Mathematical Principles in Visual Computing: Root finding

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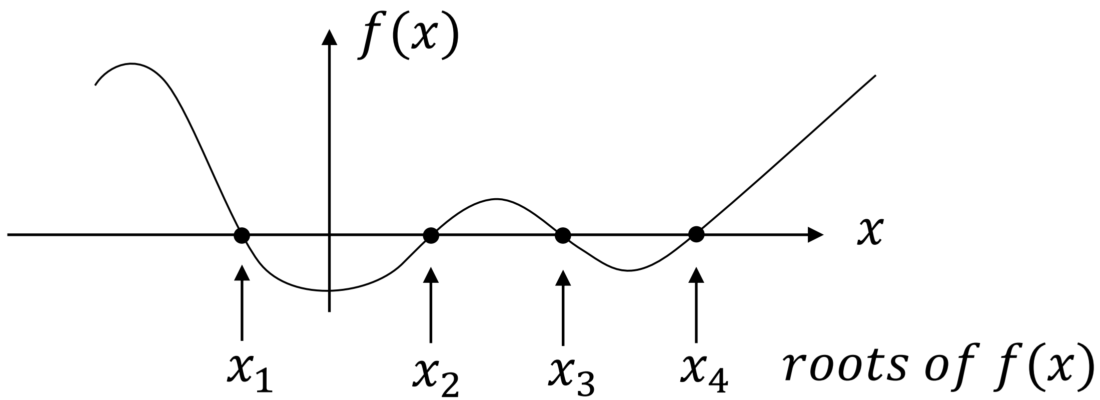
Outline

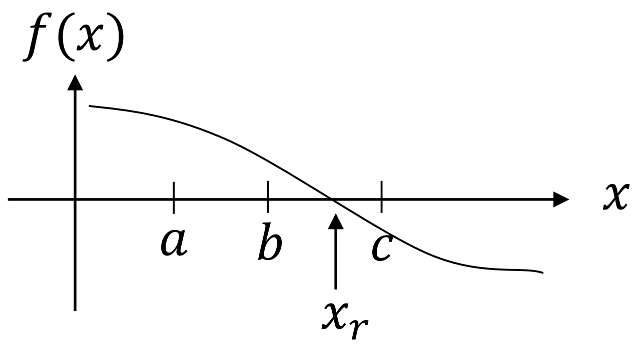
- Root finding
  - Companion Matrix
  - Sturm sequences

Root finding

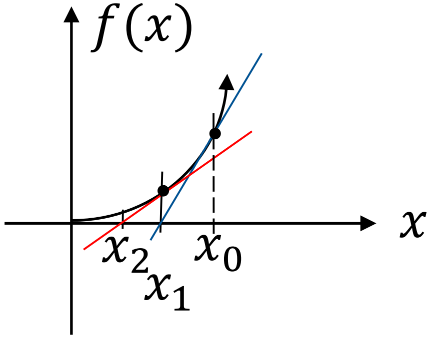
- Consider the equation f(x) = 0
- Roots of equation f(x) are the values of x which satisfy the above expression. Also referred to as the zeros of an equation



- Standard methods:
  - Bisection (look for sign changes in interval)



  - Newton-Raphson



Companion matrix

- Simple method, construct matrix of which the eigenvalues are the roots of the polynomial
- Eigenvalues of a matrix are the roots of the characteristic polynomial –> form a matrix for which the characteristic polynomial is the one to solve for.

$$p(z) = \det(zI - A)$$

$$C = \begin{bmatrix} 0 & 0 & & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & & 0 & -c_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$

$$p(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} + z^n$$

- C … nxn matrix where n is the degree of the polynomial
- Matlab: e = eig(C) … are the roots
- Finds complex roots, can be slow

Root finding with Sturm sequences

- Sturm's sequence of a univariate polynomial p is a sequence of polynomials associated with p and its derivative
- Sturm's theorem counts the number of distinct real roots and locates them in intervals.
- By subdividing the intervals containing some roots, it can isolate the roots into arbitrary small intervals, each containing exactly one root. This yields an arbitrary-precision numeric root finding algorithm for univariate polynomials.

