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# Mathematical Principles in Visual Computing: Projective Geometry – Part 2

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SS 2023

# Learning goals

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- Understand the concept of vanishing points and vanishing lines
- Understand the calculation of vanishing points
- Understand the relation between vanishing points and camera orientation and calibration

# Outline

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- Vanishing points and lines
- Applications of vanishing points

# Vanishing points

[Source: Flickr]



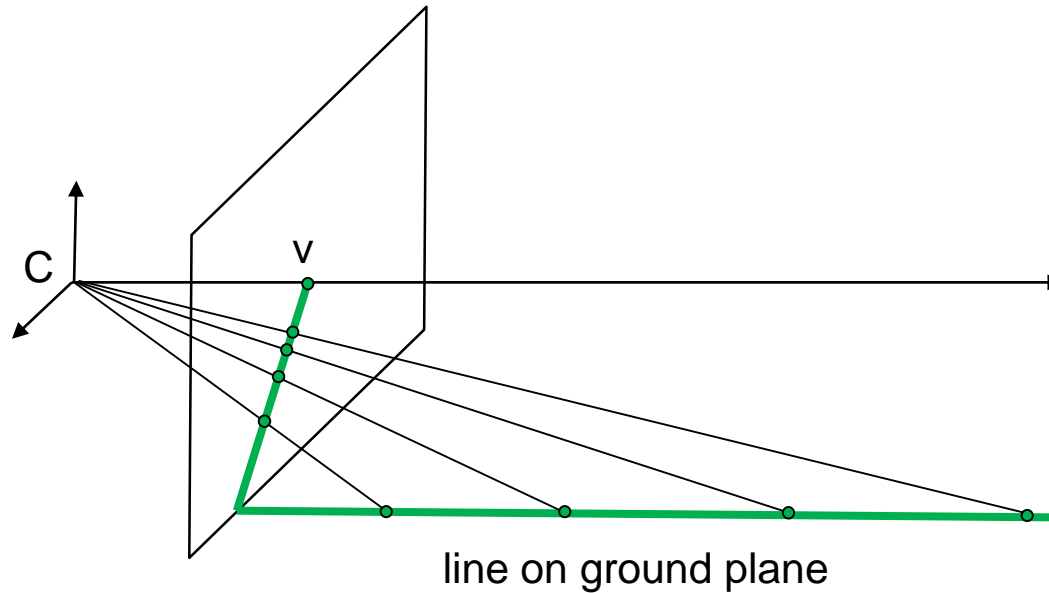
# Vanishing points

- Ideal points in the projective 3-space are located at infinity, and have homogeneous coordinates of the form  $(x, y, z, 0)$ .
- These points are also called points at infinity.
- The image of an ideal point under a projective mapping is called a vanishing point.
- Recall that two parallel lines in the projective 3-space meet at an ideal point.
- Thus the images of two or more parallel world lines converge at a vanishing point.



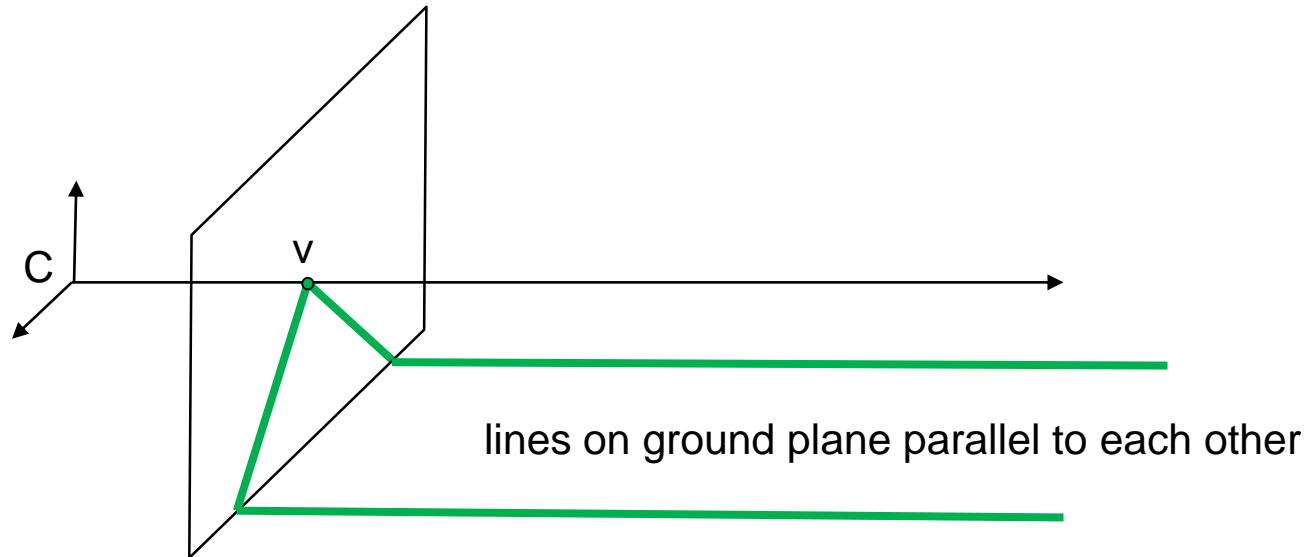
[Source: Flickr]

