
Mathematical Principles in Visual Computing: Root finding

Prof. Friedrich Fraundorfer

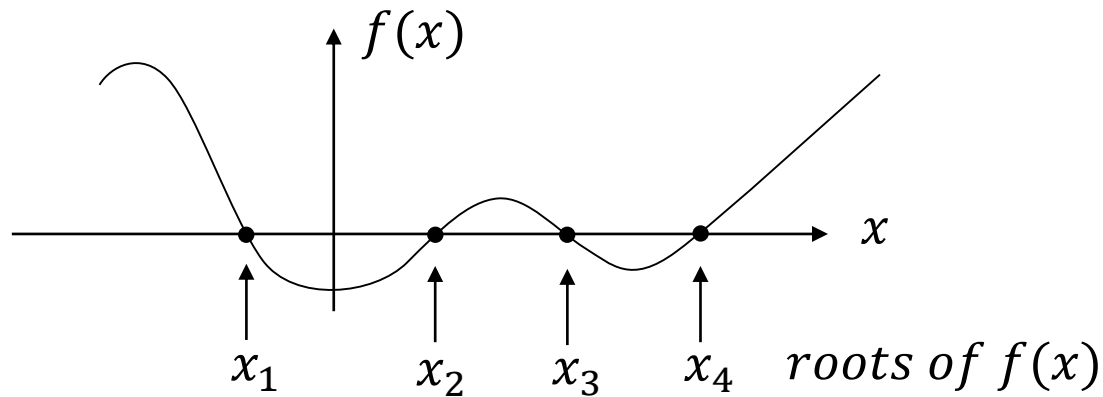
SS 2022

Outline

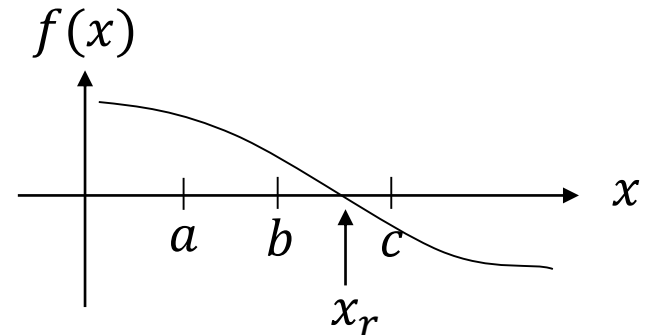
- Root finding
 - Companion Matrix
 - Sturm sequences

Root finding

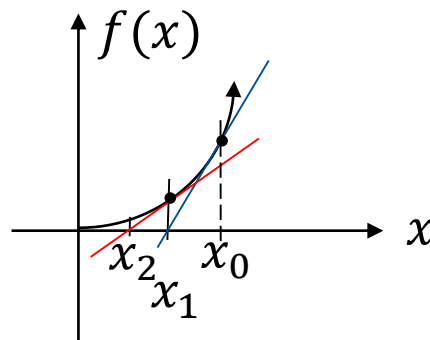
- Consider the equation $f(x) = 0$
- Roots of equation $f(x)$ are the values of x which satisfy the above expression. Also referred to as the zeros of an equation



- Standard methods:
 - Bisection (look for sign changes in interval)



- Newton-Raphson



Companion matrix

- Simple method, construct matrix of which the eigenvalues are the roots of the polynomial
- Eigenvalues of a matrix are the roots of the characteristic polynomial → form a matrix for which the characteristic polynomial is the one to solve for.

$$p(z) = \det(zI - A)$$

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}$$

$$p(z) = c_0 + c_1z + \dots + c_{n-1}z^{n-1} + z^n$$

- C ... nxn matrix where n is the degree of the polynomial
- Matlab: `e = eig(C)` ... are the roots
- Finds complex roots, can be slow

Root finding with Sturm sequences

- Sturm's sequence of a univariate polynomial p is a sequence of polynomials associated with p and its derivative
- Sturm's theorem counts the number of distinct real roots and locates them in intervals.
- By subdividing the intervals containing some roots, it can isolate the roots into arbitrary small intervals, each containing exactly one root. This yields an arbitrary-precision numeric root finding algorithm for univariate polynomials.
- Advantages:
 - Typically faster than companion matrix
 - Finds only real roots (-> again faster)

