Mathematical Principles in Visual Computing
Projective Geometry – Geometric Relations

Prof. Friedrich Fraundorfer

SS 2022
Outline

- Epipolar constraint derivation
- Stereo normal case
- Triangulation
- Camera pose estimation
Learning goals

- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation
Epipolar constraint

- The epipolar constraint is a mathematical relationship between the point correspondences of two images.
Epipolar constraint – derivation by coplanarity condition

- Vector $p$ and $t$ define a plane
- Vector $p'$ and $t$ define also a plane
- Both planes must have the same normal
- What we seek is a relation between $p$ and $p'$
Epipolar constraint – derivation by coplanarity condition

\[ \mathbf{P} = [I|0] \]

\[ \mathbf{P}' = [R|t] \]

X world point
Epipolar constraint – derivation by coplanarity condition
Epipolar constraint – derivation by coplanarity condition

$X$ world point

$[R|t]$

$(t \times p')$
Epipolar constraint – derivation by coplanarity condition

\[
p'' = Rp + t = (t \times p')
\]

X world point

[C]

[t]

[p]

[p']
Epipolar constraint – derivation by coplanarity condition

\[ t \times p' = t \times p'' \]
\[ t \times p' = t \times (Rp + t) \]
\[ \text{with } p'' = Rp + t \]
\[ t \times p' = t \times Rp + t \times t \]
\[ p'^T (t \times p') = 0 \]
\[ p'^T (t \times Rp) = 0 \]
\[ p'^T [t] x Rp = 0 \]
\[ E \]
\[ p'^T Ep = 0 \]

E is called the Essential matrix
Fundamental matrix

- \( p, p' \) from the Essential matrix derivation are in normalized coordinates
- \( x, x' \) are image coordinates, \( x = Kp \), \( x' = Kp' \)
- By replacing \( p, p' \) with \( x, x' \) one gets the Fundamental matrix

\[
\begin{align*}
p & = K^{-1}x \\
p' & = K^{-1}x'
\end{align*}
\]

\[
\begin{align*}
p'^T E p & = 0 \\
x'^T K^{-T} E K^{-1} x & = 0 \\
x'^T F x & = 0 \\
F & = K^{-T} E K^{-1}
\end{align*}
\]
Epipolar lines

- The corresponding line $l'$ to image coordinate $x$
- $l'$ is the line connecting the epipole $e'$ and the image coordinate $x'$
- Hypothesis: $l' = Fx$
- Point $x'$ must lie on $l'$, thus $x'^T l' = 0$
- Now $x'^T Fx = 0$
Stereo case

\[ R = I_{3x3} \]

\[ T = \begin{bmatrix} T_x & 0 & 0 \end{bmatrix}^T \]
Stereo case

\[ R = I_{3x3} \quad T = [T_x \ 0 \ 0]^T \]

\[ E = [T]_x R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & T_x & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ [x' \ y' \ 1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix} [x \ y \ 1] = 0 \]

\[ [x' \ y' \ 1] \begin{bmatrix} 0 \\ -T_x \\ T_x y \end{bmatrix} = 0 \]

\[ -y'T_x + T_xy = 0 \]
Triangulation

- Compute coordinates of world point X given the measurements x, x' and the camera projection matrices P and P'
Triangulation

- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)
  - $x \times (PX) = 0$ and $x' \times (P'X) = 0$
- Can be rewritten into equation system $AX = 0$ to solve for $X$
Triangulation

\[ x \times (PX) = 0 \text{ and } x' \times (P'X) = 0 \]
\[ x(P_3^TX) - (P_1^TX) = 0 \]
\[ y(P_3^TX) - (P_2^TX) = 0 \]
\[ x(P_2^TX) - y(P_1^TX) = 0 \]

\[
\begin{bmatrix}
  xP_3^T - P_1^T \\
  yP_3^T - P_2^T \\
  x'P'_3^T - P'_1^T \\
  y'P'_3^T - P'_2^T
\end{bmatrix}
\begin{bmatrix}
  X
\end{bmatrix}
= 0
\]

\[ P = \begin{bmatrix}
  P_1^T \\
  P_2^T \\
  P_3^T
\end{bmatrix} \]
Camera pose estimation

- perspective-n-point problem
- Goal is to estimate camera matrix $P$ such that $m_1 = PM_1$
- $m_1, M_1, m_2, M_2, m_3, M_3$ are known
- Algebraic linear solution requires 6 3D-2D point correspondences, minimal nonlinear solution requires only 3
Camera pose estimation

- Derivation similar to Triangulation, but now entries of $P$ are the unknowns instead of $X$
- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)
Camera pose estimation

- Derivation similar to Triangulation, but now entries of $P$ are the unknowns instead of $X$
- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)

$$x \times (PX) = 0 \text{ for all pairs } x \leftrightarrow X$$
$$y(P_3^T X) - w(P_2^T X) = 0$$
$$x(P_3^T X) - w(P_1^T X) = 0$$
$$x(P_2^T X) - y(P_1^T X) = 0$$

$$\begin{bmatrix} 0 & -wX^T & yX^T \\ -wX^T & 0 & xx^T \\ -yX^T & xx^T & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$
Recap - Learning goals

- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation