Mathematical Principles in Visual Computing: Projective Geometry

Prof. Friedrich Fraundorfer

SS 2024

Learning goals

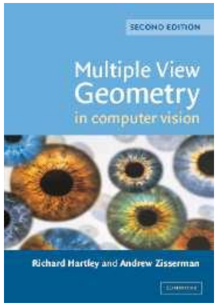
- Understand image formation mathematically
- Understand homogeneous coordinates
- Understand point, line, plane parameters and interpret them geometrically
- Understand point,line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the concept of the line and plane at infinity
- Understand the geometric interpretation of the camera matrix

Outline

- Axioms of geometry
- Differences between Euclidean and projective geometry
- 2D projective geometry
  - Homogeneous coordinates
  - Points, Lines
  - Duality
- 3D projective geometry
  - Points, Lines, Planes
  - Duality
  - Plane at infinity
  - Image formation

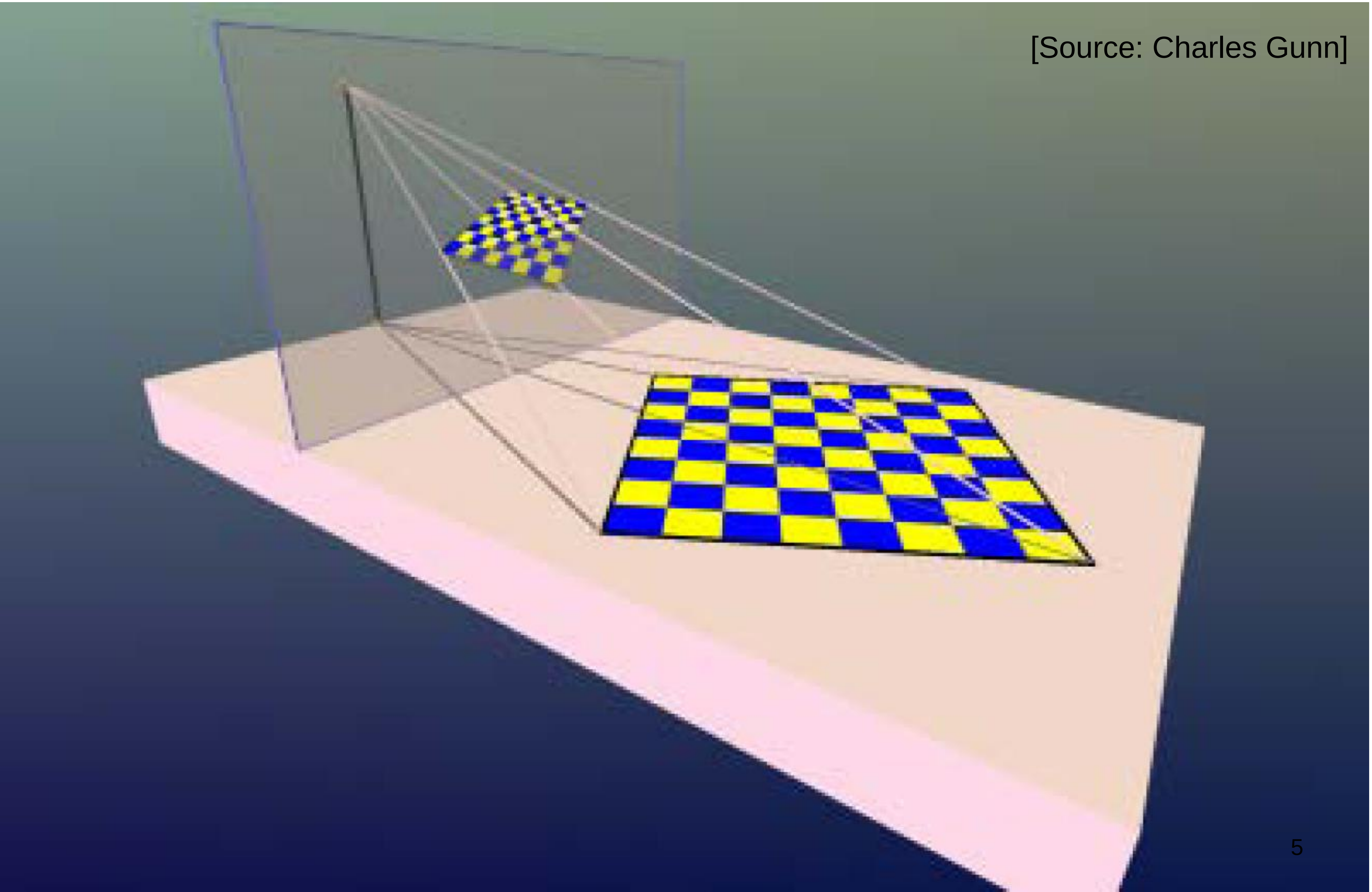
Literature

- Multiple View Geometry in Computer Vision. Richard Hartley and Andrew Zisserman. Cambridge University Press, March 2004.



- Mundy, J.L. and Zisserman, A., Geometric Invariance in Computer Vision, Appendix: Projective Geometry for Machine Vision, MIT Press, Cambridge, MA, 1992
- *Available online: www.cs.cmu.edu/~ph/869/papers/**zisser**-**mundy**.**pdf***