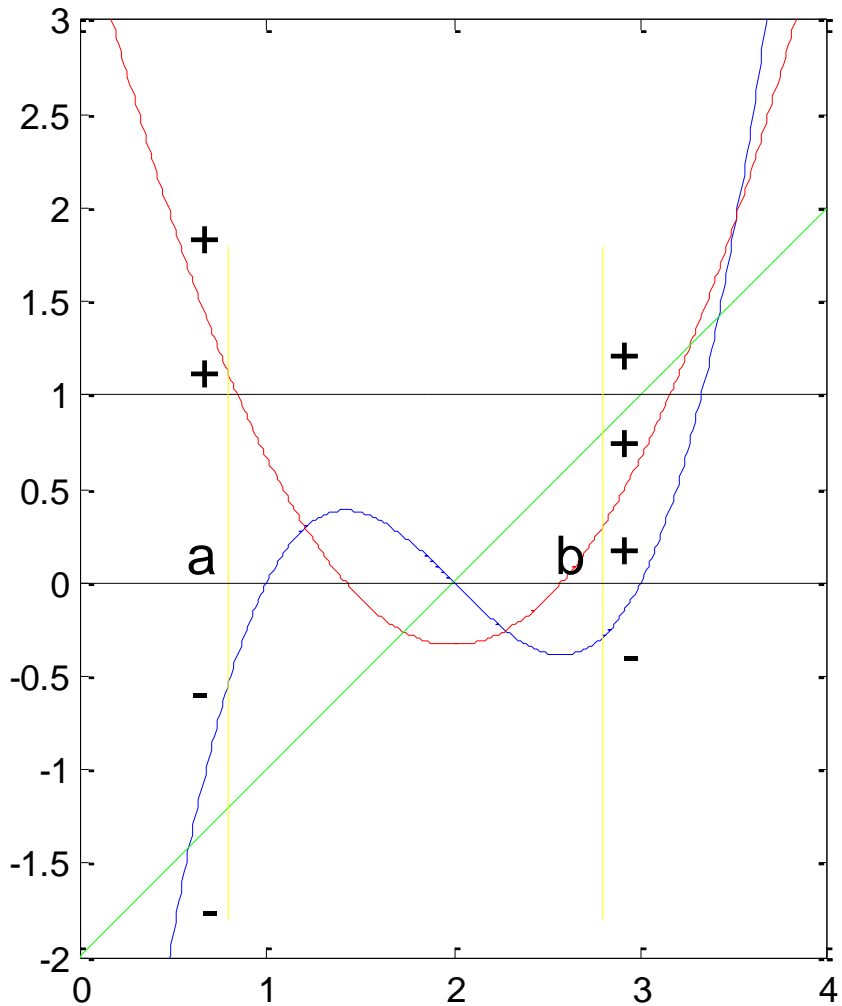
Root finding with Sturm sequences

- A Sturm chain or Sturm sequence is a finite sequence of polynomials p_0, p_1, \dots, p_m of decreasing degree
- Sturm sequence construction:
 - $p_0(z) = p(z)$... original
 - $p_1(z) = p'(z)$... derivative
 - $p_2(z) = -\text{remainder}(p_0(z), p_1(z))$ remainder of polynomial division
 - $p_3(z) = -\text{remainder}(p_1(z), p_2(z))$
 -
 - $p_n(z) = \text{constant}$

Root finding with Sturm sequences

- $\sigma(z)$ denotes the number of sign changes (ignoring zeroes) in the sequence
- Sturm's theorem then states that for two real numbers $a < b$ (bracket, interval), the number of distinct roots of p in the half-open interval $(a, b]$ is $\sigma(a) - \sigma(b)$.
- To find the number of roots between a and b , first evaluate $p_0, p_1, p_2, \dots, p_n$, at a and note the sequence of signs of the results, e.g. $+ - + + -$. The same procedure for b gives another sign sequence, e.g. $+ + + - -$, which contains just one sign change. Hence the number of roots of the original polynomial between a and b in the above example is $3 - 1 = 2$.
- Algorithm:
 - Test intervals
 - If roots are in interval split it and test again
 - Repeat until interval is small enough

Root finding with Sturm sequences



$$g_0(z) = f(z) = (z - 1)(z - 2)(z - 3)$$

$$g_1(z) = f'(z) = z^2 - 4z + 11/3$$

$$g_2(z) = -\text{rem}(g_0(z), g_1(z)) = z - 2$$

$$g_3(z) = -\text{rem}(g_1(z), g_2(z)) = 1$$

	$g_0(z)$	$g_1(z)$	$g_2(z)$	$g_3(z)$	$s(z)$
$a=0.8$	-	+	-	+	3
$b=2.8$	-	+	+	+	1

$$s(0.8) - s(2.8) = 3 - 1 = 2$$