Mathematical Principles in Visual Computing: Root finding

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Outline

- Root finding
  - Companion Matrix
  - Sturm sequences
Root finding

- Consider the equation \( f(x) = 0 \)
- Roots of equation \( f(x) \) are the values of \( x \) which satisfy the above expression. Also referred to as the zeros of an equation

- Standard methods:
  - Bisection (look for sign changes in interval)
  - Newton-Raphson
Companion matrix

- Simple method, construct matrix of which the eigenvalues are the roots of the polynomial
- Eigenvalues of a matrix are the roots of the characteristic polynomial $\rightarrow$ form a matrix for which the characteristic polynomial is the one to solve for.

$$p(z) = \det(zI - A)$$

$$C = \begin{bmatrix}
0 & 0 & \cdots & 0 & -c_0 \\
1 & 0 & \cdots & 0 & -c_1 \\
0 & 1 & \cdots & 0 & -c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -c_{n-1}
\end{bmatrix}$$

$$p(z) = c_0 + c_1z + \cdots + c_{n-1}z^{n-1} + z^n$$

- $C$ … nxn matrix where n is the degree of the polynomial
- Matlab: $e = \text{eig}(C)$ … are the roots
- Finds complex roots, can be slow
Root finding with Sturm sequences

- Sturm's sequence of a univariate polynomial $p$ is a sequence of polynomials associated with $p$ and its derivative.
- Sturm's theorem counts the number of distinct real roots and locates them in intervals.
- By subdividing the intervals containing some roots, it can isolate the roots into arbitrary small intervals, each containing exactly one root. This yields an arbitrary-precision numeric root finding algorithm for univariate polynomials.

Advantages:
- Typically faster than companion matrix
- Finds only real roots (-> again faster)
A Sturm chain or Sturm sequence is a finite sequence of polynomials $p_0, p_1, \ldots, p_m$ of decreasing degree.

Sturm sequence construction:

- $p_0(z) = p(z)$ ... original
- $p_1(z) = p'(z)$ ... derivative
- $p_2(z) = \text{remainder}(p_0(z), p_1(z))$ ... remainder of polynomial division
- $p_3(z) = \text{remainder}((p_1(z), p_2(z))$
- $\ldots$
- $p_n(z) = \text{constant}$
Root finding with Sturm sequences

- $\sigma(z)$ denotes the number of sign changes (ignoring zeroes) in the sequence.
- Sturm's theorem then states that for two real numbers $a < b$ (bracket, interval), the number of distinct roots of $p$ in the half-open interval $(a, b]$ is $\sigma(a) - \sigma(b)$.

To find the number of roots between $a$ and $b$, first evaluate $p_0, p_1, p_2, \ldots, p_n$, at $a$ and note the sequence of signs of the results, e.g. $+ - + + -$. The same procedure for $b$ gives another sign sequence, e.g. $+ + + - - .$, which contains just one sign change. Hence the number of roots of the original polynomial between $a$ and $b$ in the above example is $3 - 1 = 2$.

**Algorithm:**
- Test intervals
  - If roots are in interval split it and test again
  - Repeat until interval is small enough
Root finding with Sturm sequences

\[ g_0(z) = f(z) = (z - 1)(z - 2)(z - 3) \]
\[ g_1(z) = f'(z) = z^2 - 4z + 11/3 \]
\[ g_2(z) = -\text{rem}(g_0(z), g_1(z)) = z - 2 \]
\[ g_3(z) = -\text{rem}(g_1(z), g_2(z)) = 1 \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
 & g_0(z) & g_1(z) & g_2(z) & g_3(z) & s(z) \\
\hline
a=0.8 & - & + & - & + & 3 \\
b=2.8 & - & + & + & + & 1 \\
\hline
\end{array}
\]

\[ s(0.8) - s(2.8) = 3 - 1 = 2 \]