# Vanishing points



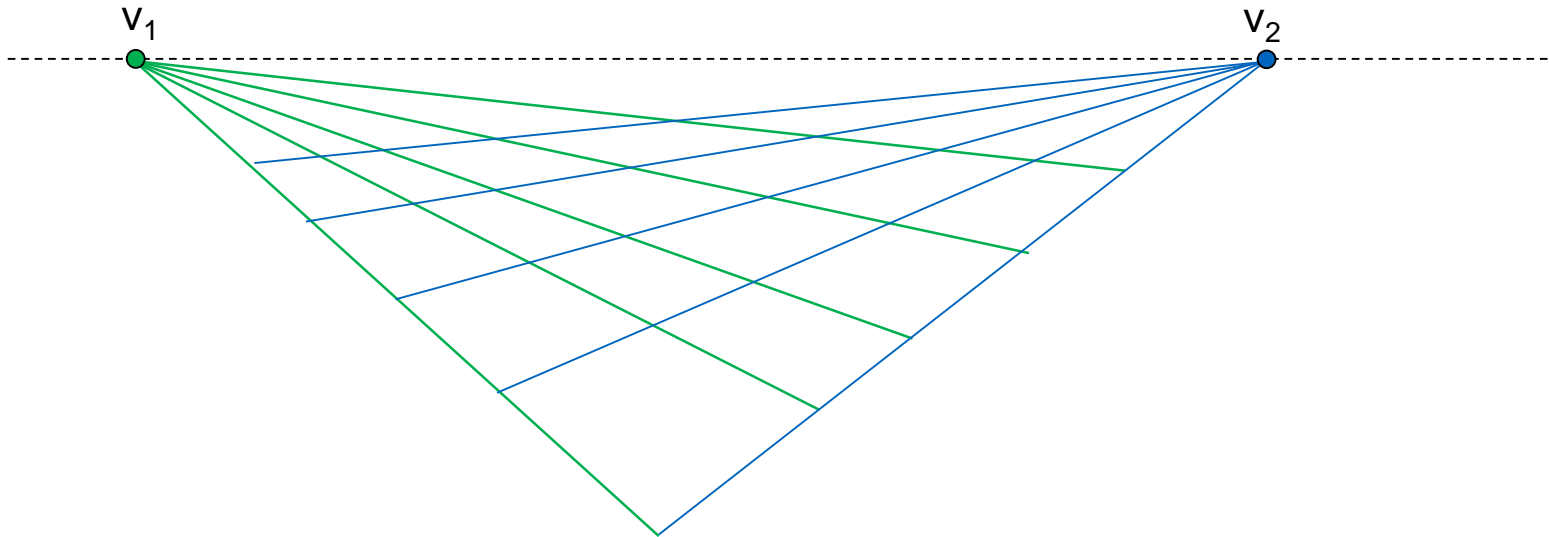
The vanishing point  $v$  is the projection of a point at infinity.  
Think of extending the line on the ground plane further and further into infinity.

