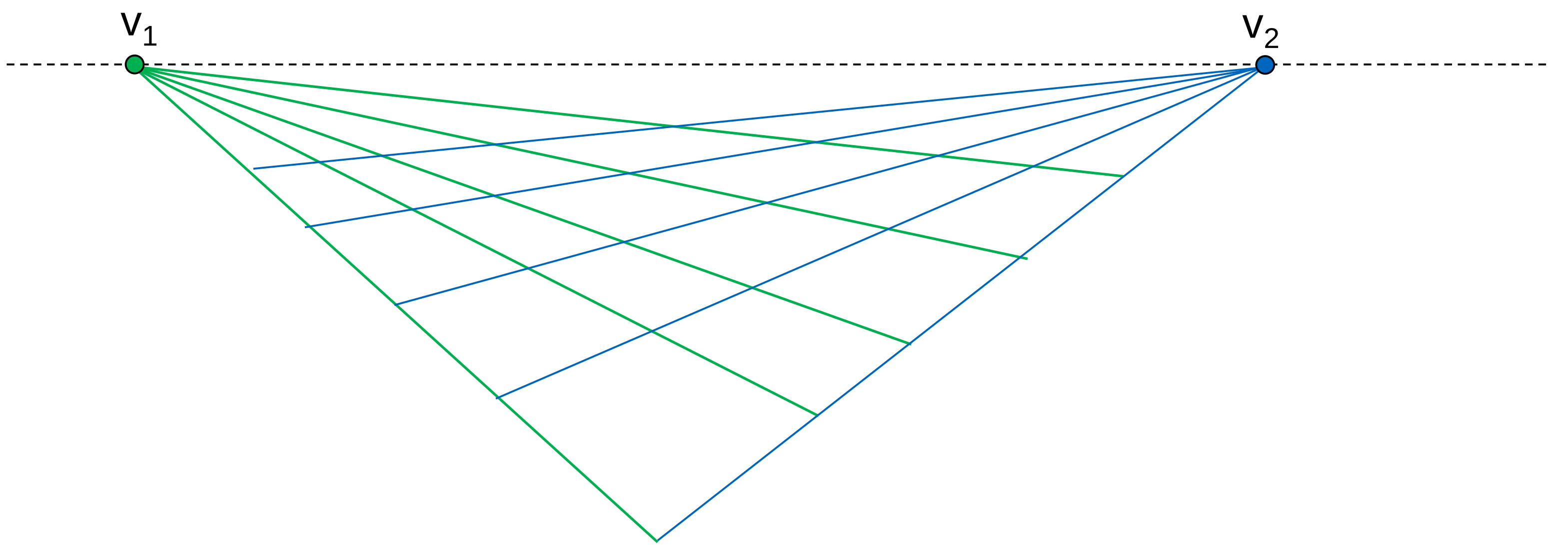
Think of extending the line on the ground plane further and further into infinity.