Motivation – Image formation

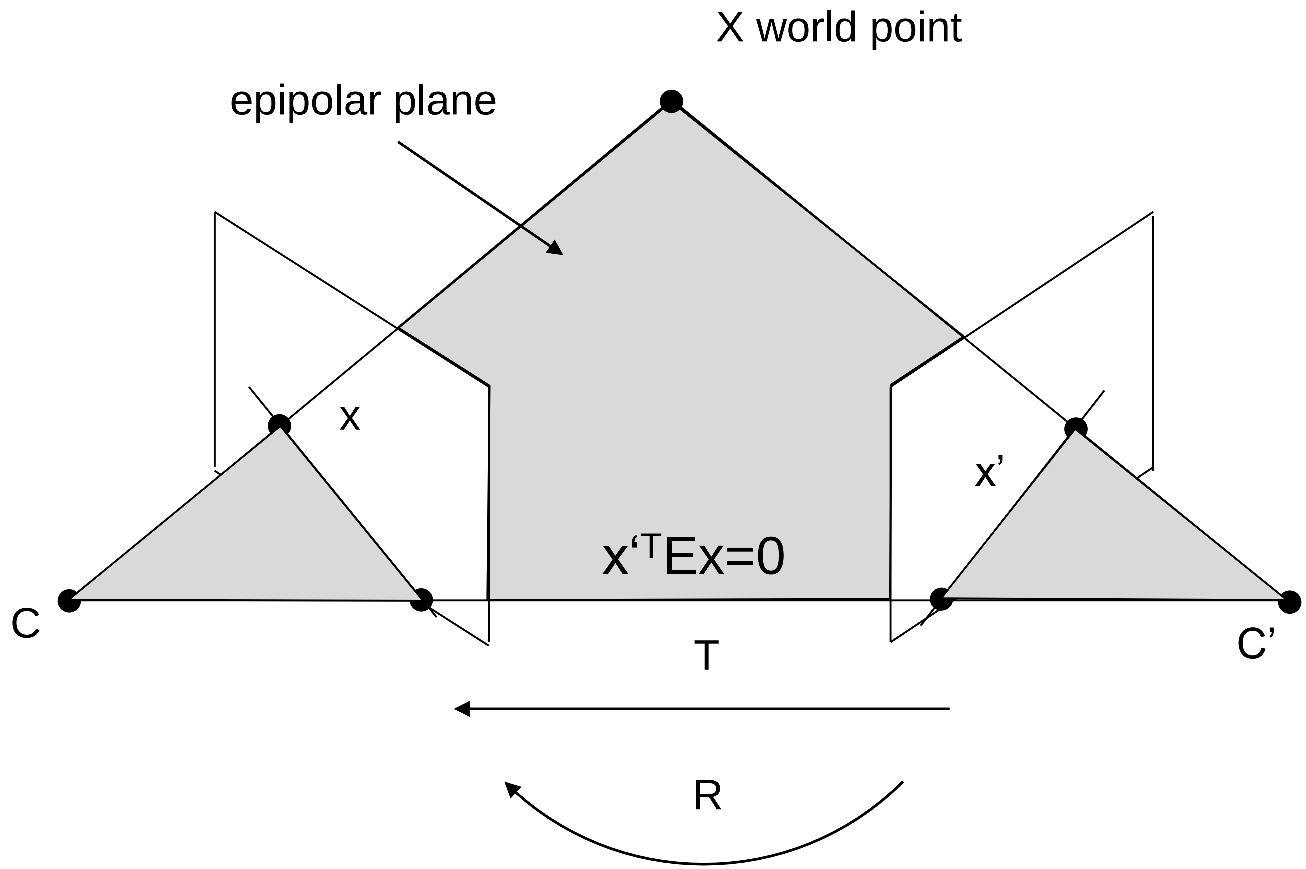


[Source: Charles Gunn]

Motivation – Parallel lines


[Source: Flickr]

Motivation – Epipolar constraint



Plane Euclidean and Projective Geometries

Euclidean

1. There exist at least three points not incident with the same line
2. Every line is incident with at least two distinct points.
3. Every point is incident with at least two distinct lines.
4. Any two distinct points are incident with one and only one line.
5. Any two distinct lines are incident with at most one point.

Projective

1. There exist a point and a line that are not incident.
2. Every line is incident with at least three distinct points.
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Main difference: The projective axioms do not allow for the possibility that two lines **don't** intersect, and the complete duality between "point" and "line".

Comments on the axioms

- The projective axioms do not allow for the possibility that two lines **don't** intersect (no parallel lines). (Axiom 5)
- Complete duality between points and lines in the projective axioms (Axiom 2 and 3).
- The projective plane may be thought of as the ordinary Euclidean plane, with an additional line called the line at infinity.
- A pair of parallel lines intersect at a unique point on the line at infinity, with pairs of parallel lines in different directions intersecting the line at infinity at different points.
- Every line (except the line at infinity itself) intersects the line at infinity at exactly one point. (Axiom 2)

Difference between Euclidean and projective geometry

Euclidean geometry

- Any two points are connected by a line.
- Most pairs of lines meet in a point.
- But parallel lines don’t meet in a point!

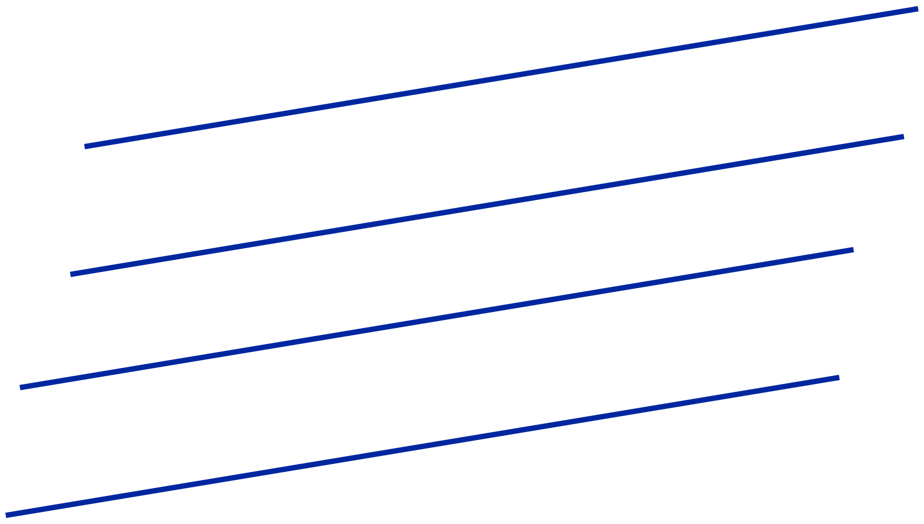
Projective geometry

- All lines intersect

All lines intersect - details

Definition: A *sheaf* of parallel lines is all the lines that are parallel to one another.

Obvious comment: Every line L belongs to exactly one sheaf (the set of lines parallel to L).