# Vanishing points



- Any two parallel lines have the same vanishing point  $\mathbf{v}$
- The vanishing point is the image of the intersection point of the two parallel lines.
- The ray from  $\mathbf{C}$  through  $\mathbf{v}$  is parallel to the lines
- An image may have more than one vanishing point

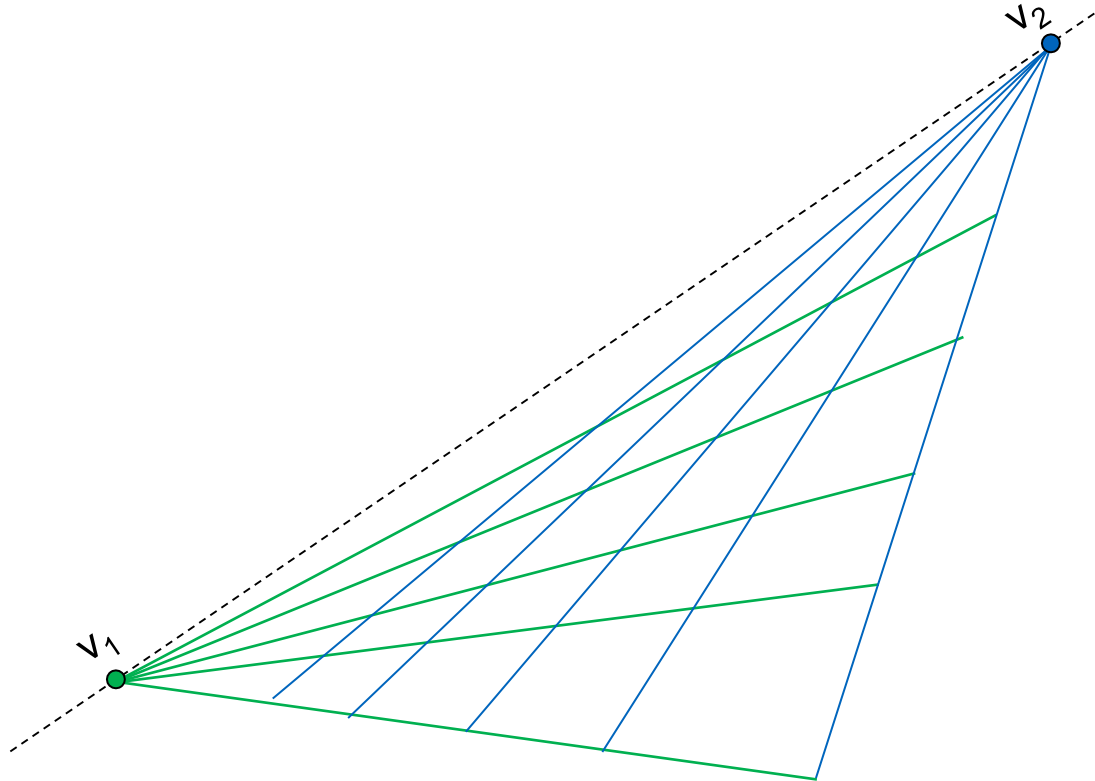
# Vanishing lines



- Multiple vanishing points
  - Any set of parallel lines on the plane define a vanishing point
  - Lines at different orientation result in a different vanishing point
  - The union of all the vanishing points from lines on the same plane is the vanishing line