Vanishing points



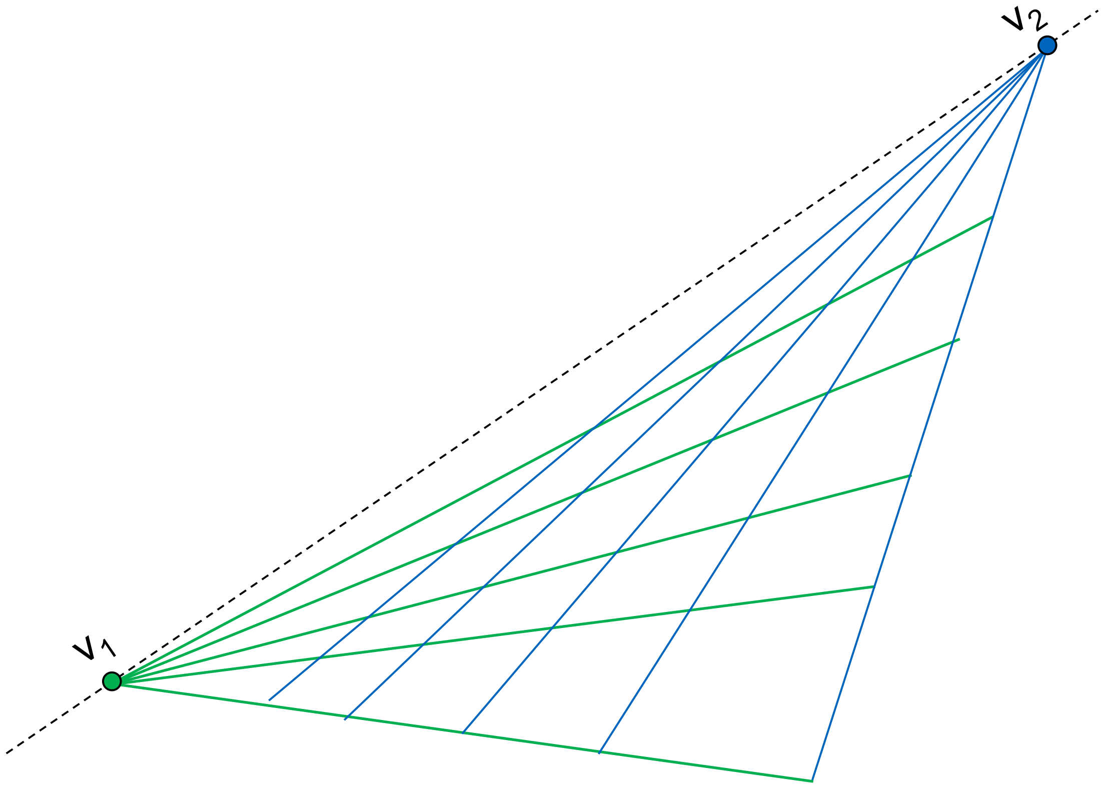
- Any two parallel lines have the same vanishing point **v**
- The vanishing point is the image of the intersection point of the two parallel lines.
- The ray from **C** through **v** is parallel to the lines
- An image may have more than one vanishing point

Vanishing lines



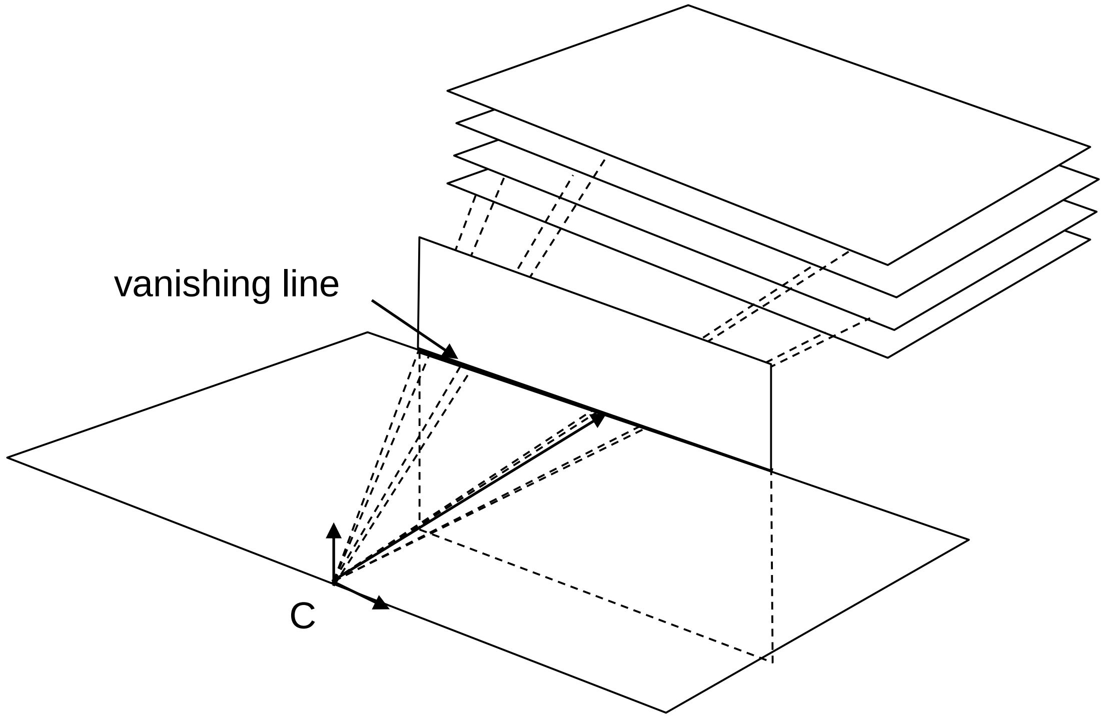
- Multiple vanishing points
  - Any set of parallel lines on the plane define a vanishing point
  - Lines at different orientation result in a different vanishing point
  - The union of all the vanishing points from lines on the same plane is the vanishing line