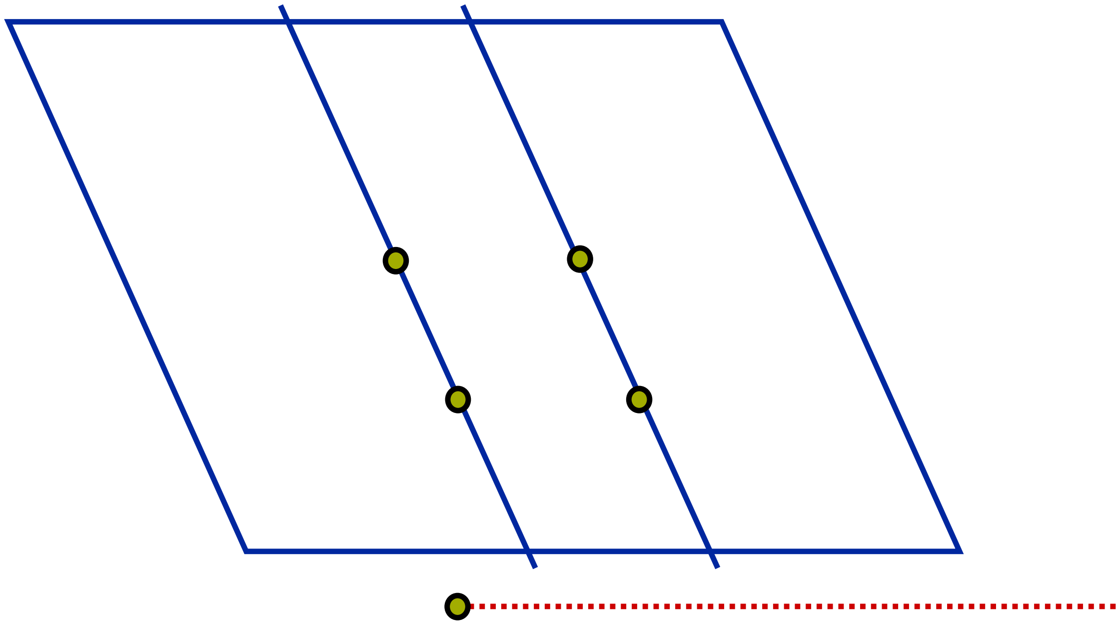
All lines intersect - details

For each sheaf *S* of parallel lines, construct a new point p “at infinity”. Assert that p lies on every line in *S.*

All the “points at infinity” together comprise the “line at infinity”

The projective plane is the regular plane plus the line at infinity.

projective plane = Euclidean plane + a new line of points



All lines intersect - details

Every pair of points *U* and *V* is connected by a single line (axiom 4).

Case 1: If *U* and *V* are ordinary points, they are connected in the usual way.

Case 2. If U is an ordinary point and V is the point on sheaf S, then the line in S through U connects U and V.

Case 3. If U and V are points at infinity they lie on the line at infinity.

All lines intersect - details

If *L* and *M* are any two lines, then they meet at a single point (axiom 5).

Case 1: *L* and *M* are ordinary, non-parallel lines: as usual.

Case 2: *L* and *M* are ordinary, parallel lines: they meet at the corresponding point at infinity.

Case 3: *L* is an ordinary line and *M* is the line at infinity: they meet at the point at infinity for *L.*

Summary

- Projective geometry extends ordinary geometry with ideal points/lines – where parallel lines meet!
- 1D: Projective line = ordinary line + ideal point
- 2D: Projective plane = ordinary plane + ideal line
- Two parallel lines intersect in an ideal point.

Projective Geometry
2D Projective Geometry

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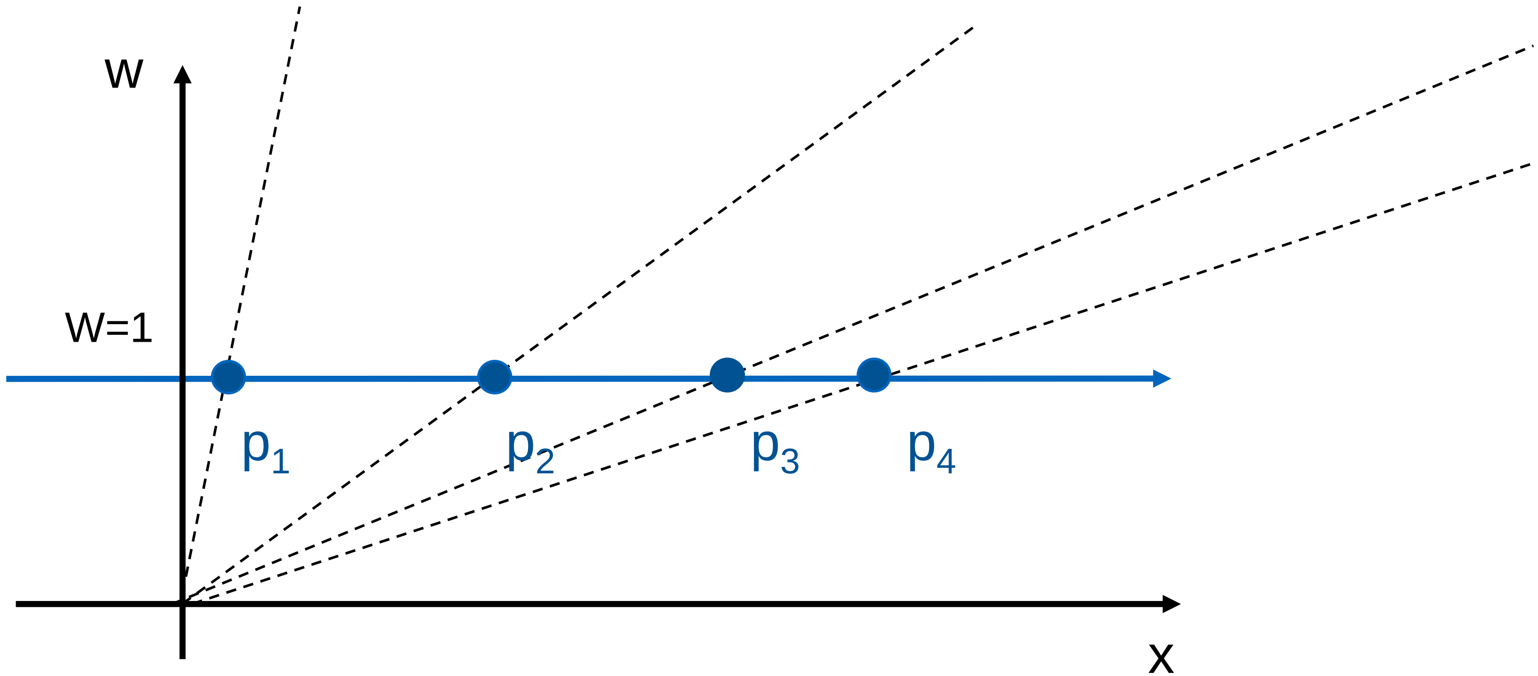
1D Euclidean geometry

Euclidean coordinate:
$p_1=[x]$

1D projective geometry



homogeneous coordinate:
p_1=[x,w] ≈ [x,1]

2D projective geometry

- Homogeneous coordinates
- Points, Lines
- Duality

Homogeneous coordinates

- projective plane = Euclidean plane + a new line of points
- The projective space associated to R3 is called the projective plane P2.

