Mathematical Principles in Visual Computing: Solving Polynomial Systems Prof. Friedrich Fraundorfer SS2022

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Polynomial Systems in Computer Vision

Many Computer Vision problems can be solved by finding the roots of a polynomial system:

- camera pose estimation from point correspondences;
- camera relative motion estimation from point correspondences;
- image distortion calibration;
- point triangulation;
- ...

Solving Polynomial Systems

- no general method;
- several mathematical tools exist. For a given problem, a tool can be more adapted than the others.

Gröbner Bases

- introduced in 1965 by Bruno Buchberger (now at the Johannes Kepler University in Linz) in his Ph.D. thesis (named after his advisor Wolfgang Gröbner) to study sets of polynomials

A Polynomial System

Let consider the following polynomial system:

$$\begin{array}{l} L_1 \\ L_2 \\ L_3 \end{array} \begin{cases} 2x^2 + y^2 - 2z + 3z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \end{cases}$$

Hint: try to remove x from the first equation

Replace L_1 by $L_1 - 2L_2$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \end{array} \begin{cases} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \end{cases}$$

A Real Polynomial System (continued)

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \end{array} \begin{cases} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \end{cases}$$

Hint: try to remove x from the second equation:

Adding $y^2 L_2 - L_3$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \end{array} \begin{cases} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \end{cases}$$

A Real Polynomial System (continued)

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \end{array} \left\{ \begin{array}{rcl} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \end{array} \right.$$

Hint: try to remove y from the first equation

Add $zL_1' - L_4$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \left\{ \begin{array}{rcl} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \\ 5z - 4z^2 + z^3 - 2 & = & 0 \end{array} \right.$$

A Real Polynomial System (continued)

$$
\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array}
\left\{
\begin{array}{rcl}
y^2 - 4z + z^2 + 5 & = & 0 \\
x^2 + z + z^2 & = & 0 \\
x^2y^2 + y^2z^2 - 2 & = & 0 \\
y^2z + 2 & = & 0 \\
5z - 4z^2 + z^3 - 2 & = & 0
\end{array}
\right.
$$

Hint: L_5 is a polynomial in z only

$5z - 4z^2 + z^3 - 2 = (z-1)^2(z-2)$

Each possible value for z gives a new polynomial system in x and y only.

Solving a Univariate Polynomial

- closed form up to degree 4;
- for higher degrees:
  - the companion matrix method: The *companion matrix* of $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ is

$$\mathbf{C} = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{n-1} \end{bmatrix}.$$

    Its eigenvalues are the roots of $p(z)$ (because $p(z)$ is the characteristic polynomial $\det(z\mathbf{I} - \mathbf{C})$ of $\mathbf{C}$).
  - Sturm's bracketing method (slightly less stable but much faster).

Two Gröbner bases

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \left\{ \begin{array}{rcl} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \\ 5z - 4z^2 + z^3 - 2 & = & 0 \end{array} \right.$$

$$\left\{ y^2 - 4z + z^2 + 5, x^2 + z + z^2, x^2y^2 + y^2z^2 - 2, y^2z + 2, 5z - 4z^2 + z^3 - 2 \right\}$$

is a Gröbner basis.

$$\left\{ y^2 - 4z + z^2 + 5, x^2 + z + z^2, 5z - 4z^2 + z^3 - 2 \right\}$$

is also a Gröbner basis.

A Gröbner basis is a set of polynomials $\{g_1, \ldots, g_t\}$, such that the system

$$\left\{ \begin{array}{rcl} g_1(x_1, \ldots, x_n) & = & 0 \\ & \ldots & \\ g_t(x_1, \ldots, x_n) & = & 0 \end{array} \right.$$

has the same solutions as the original one,

but with some specific properties that make the new system easier to solve than the original one, OR AT LEAST USEFUL to solve the original one.

Tools

We can create new equations from:

- linear combinations of existing equations. Gauss-Jordan elimination algorithm to simplify the system.

$$\left\{ \begin{array}{rcl} 2x^2+xy+y^2+1 & = & 0 \\ x^2-xy+2y^2-1 & = & 0 \end{array} \right.$$

  in matrix form:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ 1 \end{bmatrix} = 0\,.$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ 1 \end{bmatrix} = 0\,.$$

- *algebraic* combinations of existing equations.

Tools

We can create new equations from:

- linear combinations of existing equations.
- *algebraic* combinations of existing equations.
- the remainder of polynomial divisions (used by Buchberger's algorithm).