Vanishing lines



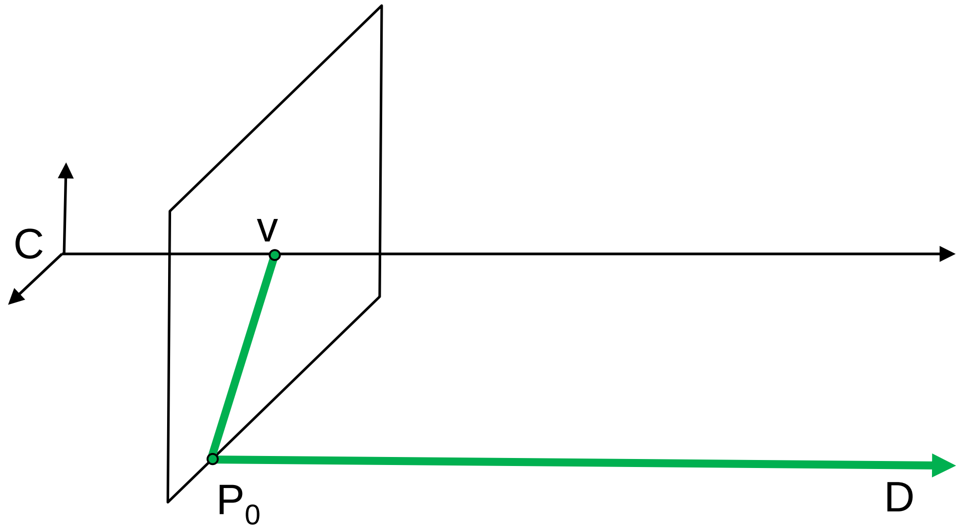
- Different planes define different vanishing lines.

Vanishing lines



- A set of parallel planes that are not parallel to the image plane intersect the image plane at a vanishing line.
- The horizon is a special vanishing line when the set of parallel planes are parallel to the ground reference.
- Anything in the scene that is above the camera will be projected above the horizon in the image.