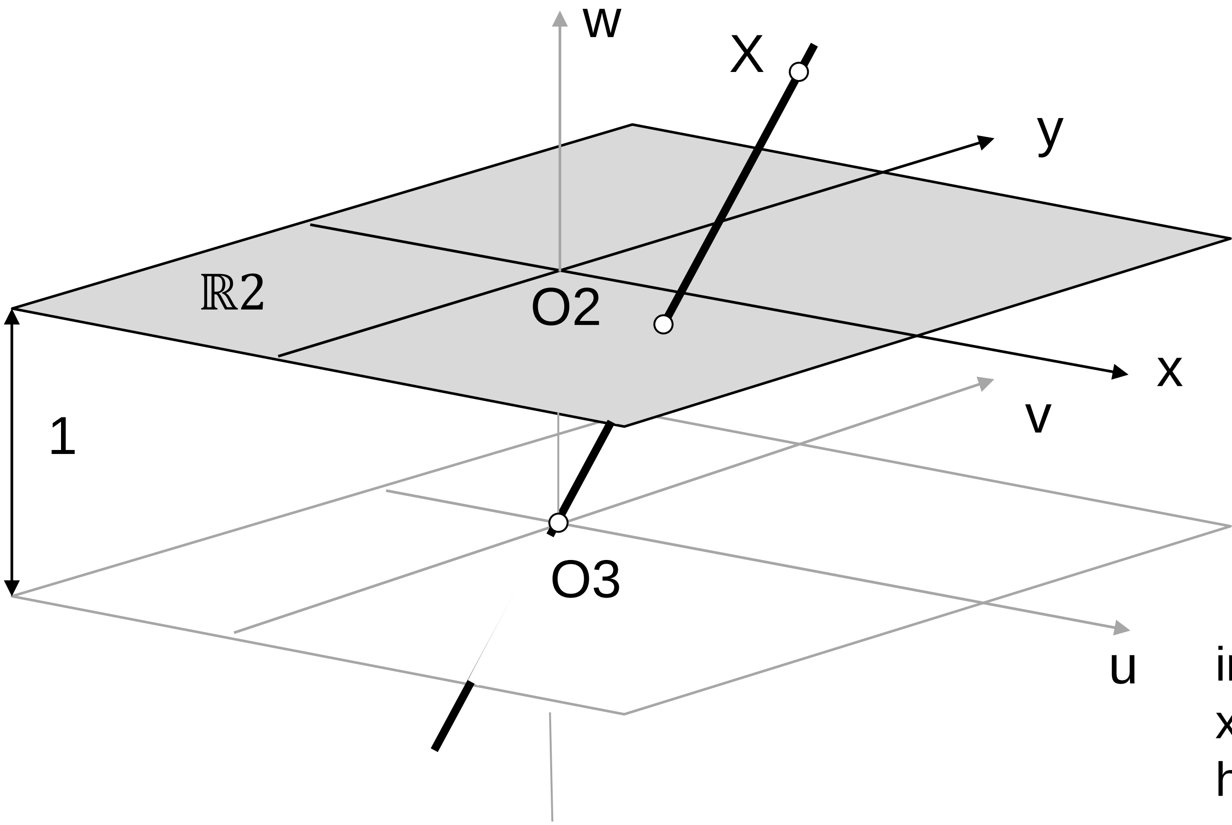
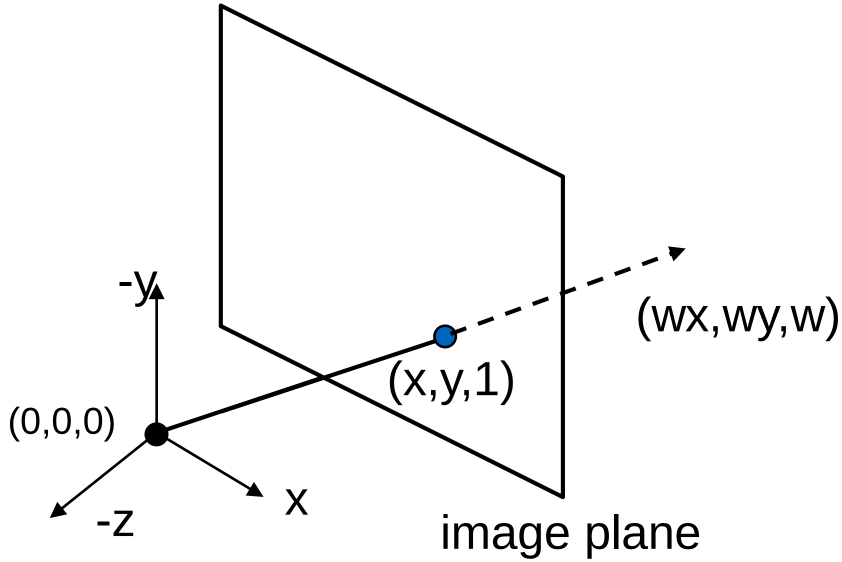


image coordinate:
x=[x,y]
homogeneous coordinate:
x=[u,v,w] ≈ [u,v,1]

Points

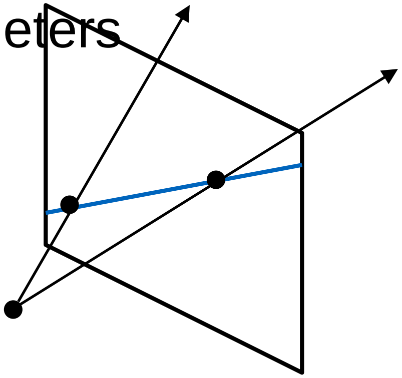
- A point in the image is a ray in projective space



- Each point (x,y) on the plane is represented by a ray (wx,wy,w)
  - all points on the ray are equivalent: $(x, y, 1) \cong (wx, wy, w)$

Lines

- A line in the image plane is defined by the equation ax + by + cz = 0 in projective space
- [a,b,c] are the line parameters



- A point [x,y,1] lies on the line if the equation ax + by + cz = 0 is satisfied
- This can be written in vector notation with a dot product:

$$0 = \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\mathbf{l}^T} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{p}}$$

- A line is also represented as a homogeneous 3-vector **l**

Calculations with lines and points

- Defining a line by two points

$$l = x \times y$$

- Intersection of two lines

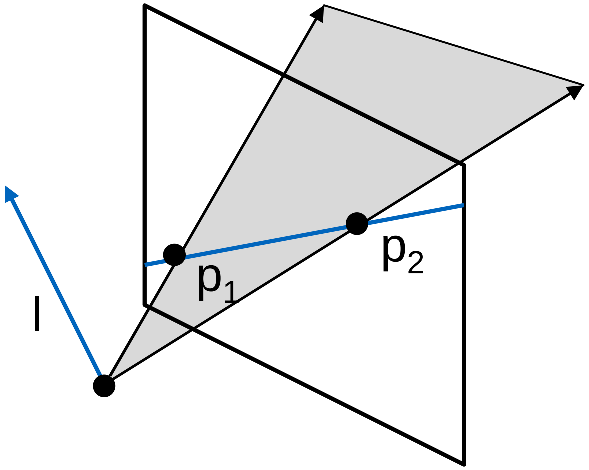
$$x = l \times m$$

- Proof:

$$l = x \times y$$
$$x^T(x \times y) = y^T(x \times y) = 0 \text{ (scalar triple product)}$$
$$x^T l = y^T l = 0$$

Geometric interpretation of line parameters [a,b,c]

- A line **l** is a homogeneous 3-vector, which is a ray in projective space
- It is $\perp$ to every point (ray) **p** on the line: **l p**=0



What is the line **l** spanned by rays $\mathbf{p_1}$ and $\mathbf{p_2}$?