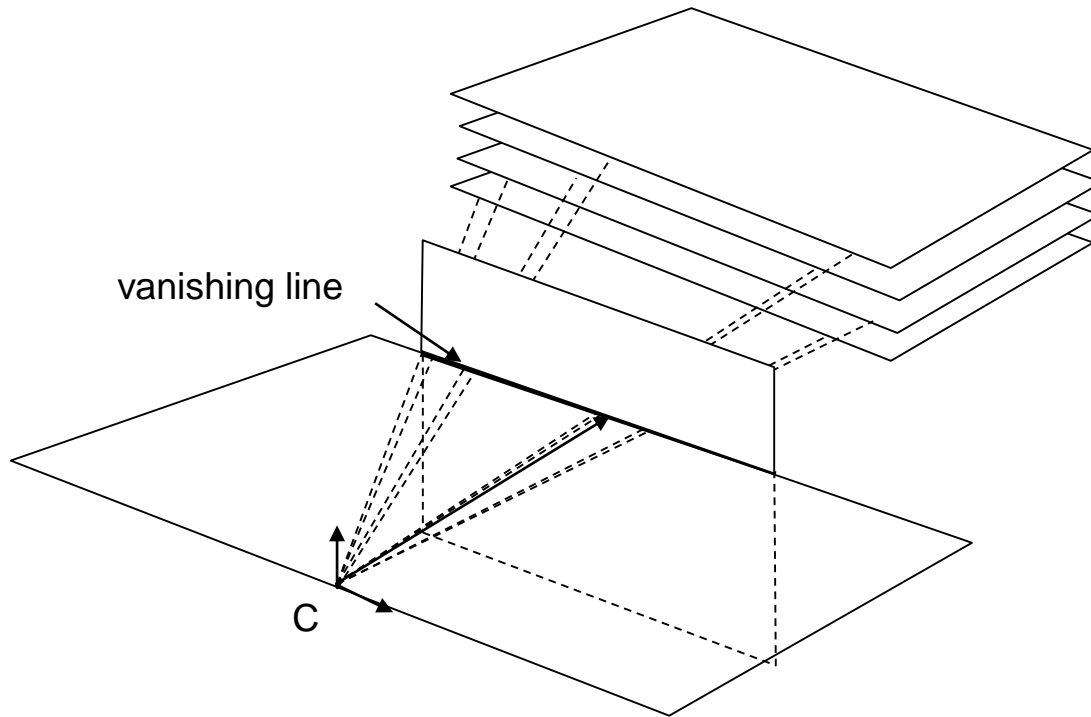


# Vanishing lines



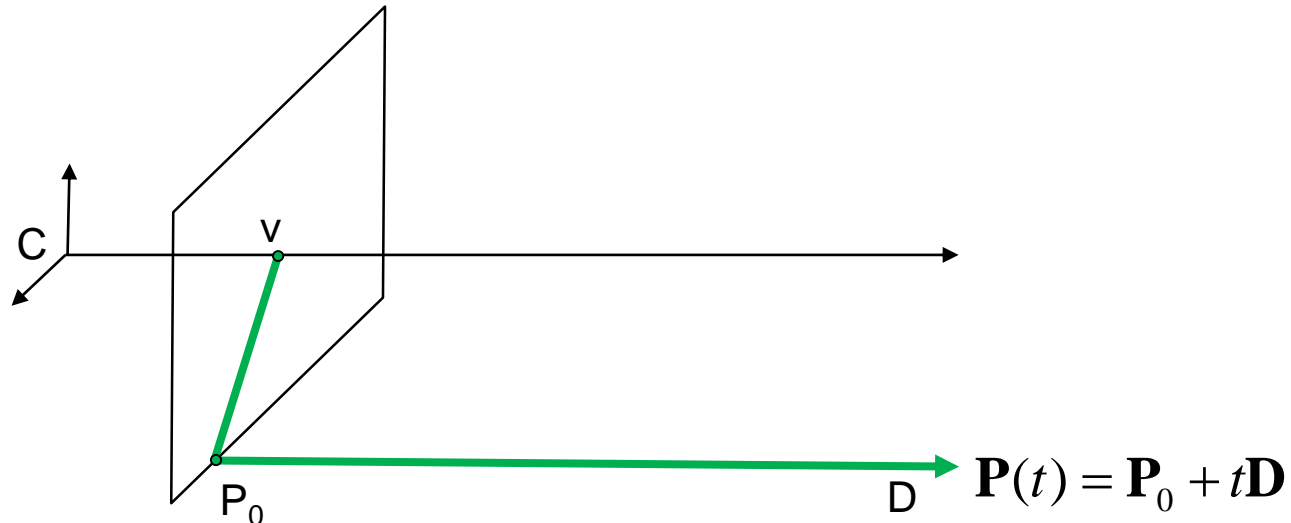
- Different planes define different vanishing lines.

# Vanishing lines



- A set of parallel planes that are not parallel to the image plane intersect the image plane at a vanishing line.
- The horizon is a special vanishing line when the set of parallel planes are parallel to the ground reference.
- Anything in the scene that is above the camera will be projected above the horizon in the image.