Computing vanishing points

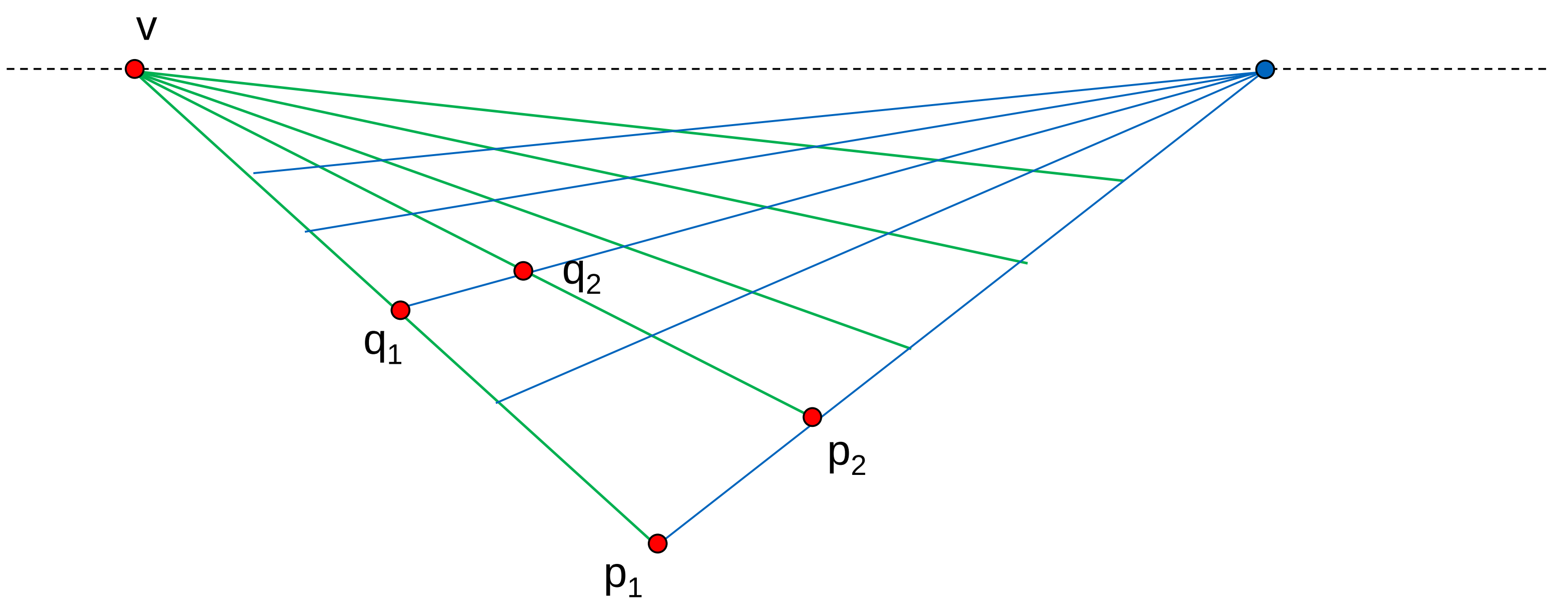


$$\mathbf{P}(t) = \mathbf{P}_0 + t\mathbf{D}$$

$$\mathbf{P}(t) = \begin{bmatrix} P_X + tD_X \\ P_Y + tD_Y \\ P_Z + tD_Z \\ 1 \end{bmatrix} \cong \begin{bmatrix} P_X / t + D_X \\ P_Y / t + D_Y \\ P_Z / t + D_Z \\ 1/t \end{bmatrix} \qquad t \to \infty \qquad \mathbf{P}_\infty \cong \begin{bmatrix} D_X \\ D_Y \\ D_Z \\ 0 \end{bmatrix}$$

- Properties $\mathbf{v} = \mathbf{\Pi P}_\infty$
  - $\mathbf{P}_\infty$ is a point at infinity, **v** is its projection
  - They depend only on line direction
  - Parallel lines $\mathbf{P}_0$ + t**D**, $\mathbf{P}_1$ + t**D** intersect at $\mathbf{P}_\infty$

Computing vanishing points (from lines)



- Intersect p_1q_1 with p_2q_2

$$v = (p_1 \times q_1) \times (p_2 \times q_2)$$

Computing vanishing points by projection

- Let P = K[I | 0] be a camera matrix. The vanishing point of lines with direction d in 3-space is the intersection v of the image plane with a ray through camera center with direction d. This vanishing point v is given by v = Kd.