- **l** is $\perp$ to $\mathbf{p_1}$ and $\mathbf{p_2}$ $\Rightarrow$ $\mathbf{l} = \mathbf{p_1} \times \mathbf{p_2}$
- **l** is the plane normal

Point and line duality

Duality principle:

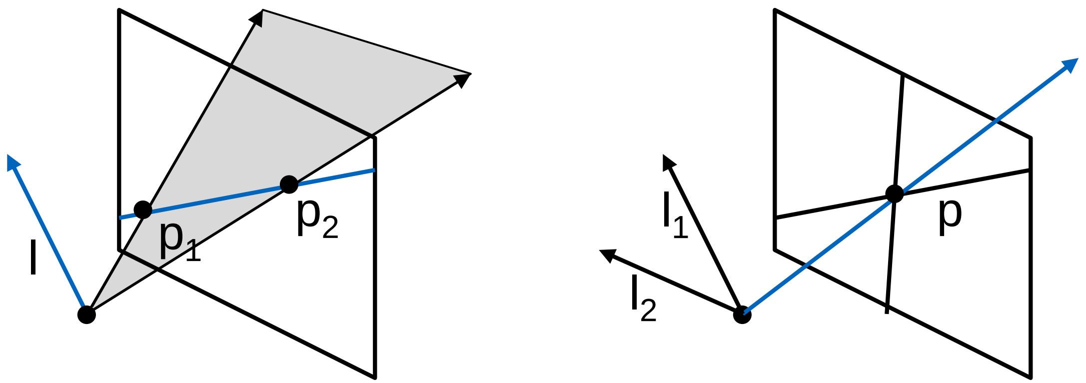
- To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Point and line duality



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- **l** is $\perp$ to $\mathbf{p_1}$ and $\mathbf{p_2}$ $\Rightarrow$ $\mathbf{l} = \mathbf{p_1} \times \mathbf{p_2}$
- **l** is the plane normal

What is the intersection of two lines $\mathbf{l_1}$ and $\mathbf{l_2}$?

- **p** is $\perp$ to $\mathbf{l_1}$ and $\mathbf{l_2}$ $\Rightarrow$ $\mathbf{p} = \mathbf{l_1} \times \mathbf{l_2}$

Points and lines are dual in projective space

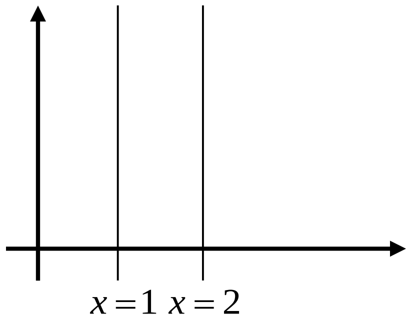
- given any formula, can switch the meanings of points and lines to get another formula

Intersection of parallel lines

- l and m are two parallel lines

$l = (a, b, c)^T$ $e.g. (-1,0,1)^T$ (a line parallel to y-axis)

$m = (a, b, d)^T$ $e.g. (-1,0,2)^T$ (another line parallel to y-axis)



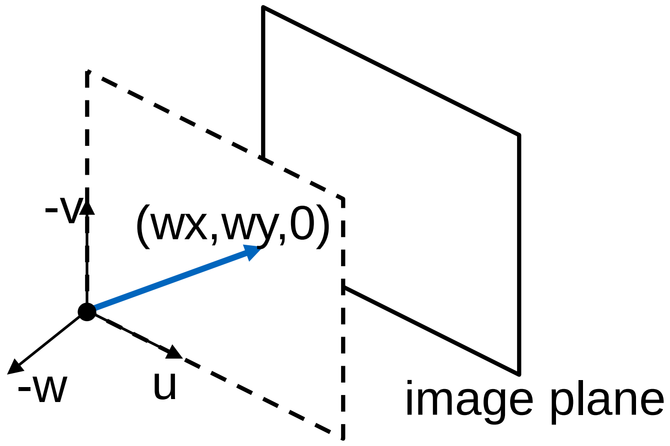
- Intersection of l and m

$x = l \times m$

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} a \\ b \\ d \end{bmatrix} = \begin{bmatrix} bd - bc \\ ac - ad \\ ab - ab \end{bmatrix} = (d - c) \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$

- A point (x,y,0) is called an ideal point, it does not lie in the image plane. But where does it lie then