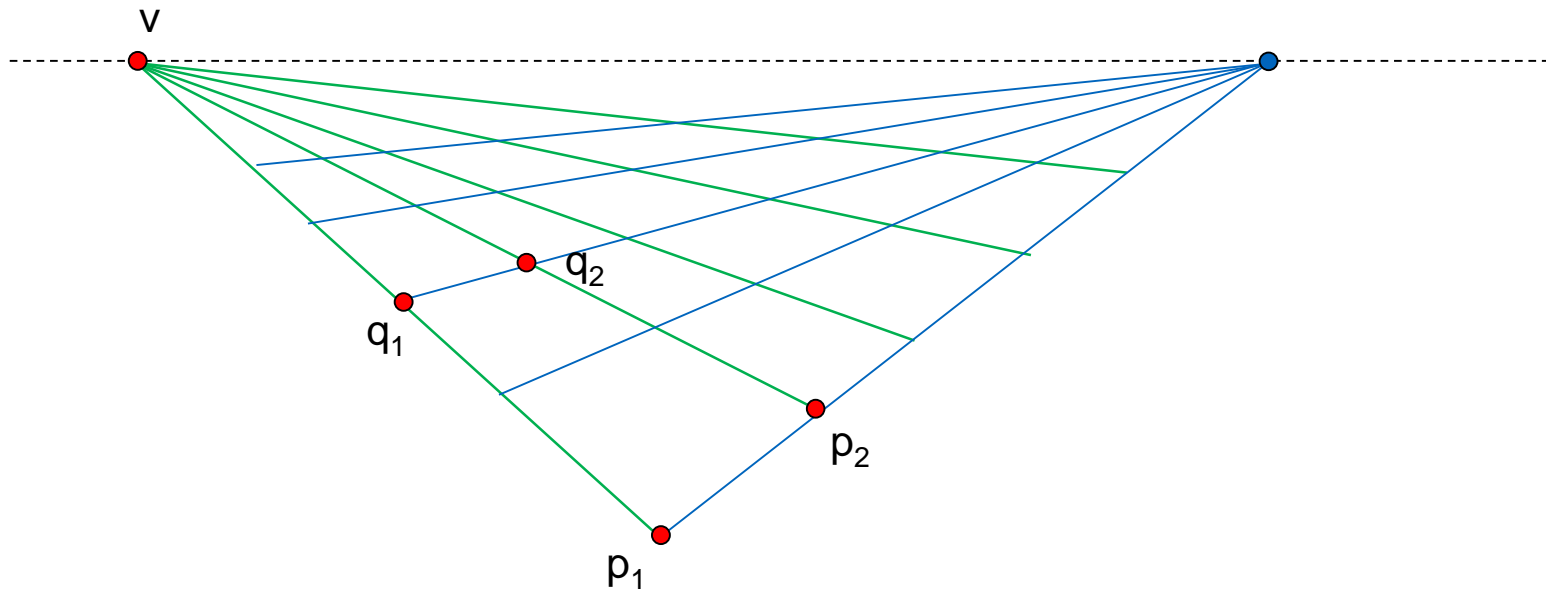
# Computing vanishing points



$$\mathbf{P}(t) = \begin{bmatrix} P_X + tD_X \\ P_Y + tD_Y \\ P_Z + tD_Z \\ 1 \end{bmatrix} \cong \begin{bmatrix} P_X / t + D_X \\ P_Y / t + D_Y \\ P_Z / t + D_Z \\ 1/t \end{bmatrix} \quad t \rightarrow \infty \quad \mathbf{P}_\infty \cong \begin{bmatrix} D_X \\ D_Y \\ D_Z \\ 0 \end{bmatrix}$$

- Properties  $\mathbf{v} = \mathbf{I}\mathbf{P}_\infty$ 
  - $\mathbf{P}_\infty$  is a point at infinity,  $\mathbf{v}$  is its projection
  - They depend only on line direction
  - Parallel lines  $\mathbf{P}_0 + t\mathbf{D}$ ,  $\mathbf{P}_1 + t\mathbf{D}$  intersect at  $\mathbf{P}_\infty$

# Computing vanishing points (from lines)



- Intersect  $p_1q_1$  with  $p_2q_2$

$$v = (p_1 \times q_1) \times (p_2 \times q_2)$$

# Computing vanishing points by projection

- Let  $P = K[ I \mid 0 ]$  be a camera matrix. The vanishing point of lines with direction  $d$  in 3-space is the intersection  $v$  of the image plane with a ray through camera center with direction  $d$ . This vanishing point  $v$  is given by  $v = Kd$ .

