Mathematical Principles in Visual Computing: Multi-Camera-Systems

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Outline

- Multi-Camera-Systems
- The generalized camera
- Plücker coordinates
- Generalized Epipolar Constraint
- Intra- and Inter-camera correspondences
- Generalized PnP
Learning goals

- Understand Plücker-line coordinates
- Understand the use of Plücker-line coordinates to describe multi-camera systems
- Understand the properties of multi-camera systems
- Understand the concept of a generalized camera
- Understand the generalized epipolar concept
- Understand the P3P extension to the generalized camera P3P
PtGrey Ladybug

- 6 cameras
- 360 field of view
- Panorama images
Automotive around view system
The generalized camera

pinhole camera

generalized camera
The generalized camera

generalized camera

mixed configuration
Multi camera systems

general non-axial camera
(locally-central)

axial camera

axial camera
Plücker coordinates

- Plücker line matrix (4x4)

- Plücker coordinates (6-vector)

\[ L = AB^T - BA^T \]

\[
\begin{bmatrix}
A_4B_1 - A_1B_4 \\
A_4B_2 - A_2B_4 \\
A_4B_3 - A_3B_4 \\
A_3B_2 - A_2B_3 \\
A_1B_3 - A_3B_1 \\
A_2B_1 - A_1B_2 \\
\end{bmatrix}
\]
Plücker coordinates

- 6-vector \( L \) consists of 2 parts
  \[
  a^T = (L_1 \ L_2 \ L_3) \quad b^T = (L_4 \ L_5 \ L_6)
  \]
  \[a^T b = 0\]

- Rigid transformation
  - Point:
    \[
    X' = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} X
    \]
  - Plücker coordinate
    \[
    \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} R \ & 0 \\ -[t]_x R & R \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
    \]
Plücker coordinates

- Line intersection of $L_1$ and $L_2$

\[
L_2^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} L_1 = a_2^T b_1 + b_2^T a_1 = 0
\]
Plücker coordinates: Geometric interpretation

$\mathbf{L} = (\mathbf{q}, \mathbf{q}')$

$\mathbf{q}' = \mathbf{P} \times \mathbf{q}$

$\mathbf{q}^T \mathbf{q}' = 0$
All points on the line $L$ are expressed by the two vectors $q \times q'$ and $\alpha q$, where $\alpha$ is a scalar.
Plücker coordinates: Pinhole camera

- Camera center is point C
- Plücker coordinate is just standard homogeneous coordinate in this case

\[ L = (q, q') \]
\[ q' = C \times q \]
\[ P = C = (0,0,0) \]
\[ L = (q, (0 0 0)') \]
Plücker coordinates: Generalized camera

- Camera centers are not at the origin (0,0,0)

  \[ L_1 = (R_1 x_1, C_1 \times R_1 x_1) \]

- \( R_1, C_1 \) … camera center and orientation in a common coordinate system
- \( x_1 \) … homogeneous (normalized) image coordinate
Relative rotations are the same for all cameras
Relative translations are all different due to the lever arm (direction as well as length).
Motion is defined as full 6DOF as compared to single camera case with 5DOF (length of translation is defined!)

\[ R_{11r} = R_{22r} \]
\[ t_{11r} \neq t_{22r} \]
Relative motion, scale estimation

\[ C_1'' \quad \lambda c_{11}' \quad C_1' \quad C_1 \quad C_2' \quad C_2'' \quad X \]

Translation
Rotation

\[ v' \quad v \quad R_1' C_2 \]

\[ v' \quad v \quad R_1' C_2 \]
Generalized epipolar constraint (GEC)

- Line correspondence $L \leftrightarrow L'$
  \[
  \begin{align*}
  L &= (q_1^T, q_1'^T)^T \\
  L' &= (q_2^T, q_2'^T)^T
  \end{align*}
  \]

- Relative motion transforms Plücker coordinates (light ray)
  \[
  L' = \begin{pmatrix}
  Rq_1 \\
  (Rq'_1 + t \times (Rq_1))
  \end{pmatrix}
  \]

- Generalized epipolar constraint (GEC) defines intersection of two light rays $([R,T]L,L')$
  \[
  q_2^T q'_1 + q_2'^T q_1 = 0 \\
  q_1 \rightarrow Rq_1 \\
  q'_1 \rightarrow Rq'_1 + t \times (Rq_1) \\
  q_2^T \left( Rq'_1 + t \times (Rq_1) \right) + q_2'^T (Rq_1) = 0 \\
  q_2^T Rq'_1 + q_2'^T [t]_x Rq_1 + q_2'^T Rq_1 = 0
  \]
Generalized epipolar constraint (GEC)

- Matrix form

\[ q_2^T R q'_1 + q_2^T [t]_x R q_1 + q'_2^T R q_1 = 0 \]

\[ L_2^T G L_1 = \begin{pmatrix} q_2 \\ q'_2 \end{pmatrix}^T \begin{bmatrix} [t]_x & R \\ R & 0 \end{bmatrix} \begin{pmatrix} q_1 \\ q'_1 \end{pmatrix} = 0 \]

\[ G = \begin{bmatrix} [t]_x & R \\ R & 0 \end{bmatrix} \]