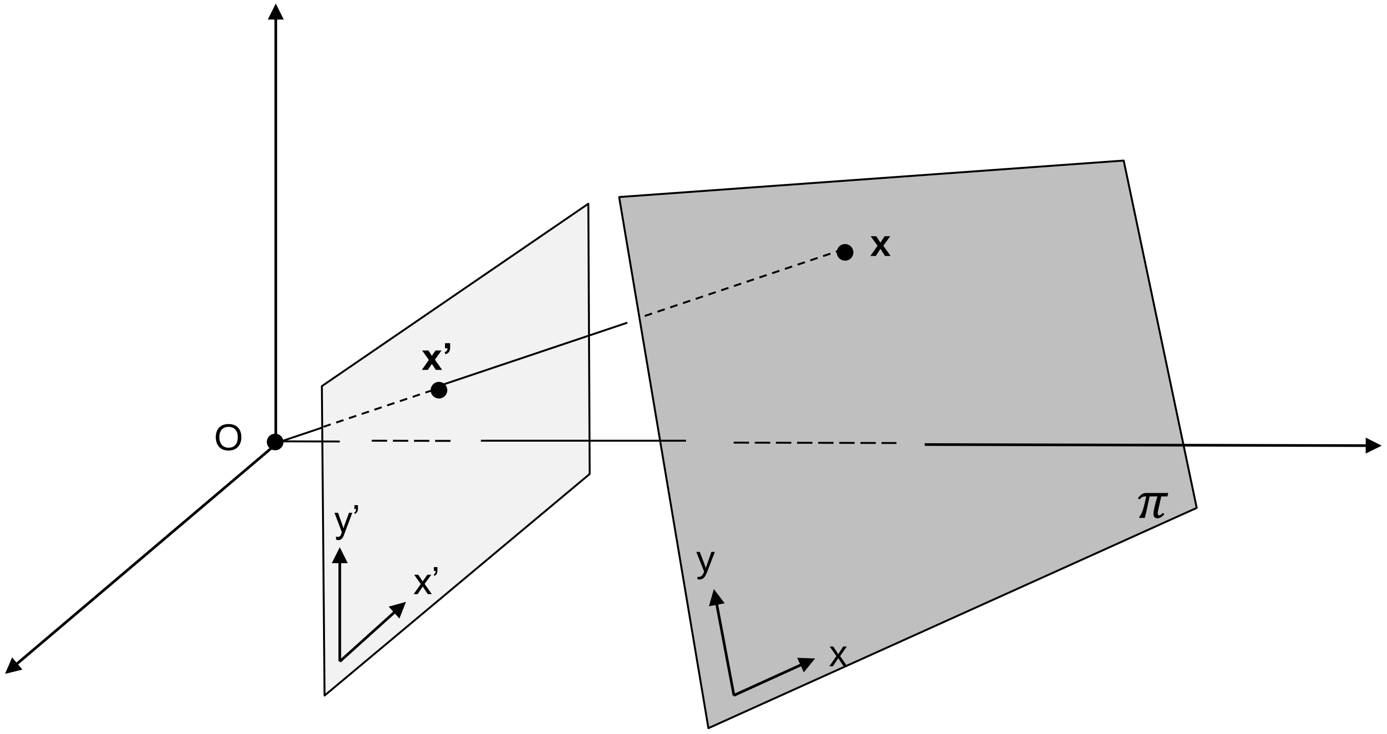
Ideal points and line at infinity



- Ideal point (“point at infinity”)
  - p ≅ (x, y, 0) – parallel to image plane
  - It has infinite image coordinates
- All ideal points lie at the line at infinity
  - l ≅ (0, 0, 1) – normal to the image plane
  - Why is it called a line at infinity?

Projective transformations

- Mapping between planes x'=Hx



Projective transformations

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $\mathrm{x}' = \mathbf{H}\,\mathrm{x}$

8DOF

projectivity=collineation=projective transformation=homography

To transform a point: **p’ = Hp**

To transform a line: **lp**=0 $\rightarrow$ **l’p’**=0

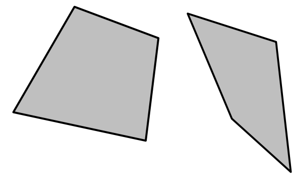
0 = **lp** = **lH^{-1}Hp** = **lH^{-1}p’** $\Rightarrow$ **l’ = lH^{-1}**

lines are transformed by postmultiplication of **H^{-1}**

Overview 2D transformations

Projective
8dof

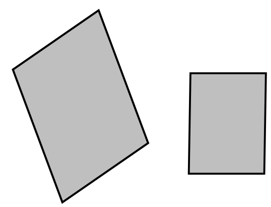
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

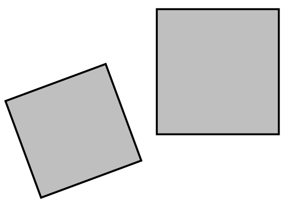
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).

Similarity
4dof

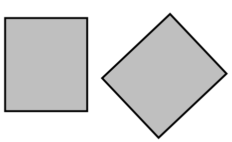
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.

Euclidean
3dof

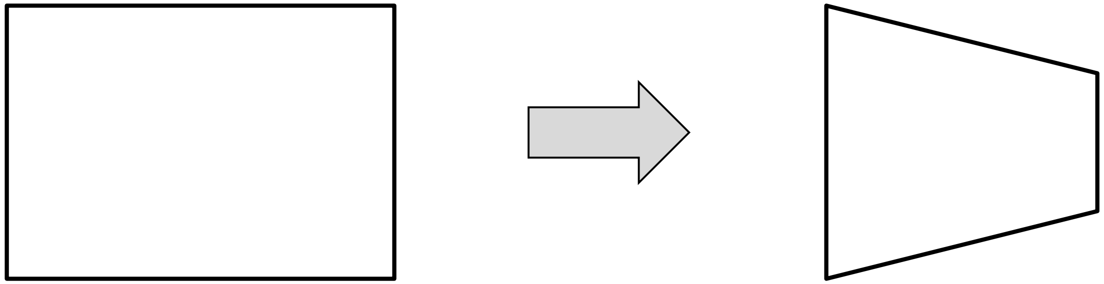
$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