Monomials

Definition. A **monomial** in $x_1, \ldots, x_n$ is a product of the form:

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots \cdot x_n^{\alpha_n},$$

where all the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers, sometimes noted $\mathbf{x}^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Examples: x, x^2, x^2y, x^2yz^3

Polynomials

Definition. A **polynomial** f in $x_1, \ldots, x_n$ with coefficients in a field k is a finite linear combination with coefficients in k of monomials. A polynomial is written in the form

$$f = \sum_{\alpha} a_\alpha \mathbf{x}^\alpha, \quad a_\alpha \in k$$

with

- a_α the **coefficient** of the monomial $\mathbf{x}^\alpha$.
- If $a_\alpha \neq 0$, then we call $a_\alpha \mathbf{x}^\alpha$ a **term** of f.

Notations: $k[x_1, \ldots, x_n]$

Notation. The set of all polynomials in $x_1, \ldots, x_n$ with coefficients in k is denoted $k[x_1, \ldots, x_n]$.

$k[x]$ is the set of polynomials in one variable: $x^2 - x \in k[x]$, $x^3 + 4x \in k[x]$.

$k[x, y]$ is the set of polynomials in two variables: $x^2 - y \in k[x, y]$, $x^3 + 2xy + y^2 \in k[x, y]$.

Definition - Leading Term $\mathrm{LT}(f)$

Definition. Given a nonzero polynomial $f \in k[x]$, let

$$f = a_0 x^m + a_1 x^{m-1} + \ldots + a_m \, ,$$

where $a_i \in k$ and $a_0 \neq 0$.

$a_0 x^m$ is called the *leading term* of f.

We will write $\mathrm{LT}(f) = a_0 x^m$.

Dividing Multivariate Polynomials?

Is there a division for polynomials in several variables?

The answer is yes, but we need to decide which term of a polynomial is the leading term.

For example, what is the leading term of $x^2 + xy + y^2$?

To decide, we will define a *monomial order*.

Monomial Order

A monomial order is any relation on the set of monomials x^α in $k[x_1, \ldots, x_n]$ satisfying:

1. $>$ is a total (linear) ordering relation:
   there is only one possible to order in increasing order under $>$ a set of monomials;
2. $>$ is compatible with multiplication:
   if $x^\alpha > x^\beta$ and x^γ is any monomial, then
   $x^\alpha x^\gamma = x^{\alpha+\gamma} > x^\beta x^\gamma = x^{\beta+\gamma}$;
3. $>$ is a well-ordering:
   every nonempty set of monomials has a smallest element under $>$.

Monomial Order on $k[x]$

The only monomial order on $k[x]$ is the degree order, given by:

$$\ldots > x^{n+1} > x^n > \ldots > x^2 > x > 1.$$

Monomial Orders on $k[x_1, \ldots, x_n]$

For polynomials in several variables, there are many choices of monomial orders.

Let's first define an order on the variables: $x_1 > x_2 > \ldots > x_n$ (this is not a monomial order), and $x > y > z$.

Monomial Orders on $k[x_1, \ldots, x_n]$ - the Lexicographic Order $>_{lex}$

Definition. The lexicographic order: analogous to the ordering of words in a dictionary.

For example, under this order $>_{lex}$:

$$x^2 >_{lex} xy^2 >_{lex} xy >_{lex} x >_{lex} y$$

Formal definition: $x^\alpha >_{lex} x^\beta$ if in the difference $\alpha - \beta$ (which belongs to $\mathbb{Z}^n$), the leftmost nonzero entry is positive.

$x^2yz^3 >_{lex} x^2z^4$ or $x^2z^4 >_{lex} x^2yz^3$?

$\to x^2yz^3 >_{lex} x^2z^4$ because $(2,1,3) - (2,0,4) = (0, \mathbf{1}, -1)$

Monomial Orders on $k[x_1, \ldots, x_n]$ - the Graded Reverse Lexicographic Order $>_{grevlex}$

Let x^α and x^β be monomials in $k[x_1, \ldots, x_n]$. $x^\alpha >_{grevlex} x^\beta$ if:

- $\sum_i^n \alpha_i > \sum_i^n \beta_i$, or if
- $\sum_i^n \alpha_i = \sum_i^n \beta_i$ and in the difference $\alpha - \beta$, the *rightmost* nonzero entry is *negative*.

Under this order $>_{grevlex}$:

$$xy^2 >_{grevlex} x^2 >_{grevlex} xy >_{grevlex} x >_{grevlex} y$$

$x^2y^2z^2 >_{grevlex} xy^4z$ or $xy^4z >_{grevlex} x^2y^2z^2$?