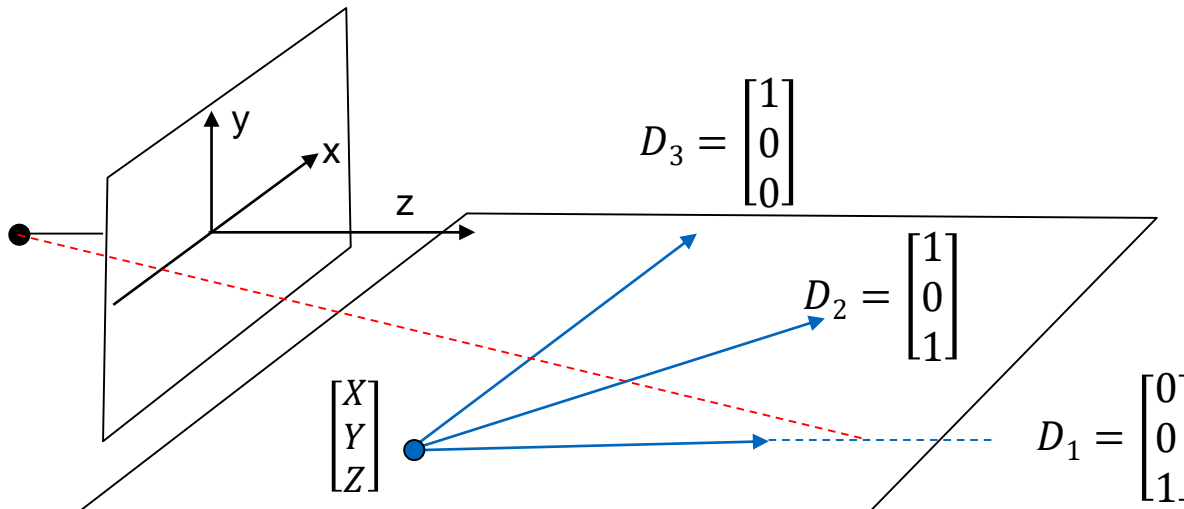
$$v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} D \\ 0 \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

- Example: Computing vanishing points of lines on a XZ plane
- (1) parallel to the Z axis, (2) at 45 deg to the Z axis
- (3) parallel to the X axis

$$D_1: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_2: \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$D_3: \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$



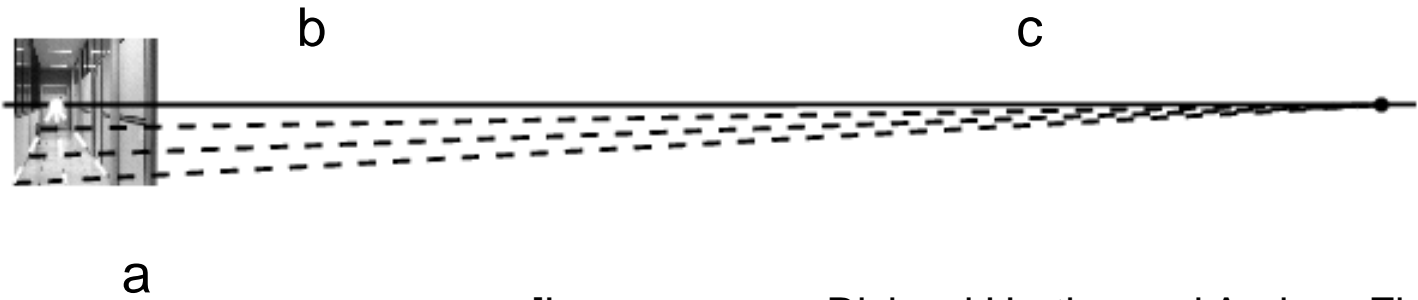
# Vanishing points and projection matrix

$$P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} = [p_1 \quad p_2 \quad p_3 \quad p_4]$$

- $p_1 = P[1 \ 0 \ 0 \ 0]^T = v_x$  (X vanishing point)
- similarly,  $p_2 = v_y$ ,  $p_3 = v_z$
- $p_4 = P[0 \ 0 \ 0 \ 1]^T =$  projection of world origin O

$$P = [\mathbf{v}_X \quad \mathbf{v}_Y \quad \mathbf{v}_Z \quad \mathbf{o}]$$

# Real vanishing points



[Image source: Richard Hartley and Andrew Zisserman]

Vanishing point of a line parallel to a plane lies on the vanishing line of the plane