- Advantages:
  - Typically faster than companion matrix
  - Finds only real roots (-> again faster)

Root finding with Sturm sequences

- A Sturm chain or Sturm sequence is a finite sequence of polynomials $p_0, p_1, \ldots, p_m$ of decreasing degree

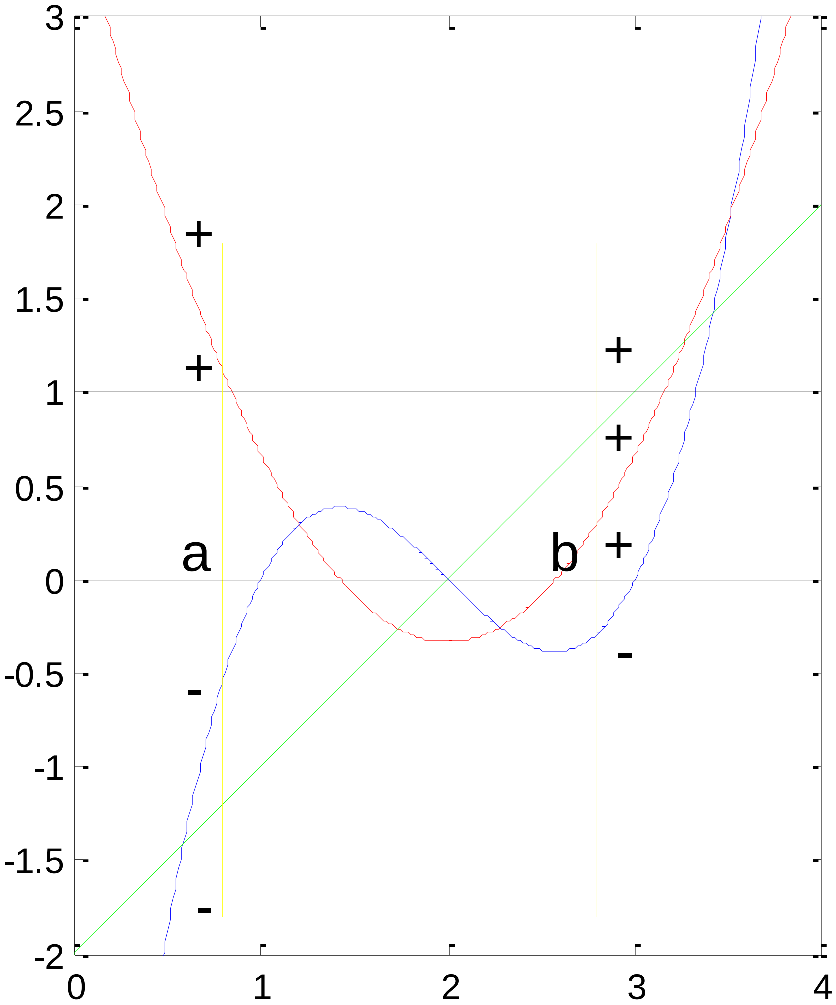
- Sturm sequence construction:
  - $p_0(z)=p(z)$ … original
  - $p_1(z) = p'(z)$ ... derivative
  - $p_2(z) = -\text{remainder}(p_0(z), p_1(z))$ …. remainder of polynomial division
  - $p_3(z) = -\text{remainder}((p_1(z), p_2(z))$
  - ….
  - $p_n(z) = \text{constant}$

Root finding with Sturm sequences

- σ(z) denotes the number of sign changes (ignoring zeroes) in the sequence
- Sturm's theorem then states that for two real numbers a < b (bracket, interval), the number of distinct roots of p in the half-open interval (a, b] is σ(a) − σ(b).

- To find the number of roots between a and b, first evaluate p_0, p_1, p_2,.., p_n, at a and note the sequence of signs of the results, e g. + − + + −. The same procedure for b gives another sign sequence, e.g. + + + − −., which contains just one sign change. Hence the number of roots of the original polynomial between a and b in the above example is 3 − 1 = 2.
- Algorithm:
  - Test intervals
  - If roots are in interval split it and test again
  - Repeat until interval is small enough

Root finding with Sturm sequences



$$g_0(z) = f(z) = (z-1)(z-2)(z-3)$$

$$g_1(z) = f'(z) = z^2 - 4z + 11/3$$

$$g_2(z) = -rem\big(g_0(z), g_1(z)\big) = z - 2$$

$$g_3(z) = -rem\big(g_1(z), g_2(z)\big) = 1$$

	g0(z)	g1(z)	g2(z)	g3(z)	s(z)
a=0.8	-	+	-	+	3
b=2.8	-	+	+	+	1

$$s(0.8) - s(2.8) = 3 - 1 = 2$$