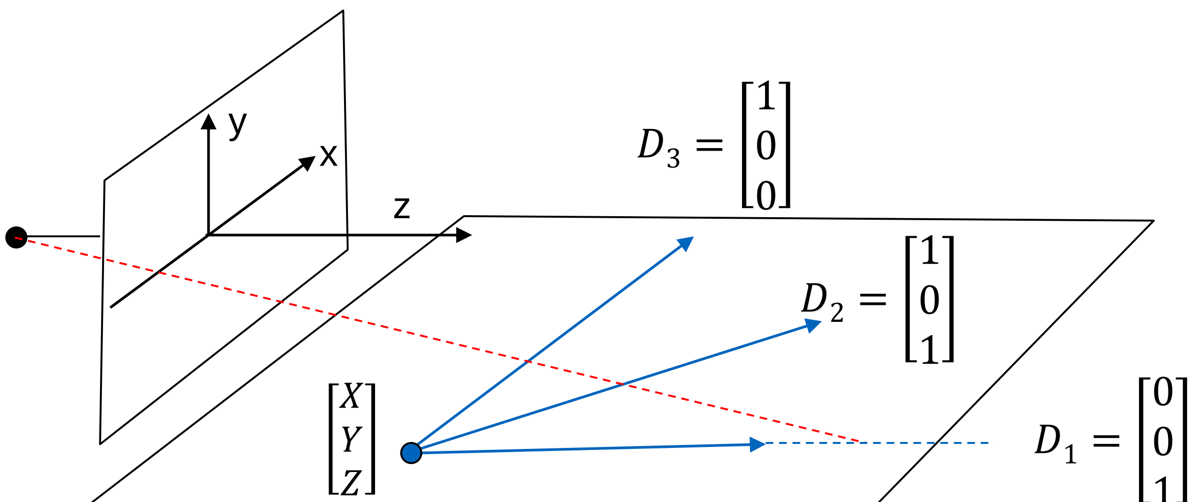
$$v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{D} \\ 0 \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

- Example: Computing vanishing points of lines on a XZ plane
- (1) parallel to the Z axis, (2) at 45 deg to the Z axis
  (3) parallel to the X axis

$$D_1 : \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_2 : \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$D_3 : \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