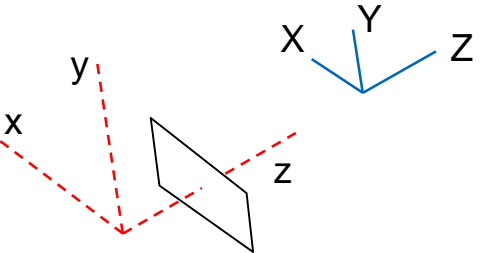
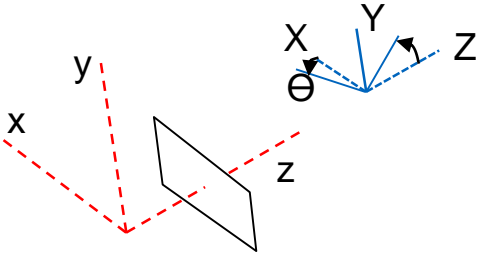
# Image rectification using vanishing points



before rectification



after rectification





# Camera rotation from vanishing points

- Two images of a scene obtained by the same camera from different position and orientation.
- The images of the points at infinity, **the vanishing points**, are not affected by the camera translation, but are affected only by the camera rotation
- Vanishing points  $v_i$  and  $v_i'$  have the following directions  $d_i$ ,  $d_i'$

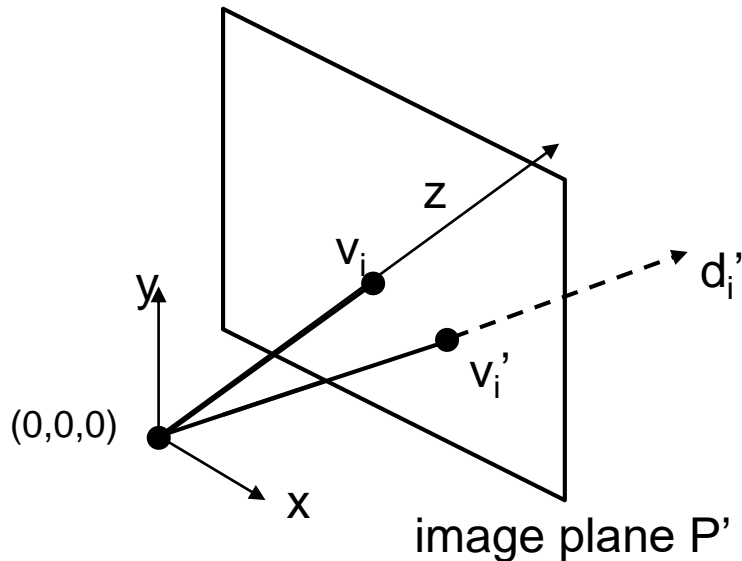
$$d_i = K^{-1}v_i / \|K^{-1}v_i\|$$
$$d_i' = K^{-1}v_i' / \|K^{-1}v_i'\|$$

- The directions are related by a rotation matrix:

$$d_i' = R d_i$$

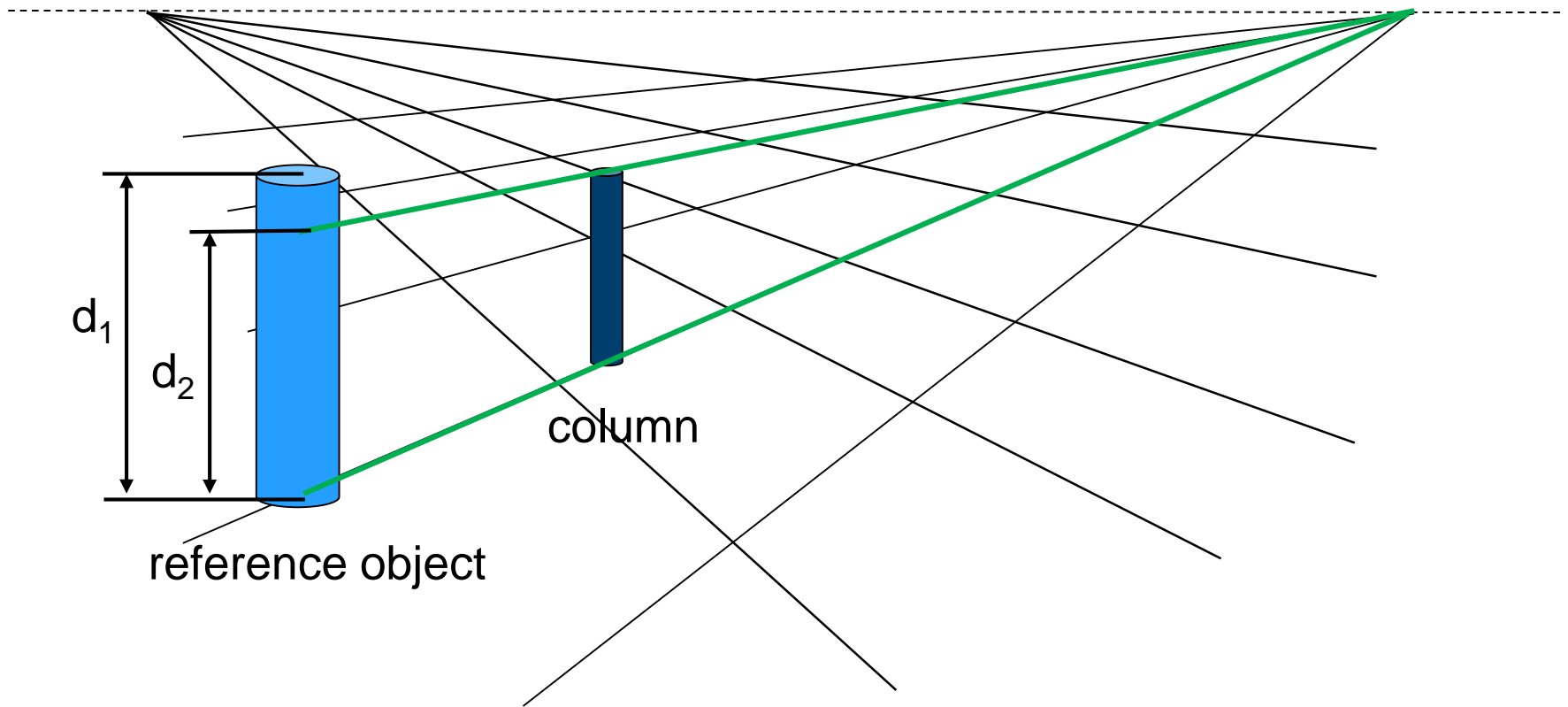
- If the directions are known, the rotation matrix can be computed from two directions

# Camera rotation from vanishing points



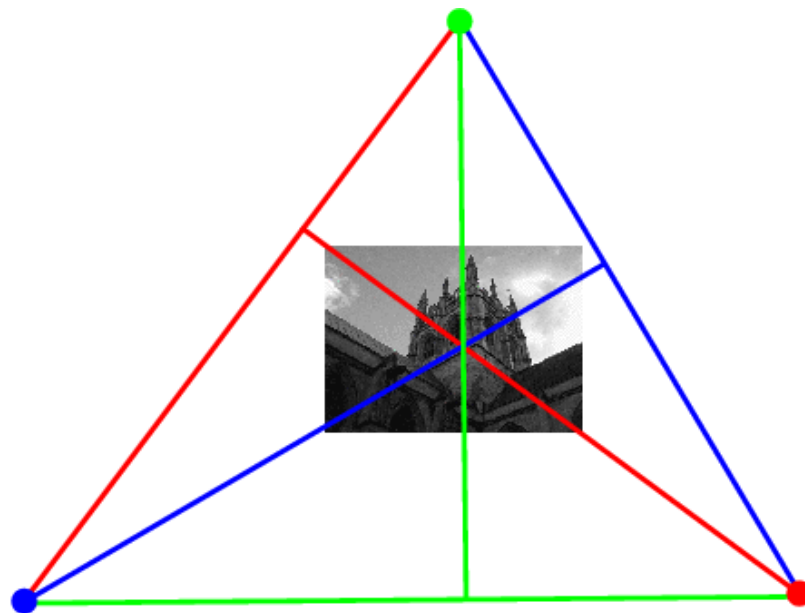
- First camera is aligned with world coordinate system.  $d_i = [0 \ 0 \ 1]$
- Second camera deviates,  $d_i' = [d_x, d_y, d_z]$  and can be computed from the image coordinates of  $v_i'$
- The rotation that aligns the second image with the first image can be computed from  $d_i$  and  $d_i'$  and 1 more direction.

# Measuring heights using vanishing points



$$\text{Height column} = \text{height of reference object} * d_2 / d_1$$

# Camera calibration from orthogonal vanishing points



[Image source: Richard Hartley and Andrew Zisserman]

## Recap - Learning goals

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