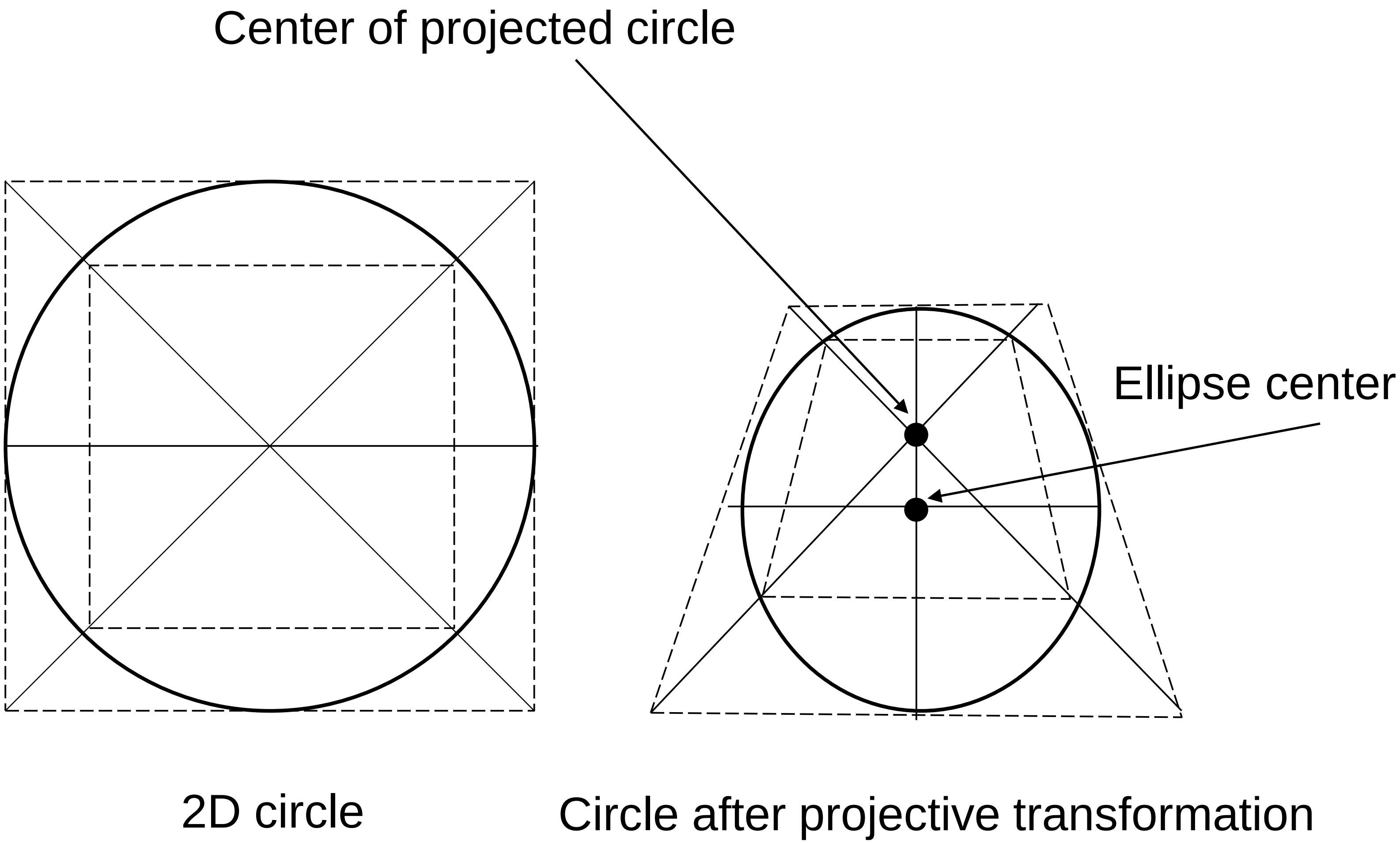
Effects of projective transformations

- Foreshortening effects can be imaged easily with primitive shapes



- But, how does a circle get transformed?

Effects of projective transformations



Projective Geometry
3D Projective Geometry

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3D projective geometry

- Points, Lines, Planes
- Duality
- Plane at infinity
- Image formation

3D projective geometry

- The concepts of 2D generalize naturally to 3D
  - The axioms of geometry can be applied to 3D as well
- 3D projective space = 3D Euclidean space + plane at infinity
  - Not so simple to visualize anymore (4D space)
- Entities are now points, lines and planes
  - Projective 3D points have four coordinates: **P** = (x,y,z,w)
- Points, lines, and planes lead to more intersection and joining options that in the 2D case

Planes

- Plane equation

$$\Pi_1 X + \Pi_2 Y + \Pi_3 Z + \Pi_4 = 0$$
$$\Pi^T X = 0$$

  - Expresses that point X is on plane Π

- Plane parameters

$$\Pi = [\Pi_1, \Pi_2, \Pi_3, \Pi_4]$$

  - Plane parameters are normal vector + distance from origin

Join and incidence relations with planes

- A plane is defined uniquely by the join of three points, or the join of a line and point in general position
- Two distinct planes intersect in a unique line
- Three distinct planes intersect in a unique point

Three points define a plane

- X1,X2,X3 are three distinct points, each has to fullfil the incidence equation. Equations can be stacked.

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \Pi = 0 \; (3x4)(4x1)$$

  - Plane parameters are the solution vector to this linear equation system (e.g. SVD)

- Points and planes are dual

$$\begin{bmatrix} \Pi_1^T \\ \Pi_2^T \\ \Pi_3^T \end{bmatrix} X = 0$$

Lines

- Lines are complicated
- Lines and points are not dual in 3D projective space
- Lines are represented by a 4x4 matrix, called Plücker matrix
- Computation of the line matrix from two points A,B

$$L = AB^T - BA^T \ (4x4) \ matrix$$

- Matrix is skew-symmetric
- Example line of the x-axis
  - $x1 = [0\ 0\ 0\ 1]^T$
    $x2 = [1\ 0\ 0\ 1]^T$
    $L = x1*x2^T - x2*x1^T$

    $$L = \begin{matrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Lines

- Points and planes are dual, we can get new equations by substituting points with planes

$$L = AB^T - BA^T \ (A, B \ are \ points)$$
$$L^* = PQ^T - QP^T \ (P, Q \ are \ planes)$$

- The intersection of two planes P,Q is a line
- Lines are self dual, the same line L has a dual representation L*
- The matrix L can be directly computed from the entries of L*

$$\begin{aligned} \ell_{12} &= \ell^*_{34} \\ \ell_{13} &= \ell^*_{42} \\ \ell_{14} &= \ell^*_{23} \\ \ell_{23} &= \ell^*_{14} \\ \ell_{42} &= \ell^*_{13} \\ \ell_{34} &= \ell^*_{12} \end{aligned}$$

Point, planes and lines

- A plane can be defined by the join of a point X and a line L

$$\Pi = L^* X$$

- A point can be defined by the intersection of a plane with a line L

$$\mathrm{X} = \mathrm{L}\Pi$$

Plane at infinity

- Parallel lines and parallel planes intersect at Π_∞
- Plane parameters of Π_∞

$$\Pi_\infty = (0,0,0,1)^T$$

- It is a plane that contains all the direction vectors $D = (x1, x2, x3, 0)^T$, vectors that originate from the origin of 4D space
- Try to imagine an extension of the 2D case (see illustration below) to the 3D case…

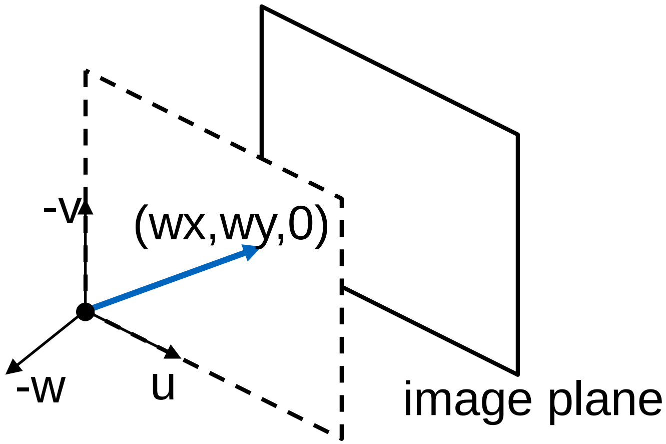


Image formation

- Projection of points in 3D onto an image plane, often called perspective projection
- Mapping 3D projective space onto 2D projective space
- A projection onto a space of one lower dimension can be achieved by eliminating one of the coordinates
- General projective transformation in 3D is a 4x4 matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

