Mathematical Principles in Visual Computing

Projective Geometry – Geometric Relations

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Outline

- Epipolar constraint derivation
- Stereo normal case
- Triangulation
- Camera pose estimation
Learning goals

- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation
The epipolar constraint is a mathematical relationship between the point correspondences of two images.
Epipolar constraint – derivation by coplanarity condition

- Vector $p$ and $t$ define a plane
- Vector $p'$ and $t$ define also a plane
- Both planes must have the same normal
- What we seek is a relation between $p$ and $p'$
Epipolar constraint – derivation by coplanarity condition

\[ P = [I|0] \quad \text{and} \quad P' = [R|t] \]
Epipolar constraint – derivation by coplanarity condition
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\[ p'' = Rp + t \]

\( X \) world point

\( [R|t] \)
Epipolar constraint – derivation by coplanarity condition

\[ t \times p' = t \times p'' \]

\[ t \times p' = t \times (Rp + t) \]

with \( p'' = Rp + t \)

\[ t \times p' = t \times Rp + t \times t \]

\[ p'^T (t \times p') = 0 \]

\[ p'^T (t \times Rp) = 0 \]

\[ p'^T [t]_x Rp = 0 \]

\[ E \]

\[ p'^T Ep = 0 \]

E is called the Essential matrix
Fundamental matrix

- $p, p'$ from the Essential matrix derivation are in normalized coordinates
- $x, x'$ are image coordinates, $x = Kp$, $x' = Kp'$
- By replacing $p, p'$ with $x, x'$ one gets the Fundamental matrix

\[ p = K^{-1}x \]
\[ p' = K^{-1}x' \]

\[ p'^T E p = 0 \]
\[ x'^T K^{-T} E K^{-1} x = 0 \]
\[ x'^T F x = 0 \]
\[ F = K^{-T} E K^{-1} \]
Epipolar lines

- The corresponding line $l'$ to image coordinate $x$
- $l'$ is the line connecting the epipole $e'$ and the image coordinate $x'$
- Hypothesis: $l' = Fx$
- Point $x'$ must lie on $l'$, thus $x'^Tl' = 0$
- Now $x'^TFx = 0$

![Diagram](image)
Stereo case

\[ R = I_{3 \times 3} \quad T = [T_x \quad 0 \quad 0]^T \]
Stereo case

\[ R = I_{3 \times 3} \]
\[ T = [T_x \ 0 \ 0]^T \]

\[ E = [T]_x R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix} \]

\[ \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \]

\[ \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -T_x \\ -T_x & 0 \end{bmatrix} = 0 \]

\[-y'T_x + T_xy = 0\]
Triangulation

- Compute coordinates of world point X given the measurements x, x' and the camera projection matrices P and P'
Triangulation

- Condition: Measurement vector \( x \) needs to have the same direction as projection of \( X \) (cross-product equals 0)
  - \( x \times (PX) = 0 \) and \( x' \times (P'X) = 0 \)
  - Can be rewritten into equation system \( AX = 0 \) to solve for \( X \)
Triangulation

\[ x \times (PX) = 0 \text{ and } x' \times (P'X) = 0 \]
\[ x(P_3^T X) - (P_1^T X) = 0 \]
\[ y(P_3^T X) - (P_2^T X) = 0 \]
\[ x(P_2^T X) - y(P_1^T X) = 0 \]

\[
\begin{bmatrix}
  xP_3^T - P_1^T \\
  yP_3^T - P_2^T \\
  x'P_3'^T - P_1'^T \\
  y'P_3'^T - P_2'^T
\end{bmatrix}X = 0
\]

\[
P = \begin{bmatrix}
P_1^T \\
P_2^T \\
P_3^T
\end{bmatrix}
\]
Camera pose estimation

- perspective-n-point problem
- Goal is to estimate camera matrix $P$ such that $m_1 = PM_1$
- $m_1, M_1, m_2, M_2, m_3, M_3$ are known
- Algebraic linear solution requires 6 3D-2D point correspondences, minimal nonlinear solution requires only 3
Camera pose estimation

- Derivation similar to Triangulation, but now entries of P are the unknowns instead of X
- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)
Camera pose estimation

- Derivation similar to Triangulation, but now entries of $P$ are the unknowns instead of $X$
- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)

\[ x \times (PX) = 0 \text{ for all pairs } x \leftrightarrow X \]
\[ y(P_3^TX) - w(P_2^TX) = 0 \]
\[ x(P_3^TX) - w(P_1^TX) = 0 \]
\[ x(P_2^TX) - y(P_1^TX) = 0 \]

\[
\begin{bmatrix}
0 & -wX^T & yX^T \\
-wX^T & 0 & xX^T \\
-yX^T & xX^T & 0
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
P_3
\end{bmatrix}
= 0
\]
Recap - Learning goals

- Understand and derive the epipolar constraint
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