$\rightarrow xy^4z >_{grevlex} x^2y^2z^2$ because $1+4+1 = 2+2+2$ and $(1,4,1) - (2,2,2) = (-1, 2, -\mathbf{1})$

Monomial Orders

$$x^3y^2z >_{lex} x^2y^6z^8$$

$$x^2y^6z^8 >_{grevlex} x^3y^2z$$

$$x^2y^2z^2 >_{lex} xy^4z$$

$$xy^4z >_{grevlex} x^2y^2z^2$$

Division in $k[x_1, \ldots, x_n]$

Let $F = (f_1, \ldots, f_s)$ be an *ordered* s-tuple of polynomials in $k[x_1, \ldots, x_n]$.

Then every $f \in k[x_1, \ldots, x_n]$ can be written as

$$f = a_1 f_1 + \ldots + a_s f_s + r\,,$$

r is called *a* remainder of f on division by F.

- Notation: $r = \overline{f}^F$;
- there exists an algorithm to compute the a_i's and r.

Division in $k[x_1, \ldots, x_n]$: Matlab example

```
syms x y
F = [x^2,y]
f = x*y^2+x^2*y+y^2+x

[r,q] = polynomialReduce(f,F)

r = x

q = [y, y + x*y]
```

$f = yf_1 + (y + xy)f_2 + r$

Why Several Orders?

Computing Gröbner bases with $>_{grevlex}$ is usually more efficient.

Computing Gröbner bases with $>_{lex}$ yields a polynomial system that can be easily solved.

Important property

If

- we use the monomial order $>_{lex}$ to compute a Gröbner basis and
- the solution set is finite,

then a univariate polynomial (in the last variable) is in the basis.

For example, the Gröbner basis for $\langle x^2 - y^2 + 1,\ xy - 1 \rangle$ is $\langle y^4 - y^2 - 1,\ x - y^3 + y \rangle$.

The system

$$\begin{cases} x^2 - y^2 + 1 & = & 0 \\ xy - 1 & = & 0 \end{cases}$$

has the same solutions as the system:

$$\begin{cases} y^4 - y^2 - 1 & = & 0 \\ x - y^3 + y & = & 0 \end{cases}$$

but the latter is much simpler to solve.

A More Ugly Example

A Gröbner basis for

$$\begin{cases} x^2 - 2xz + 5 & = & 0 \\ xy^2 + yz + 1 & = & 0 \\ 3y^2 - 8xz & = & 0 \end{cases}$$

under $>_{lex}$ is

$$\{-81 + 4320z - 86400z^2 + 766272z^3 - 2513488z^4 - 295680z^5 - 242496z^6 + 61440z^8, -2472389942760 + 1450790919y + 987224793696000z - 1312504296363936z^2 + 575639991700688z^3 + 711670127441280z^4 + 549519027506496z^5 - 103266809856000z^6 - 139421921341440z^7, 6503592729600 + 1450790919x - 257416379643438z + 3400639490020320z^2 - 1485707991955148 0z^3 - 1835782187164800z^4 - 1418473727285760z^5 + 263479449600000z^6 + 359882180198400z^7\}$$

Algorithms to Compute a Gröbner basis: Buchberger Algorithm

- S-polynomial

$$S(f_1, f_2) = \frac{x^b}{LT(f_1)} f_1 - \frac{x^b}{LT(f_2)} f_2$$

x^b is the least common multiple (LCM) of the leading monomials $LM(f_1)$ and $LM(f_2)$. $LT(f_1)$ and $LT(f_2)$ are the leading terms of polynomials f_1, f_2

Algorithms to Compute a Gröbner basis: Buchberger Algorithm

- S-polynomial example $>_{lex}$

$$f_1 = y - x^2$$
$$f_2 = z - x^3$$

$$S(f_1, f_2) = \frac{x^b}{LT(f_1)} f_1 - \frac{x^b}{LT(f_2)} f_2 \quad (1)$$
$$= \frac{x^3}{-x^2}(y - x^2) - \frac{x^3}{-x^3}(z - x^3) \quad (2)$$
$$= -xy + x^3 + z - x^3 \quad (3)$$
$$= -xy + z \quad (4)$$

$$LT(f_1) = -x^2 \;\; LT(f_2) = -x^3$$
$$LM(f_1) = x^2 \;\; LM(f_2) = x^3$$
$$LCM(LM(f_1), LM(f_2)) = x^3$$

Algorithms to Compute a Gröbner basis: Buchberger Algorithm

Input: $F = (f_1, ..., f_s)$
Output: Gröbner basis $G = (g_1, ..., g_t)$
G:=F
REPEAT
    G':=G
    FOREACH $f_1, f_2 \in G'$ with $f_1 \neq f_2$, DO
        $s := \overline{S(f_1, f_2)}^{G'}$
        IF $s \neq 0$, THEN $G := G \cup s$
UNTIL G = G'

Discussion

Unfortunately, computation of Gröbner bases under the lexicographic ordering ($>_{lex}$) is often intractable for real problems.