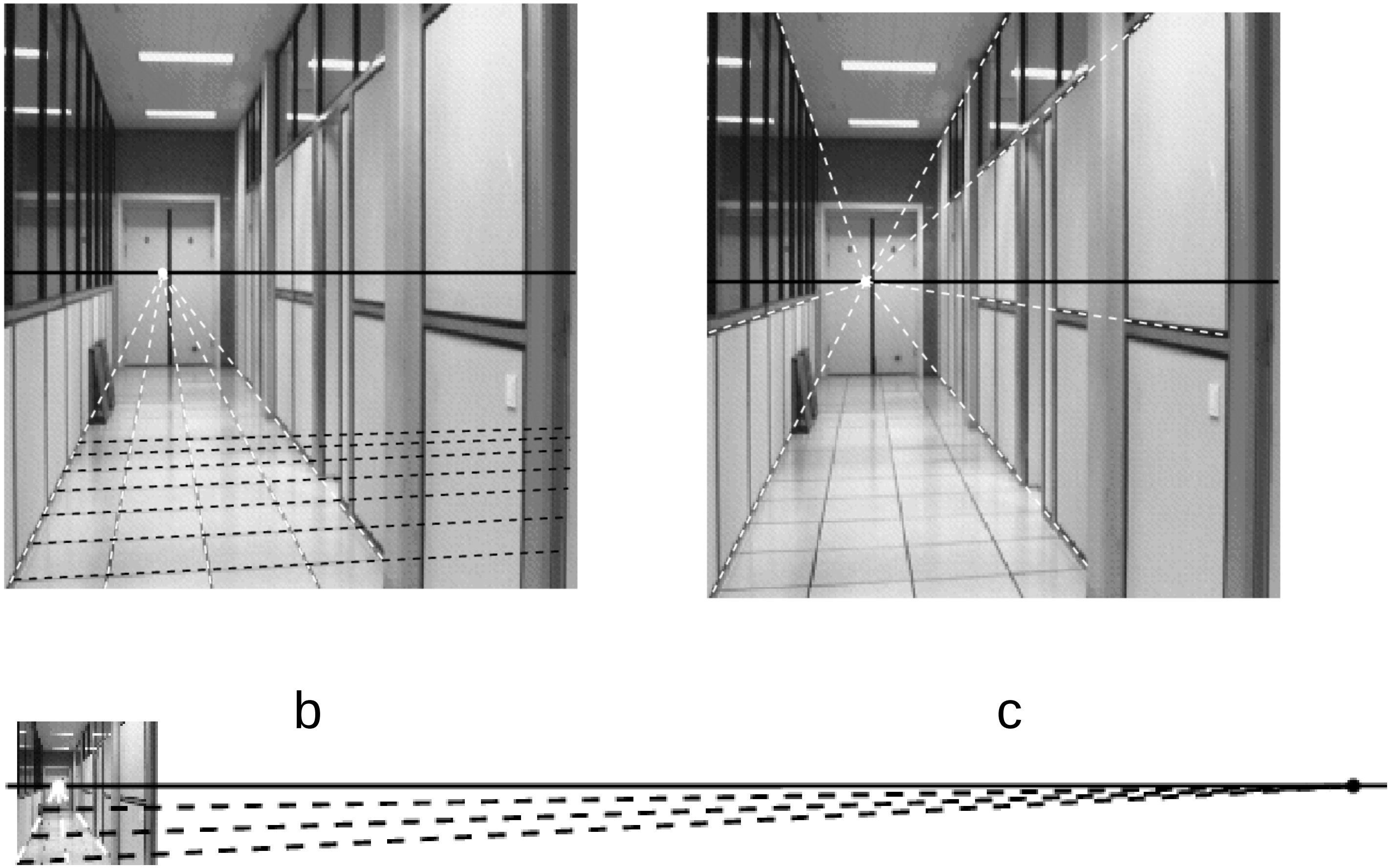
Vanishing points and projection matrix

$$\begin{array}{c} P = \begin{bmatrix} \boxed{\begin{matrix} * \\ * \\ * \end{matrix}} & \boxed{\begin{matrix} * \\ * \\ * \end{matrix}} & \boxed{\begin{matrix} * \\ * \\ * \end{matrix}} & \boxed{\begin{matrix} * \\ * \\ * \end{matrix}} \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \end{bmatrix} \\ \quad\; p_1 \quad p_2 \quad p_3 \quad p_4 \end{array}$$

- $p_1 = P\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$ = v_x (X vanishing point)
- similarly, $p_2 = v_Y$, $p_3 = v_Z$
- $p_4 = P\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T = \text{projection of world origin O}$

$$P = \begin{bmatrix} \mathbf{v}_X & \mathbf{v}_Y & \mathbf{v}_Z & \mathbf{o} \end{bmatrix}$$

Real vanishing points



b c

a

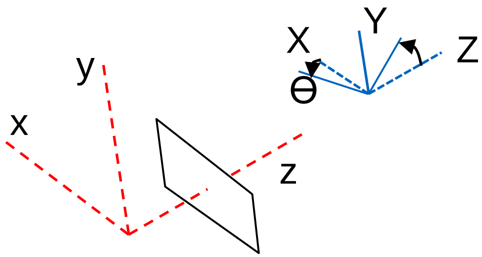
[Image source: Richard Hartley and Andrew Zisserman]

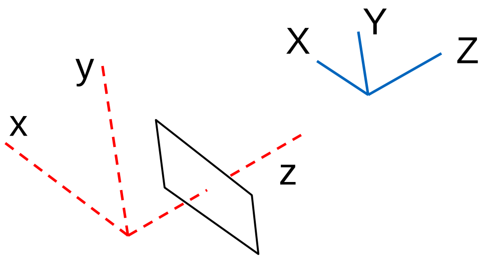
Vanishing point of a line parallel to a plane lies on the vanishing line of the plane