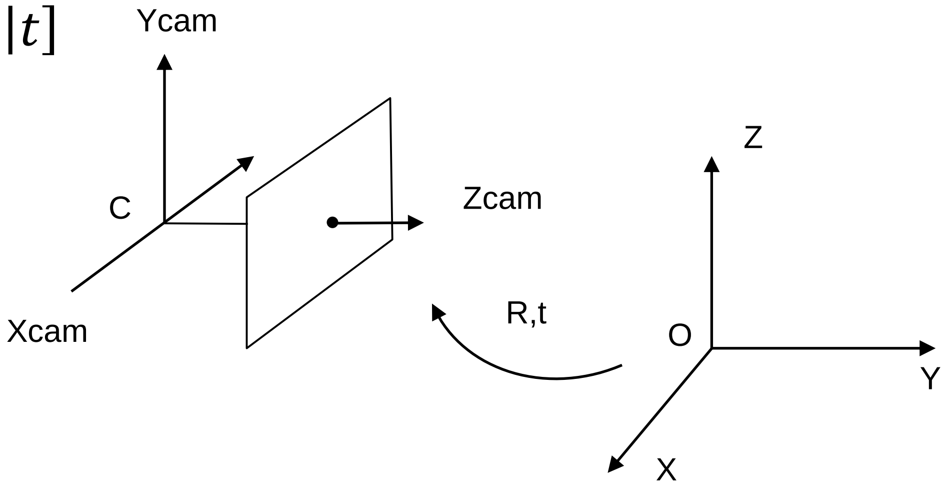
- Image projection from 3D to 2D

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

- The coordinate x4 is dropped

Camera matrix (calibrated camera)

$$P = KR[I|-C] = K[R|-RC] = K[R|t]$$
$$t = -RC$$



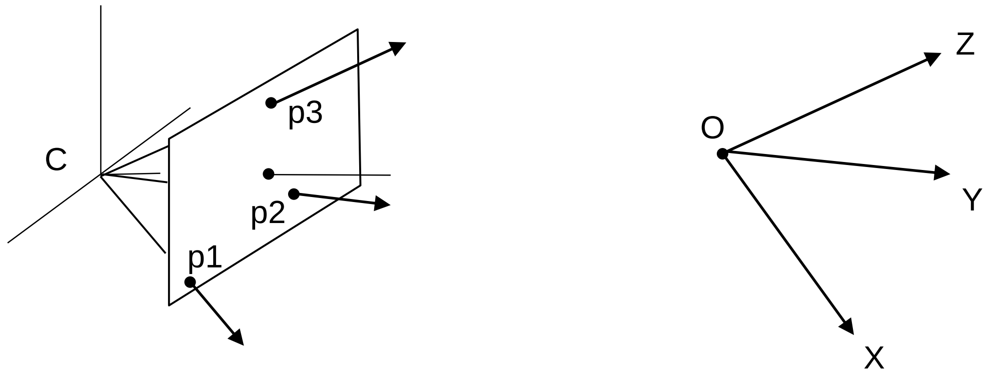
- Camera matrix is a coordinate transformation and then a projection
- C … 3x1 coordinate of the camera center in world coordinate
- R … 3x3 rotation matrix representing the orientation of the camera coordinate frame
- K … 3x3 calibration matrix

Geometric interpretation of camera matrix entries

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = [p_1 \quad p_2 \quad p_3 \quad p_4] = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}$$

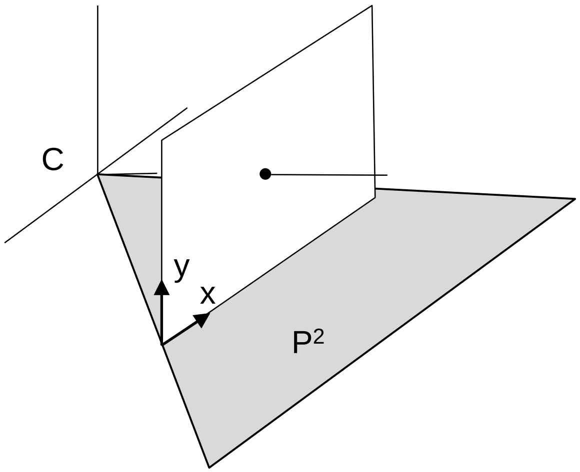
- Columns (3x1 vectors) are unit vectors of the coordinate system
- Rows transposed (1x4 vectors) are planes spanned by coordinate system axis

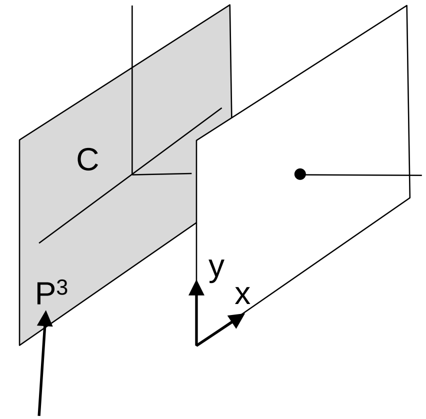
Geometric interpretation of camera matrix entries



- p1,p2,p3 are the images of the axis directions (vanishing points)
- x-axis … D=(1,0,0,0) is imaged as p1=PD

Geometric interpretation of camera matrix entries

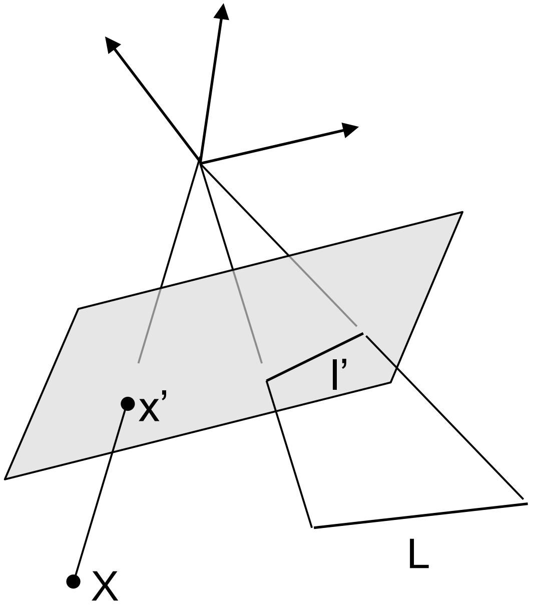




- P^{3T} is the principal plane, the plane parallel to the image plane through the camera center
- $PX = (x, y, 0)$ all points on the principal plane have the following coordinates
- Plane condition, $P^{3T}X = 0$ (point X is on the plane if it fulfils the condition)
- Points on P2, $P^{2T}X = 0$, all points have coordinates $PX = (x, 0, z)$
- Plane P2 is defined by the line y=0 and the camera center

Point and line projection

- Point projection x = PX
- Line projection is more involved (line l is a 4x4 matrix)
- Therefore indirect projection:
  $l' = x' \times y' = PX \times PY$
  $L = \overline{X\,Y}$



Recap - Learning goals

- Understand image formation mathematically
- Understand homogeneous coordinates
- Understand points, line, plane parameters and interpret them geometrically
- Understand point,line, plane interactions geometrically
- Analytical calculations with lines, points and planes
- Understand the difference between Euclidean and projective space
- Understand the concept of the line and plane at infinity
- Understand the geometric interpretation of the camera matrix