- G … generalized essential matrix (6x6)
- G contains essential matrix! (E = [t]_x R)
Linear algorithm for G

\[ L^T_2 G L_1 = \begin{pmatrix} q_2 \\ q'_2 \end{pmatrix}^T \begin{bmatrix} [t]_x R & R \\ R & 0 \end{bmatrix} \begin{pmatrix} q_1 \\ q'_1 \end{pmatrix} = 0 \]

\[ L'^T G L = \begin{pmatrix} x' \\ v' \times x' \end{pmatrix}^T \begin{bmatrix} E & R \\ R & 0 \end{bmatrix} \begin{pmatrix} x \\ v \times x \end{pmatrix} = 0 \]

- E is 3x3, R is 3x3, i.e. in total 18 unknowns
- Each line correspondence gives 1 equation like

\[ x'^T E x + (v' \times x')^T R x + x'^T R (v \times x) = 0 \]

- Linear solution needs 17 point correspondences to compute G
Linear algorithm for G

\[
A_i^T y = \begin{bmatrix}
x'_1x_1 \\
x'_2x_2 \\
x'_3x_3 \\
x'_1x_3 \\
x'_2x_1 \\
x'_2x_2 \\
x'_2x_3 \\
x'_3x_1 \\
x'_3x_2 \\
x'_3x_3 \\
(\nu'_2x'_3 - \nu'_3x'_2)x_1 + x'_1(\nu_2x_3 - \nu_3x_2) \\
(\nu'_2x'_3 - \nu'_3x'_2)x_2 + x'_1(\nu_3x_1 - \nu_1x_3) \\
(\nu'_2x'_3 - \nu'_3x'_2)x_3 + x'_1(\nu_1x_2 - \nu_2x_1) \\
(\nu'_3x'_1 - \nu'_1x'_3)x_1 + x'_2(\nu_2x_3 - \nu_3x_2) \\
(\nu'_3x'_1 - \nu'_1x'_3)x_2 + x'_2(\nu_3x_1 - \nu_1x_3) \\
(\nu'_3x'_1 - \nu'_1x'_3)x_3 + x'_2(\nu_1x_2 - \nu_2x_1) \\
(\nu'_1x'_2 - \nu'_2x'_1)x_1 + x'_3(\nu_2x_3 - \nu_3x_2) \\
(\nu'_1x'_2 - \nu'_2x'_1)x_2 + x'_3(\nu_3x_1 - \nu_1x_3) \\
(\nu'_1x'_2 - \nu'_2x'_1)x_3 + x'_3(\nu_1x_2 - \nu_2x_1) \\
\end{bmatrix}^T
\begin{bmatrix}
E_{11} \\
E_{12} \\
E_{13} \\
E_{21} \\
E_{22} \\
E_{23} \\
E_{31} \\
E_{32} \\
E_{33} \\
R_{11} \\
R_{12} \\
R_{13} \\
R_{21} \\
R_{22} \\
R_{23} \\
R_{31} \\
R_{32} \\
R_{33}
\end{bmatrix}
\]
Two types of correspondences

- Intra-camera correspondences: Correspondences from the same camera
- Inter-camera correspondences: Correspondences from different cameras
Two types of correspondences

- Problem when rotation is identity and there are only intra-camera correspondences. \( R = I \), \( t'_{C} = t_{C} \)

\[
x'^{T}Ex + (t'_{C} \times x')^{T}Rx + x'^{T}R(t_{C} \times x) = 0
\]

\[
x'^{T}Ex = 0
\]

- Scale not observable anymore
Generalized PnP

- Very similar to perspective PnP
- Use Plücker coordinates
- But: How many points needed? How many solutions?
- We will show that 3 points are needed and this yields 8 solutions.
Generalized PnP

Three 2D-3D correspondences

Express rays as Plücker lines

Find unknown depths

Absolute orientation

$R, t$
Generalized PnP

Plücker coordinates

\[ l_i = (u_i, u'_i) \]

\[ X_i^V = u_i \times u'_i + \lambda_i u_i \]

\[ d_{ij} = \| X_i - X_j \|^2 = \| X_i^V - X_j^V \|^2 \quad \text{depths} \]

\[ d_{ij} = \| X_i - X_j \|^2 = \| (u_i \times u'_i + \lambda_i u_i) - (u_j \times u'_j + \lambda_j u_j) \|^2 \]
Generalized PnP

- 3 distances lead to 3 equations with unknown depths (lambda)

\[ k_{11} \lambda_1^2 + (k_{12} \lambda_2 + k_{13}) \lambda_1 + (k_{14} \lambda_2^2 + k_{15} \lambda_2 + k_{16}) = 0 \]
\[ k_{21} \lambda_1^2 + (k_{22} \lambda_3 + k_{23}) \lambda_1 + (k_{24} \lambda_3^2 + k_{25} \lambda_3 + k_{26}) = 0 \]
\[ k_{31} \lambda_2^2 + (k_{32} \lambda_3 + k_{33}) \lambda_2 + (k_{34} \lambda_3^2 + k_{35} \lambda_3 + k_{36}) = 0 \]

- Eliminating variables using resultants (determinant of Sylvester matrix) leads to an 8 degree polynomial

\[ A \lambda_3^8 + B \lambda_3^7 + C \lambda_3^6 + D \lambda_3^5 + E \lambda_3^4 + F \lambda_3^3 + G \lambda_3^2 + H \lambda_3 + I = 0 \]

- No closed form solution possible (Root solving with companion matrix or Sturm bracket method)
Recap - Learning goals

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