Image rectification using vanishing points

before rectification

after rectification





Camera rotation from vanishing points

- Two images of a scene obtained by the same camera from different position and orientation.
- The images of the points at infinity, **the vanishing points**, are not affected by the camera translation, but are affected only by the camera rotation
- Vanishing points v_i and v_i' have the following directions d_i, d_i'

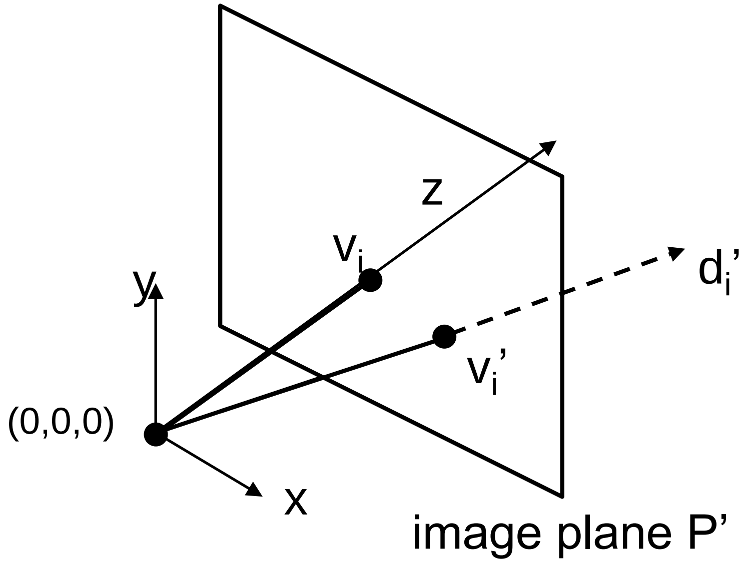
$$d_i = K^{-1}v_i / \|K^{-1}v_i\|$$
$$d_i' = K^{-1}v_i' / \|K^{-1}v_i'\|$$

- The directions are related by a rotation matrix:

$$d_i' = Rd_i$$

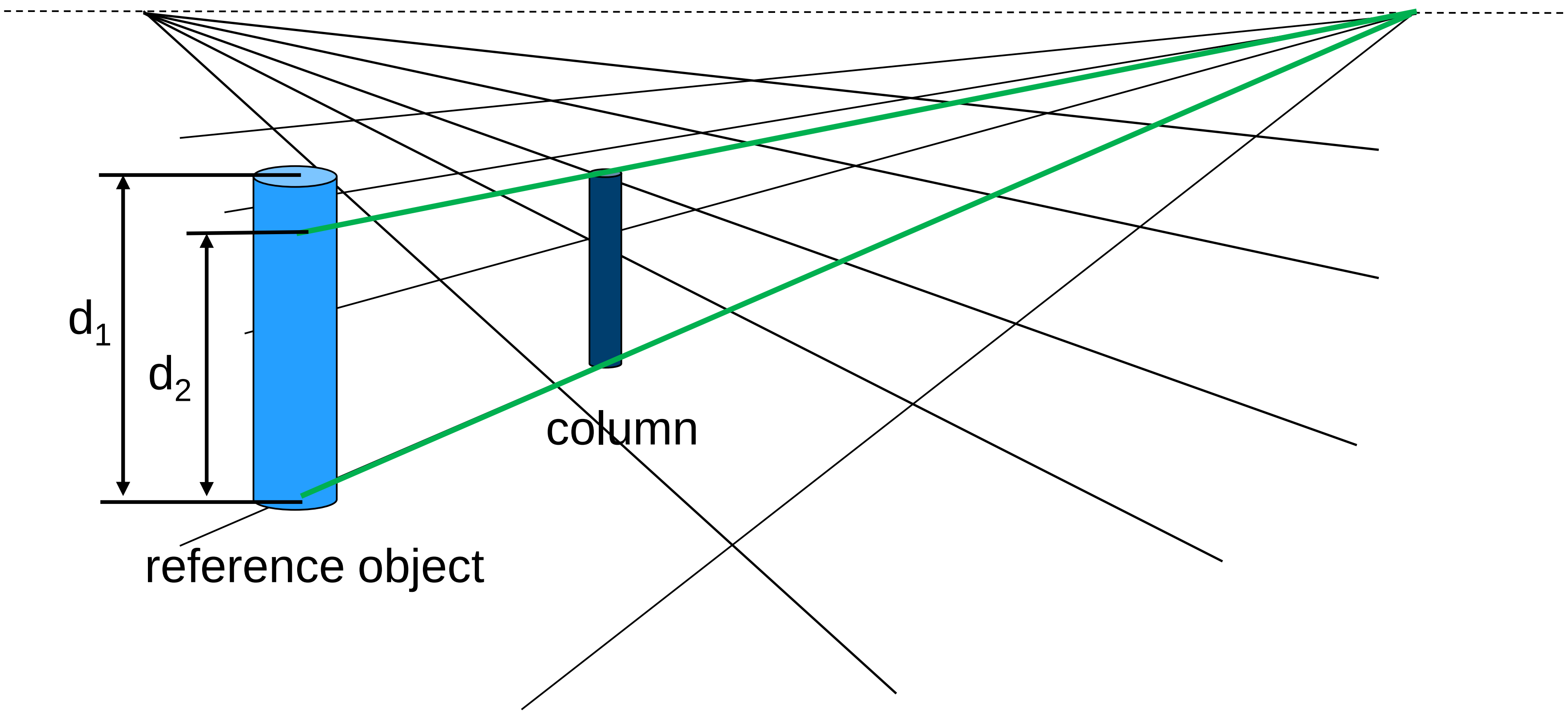
- If the directions are known, the rotation matrix can be computed from two directions

