# Mathematical Principles in Visual Computing: Sylvester Resultant
# Prof. Friedrich Fraundorfer
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Slides by Vincent Lepetit

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# Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown $x$:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$$

Sylvester matrix of $f$ and $g$:

$$\mathrm{Syl}(f,g) = \begin{bmatrix} a_m & & \ldots & & & a_0 & & \\ & \ddots & & & & & \ddots & \\ & & a_m & & \ldots & & & a_0 \\ b_n & & \ldots & & b_0 & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & b_n & & \ldots & & b_0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} m \text{ rows} \end{matrix}$$

$f$ and $g$ have a common root if and only if $\det(\mathrm{Syl}(f,g)) = 0$.

$\det(\mathrm{Syl}(f,g))$ is called the Sylvester Resultant of $f$ and $g$.

# Using the Sylvester Resultant

to solve a polynomial system of two equations and TWO unknowns:

$$p_1(x,y) = 6x^2 + 3xy - xy^2 + y + 1 \qquad \begin{cases} p_1(x,y) = 0 \\ p_2(x,y) = 0 \end{cases}$$
$$p_2(x,y) = x^2y + 5x + 4y - 1$$

Let's consider $y$ as a constant, and write these two polynomials as polynomials in $x$:

$$p_{1,y}(x) = 6x^2 + (3y - y^2)x + (y+1)$$
$$p_{2,y}(x) = yx^2 + 5x + (4y - 1)$$

$$\mathrm{Syl}(p_{1,y}, p_{2,y}) = \begin{bmatrix} 6 & (3y - y^2) & (y+1) & 0 \\ 0 & 6 & (3y - y^2) & (y+1) \\ y & 5 & (4y-1) & 0 \\ 0 & y & 5 & (4y-1) \end{bmatrix}$$

# Using the Sylvester Resultant

$$p_1(x,y) = 6x^2 + 3xy - xy^2 + y + 1$$
$$p_2(x,y) = x^2y + 5x + 4y - 1$$

$$\begin{cases} p_1(x,y) = 0 \\ p_2(x,y) = 0 \end{cases}$$

$$p_{1,y}(x) = 6x^2 + (3y - y^2)x - xy^2 + (y+1)$$
$$p_{2,y}(x) = yx^2 + 5x + (4y - 1)$$

$p_{1,y}$ and $p_{2,y}$ should have roots in common, and $\det(\mathrm{Syl}(p_{1,y}, p_{2,y})) = 0$:

$$\det(\mathrm{Syl}(p_{1,y}, p_{2,y})) = \begin{vmatrix} 6 & (3y-y^2) & (y+1) & 0 \\ 0 & 6 & (3y-y^2) & (y+1) \\ y & 5 & (4y-1) & 0 \\ 0 & y & 5 & (4y-1) \end{vmatrix} = 0$$

$\det(\mathrm{Syl}(p_{1,y}, p_{2,y}))$ is a polynomial in $y$ (only)!

$$\det(\mathrm{Syl}(p_{1,y}, p_{2,y})) = 4y^6 - 20y^5 + 153y^4 - 310y^3 + 781y^2 - 276y + 36$$

$\rightarrow$ First solve for $y$ (we will see later how it can be done).