Mathematical Principles in Visual Computing: Sylvester Resultant
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Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a single unknown $x$:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$$

Sylvester matrix of $f$ and $g$:

$$\text{Syl}(f, g) = \begin{bmatrix}
    a_m & \ldots & a_0 \\
    \ldots & \ddots & \ldots \\
    b_n & \ldots & b_0 \\
    \ldots & \ldots & \ldots \\
    b_n & \ldots & b_0
\end{bmatrix} \begin{cases}
    n \text{ rows} \\
    m \text{ rows}
\end{cases}$$

$f$ and $g$ have a common root if and only if $\det(\text{Syl}(f, g)) = 0$. $\det(\text{Syl}(f, g))$ is called the Sylvester Resultant of $f$ and $g$. 
Using the Sylvester Resultant

to solve a polynomial system of two equations and TWO unknowns:

\[
p_1(x, y) = 6x^2 + 3xy - xy^2 + y + 1 \quad \left\{ \begin{array}{l} p_1(x, y) = 0 \\ p_2(x, y) = 0 \end{array} \right. \\
p_2(x, y) = x^2y + 5x + 4y - 1
\]

Let’s consider \( y \) as a constant, and write these two polynomials as polynomials in \( x \):

\[
p_{1,y}(x) = 6x^2 + (3y - y^2)x + (y + 1) \\
p_{2,y}(x) = yx^2 + 5x + (4y - 1)
\]

\[
\text{Syl}(p_{1,y}, p_{2,y}) = \begin{bmatrix} 6 & (3y - y^2) & (y + 1) & 0 \\ 0 & 6 & (3y - y^2) & (y + 1) \\ y & 5 & (4y - 1) & 0 \\ 0 & y & 5 & (4y - 1) \end{bmatrix}
\]
Using the Sylvester Resultant

\[ p_1(x, y) = 6x^2 + 3xy - xy^2 + y + 1 \]
\[ p_2(x, y) = x^2y + 5x + 4y - 1 \]

\[ \left\{ \begin{array}{l} p_1(x, y) = 0 \\ p_2(x, y) = 0 \end{array} \right. \]

\[ p_{1,y}(x) = 6x^2 + (3y - y^2)x - xy^2 + (y + 1) \]
\[ p_{2,y}(x) = yx^2 + 5x + (4y - 1) \]

\( p_{1,y} \) and \( p_{2,y} \) should have roots in common, and
\[ \det(Syl(p_{1,y}, p_{2,y})) = 0: \]

\[
\begin{vmatrix}
6 & (3y - y^2) & (y + 1) & 0 \\
0 & 6 & (3y - y^2) & (y + 1) \\
y & 5 & (4y - 1) & 0 \\
0 & y & 5 & (4y - 1)
\end{vmatrix} = 0
\]

\[ \det(Syl(p_{1,y}, p_{2,y})) \text{ is a polynomial in } y \text{ (only)} \]

\[ \det(Syl(p_{1,y}, p_{2,y})) = 4y^6 - 20y^5 + 153y^4 - 310y^3 + 781y^2 - 276y + 36 \]

\( \rightarrow \) First solve for \( y \) (we will see later how it can be done).