Camera rotation from vanishing points



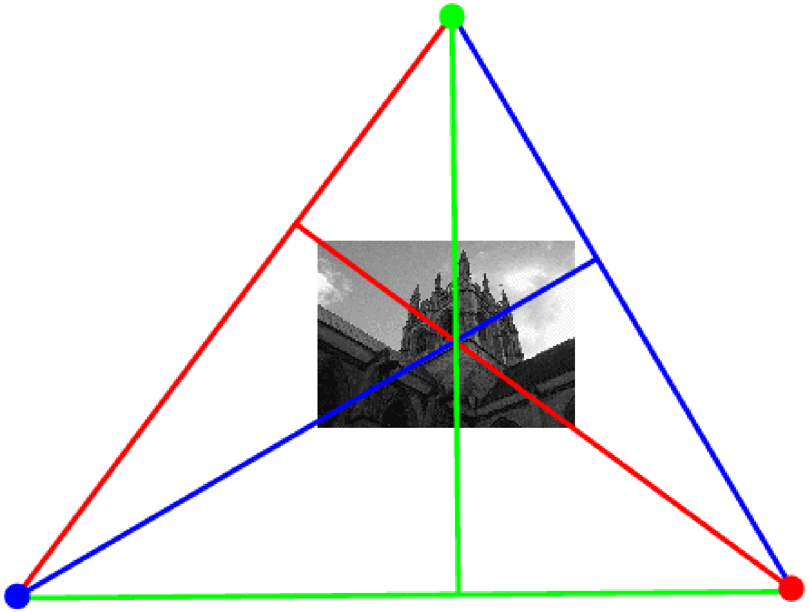
- First camera is aligned with world coordinate system. d_i = [0 0 1]
- Second camera deviates, d_i' = [d_x,d_y,d_z] and can be computed from the image coordinates of v_i'
- The rotation that aligns the second image with the first image can be computed from d_i and d_i' and 1 more direction.

Measuring heights using vanishing points



Height column = height of reference object*d_2/d_1

Camera calibration from orthogonal vanishing points



[Image source: Richard Hartley and Andrew Zisserman]

Recap - Learning goals

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