Using the graded reverse lexicographical ordering ($>_{grevlex}$) usually yields more tractable computations.

Unfortunately, the resulting polynomial system is not necessarily easy to solve.

Fortunately, other properties of Gröbner bases can be used to find the solutions.

$>_{lex}$ versus $>_{grevlex}$: Example

Computing a Gröbner basis for

$$\left\{\begin{array}{rcl} d_1^2 + Ad_1d_2 + d_2^2 - F^2 & = & 0 \\ d_1^2 + Bd_1d_3 + d_3^2 - F^2 & = & 0 \\ d_2^2 + Cd_2d_3 + d_3^2 - G^2 & = & 0 \\ d_2^2 + Dd_2d_4 + d_4^2 - F^2 & = & 0 \\ d_3^2 + Ed_3d_4 + d_4^2 - F^2 & = & 0 \end{array}\right.$$

under $>_{grevlex}$: less than a second (but 130 polynomials in a 96Kb text file).

under $>_{lex}$: more than a week

Properties of $>_{grevlex}$ Gröbner basis

Example:

$$
\begin{aligned}
\{ \quad & 3y^2 - 8xz, \\
& x^2 - 2xz + 5, \\
& 160z^3 - 160xz + 415yz + 12x - 30y - 224z + 15, \\
& 240yz^2 - 9xy + 1600xz + 18yz + 120z^2 - 120x + 240z, \\
& 16xz^2 + 3yz - 40z + 3, \\
& 40xyz - 3xy + 6yz + 40z^2 \}
\end{aligned}
$$

Each term has a leading term which does not appear in any other equation

All other monomials are not divisible by the leading terms and are called basis monomials. The number of basis monomials is equal to the number of solutions.

Basis monomials are : $\{ 1, x, y, z, xy, xz, yz, z^2\}$

Leading terms are: $\{y^2, x^2, z^3, yz^2, xz^2, xyz\}$

Number of solutions: 8

Using Gröbner basis to solve polynomial equation systems

The Action matrix method:

- Calculation of a Gröbner basis using $>_{grevlex}$ ordering
- Calculation of the so called Action matrix
- Calcuation of the eigenvalues of the Action matrix to find solutions of one variable
- Backsubstitution into equations to find solutions to other variables

Action matrix example

The Gröbner basis G under $>_{grevlex}$ for

$$\begin{cases} x^2 - 2xz + 5 & = & 0 \\ xy^2 + yz + 1 & = & 0 \\ 3y^2 - 8xz & = & 0 \end{cases}$$

is $\{ \quad 3y^2 - 8xz,\ x^2 - 2xz + 5,$
$160z^3 - 160xz + 415yz + 12x - 30y - 224z + 15,$
$240yz^2 - 9xy + 1600xz + 18yz + 120z^2 - 120x + 240z,$
$16xz^2 + 3yz - 40z + 3,\ 40xyz - 3xy + 6yz + 40z^2 \quad \}$

with leading terms $\{y^2, x^2, z^3, yz^2, xz^2, xyz\}$. Leading terms do only appear in one equation each.

The monomials in B are the monomials that are not divisible by the leading terms: $1,\ x,\ y,\ z,\ xy,\ xz,\ yz,\ z^2$

Action matrix example

To compute the Action matrix we need to decide on one variable for which we calculate the Action matrix. If we select the variable x we call the Action matrix M_x.

All monomials of the basis B need to be multiplied with the variable x and we need to find an expression for these terms in terms of basis monomials only.

The basis monomials are $\{[1], [x], [y], [z], [xy], [xz], [yz], [z^2]\}$

$$
\begin{aligned}
x.1 &= x = \\
x.x &= x^2 = \\
x.y &= xy = \\
x.z &= xz = \\
x.xy &= x^2y = \\
x.xz &= x^2z = \\
x.yz &= xyz = \\
x.z^2 &= xz^2 =
\end{aligned}
$$

Action matrix example

To find an expression for these terms in terms of basis monomials only we can divide the term by the Gröbner basis. The remainder of this division is an expression in the term of basis monomials then.

$$\overline{x^2}^G = -5 + 2xz$$

Action matrix example

The division can be done in Matlab using the function polynomialReduce

$$\overline{x^2}^G = -5 + 2xz$$

```
syms x y z
G = [3*y^2-8*x*z, x^2-2*x*z+5, 160*z^3-160*x*z+415*y*z+12*x-30*y-224*z+15,
        240*y*z^2-9*x*y+1600*x*z+18*y*z+120*z^2-120*x+240*z,
        16*x*z^2+3*y*z-40*z+3, 40*x*y*z-3*x*y+6*y*z+40*z^2]
[r,q] = polynomialReduce(x^2,G);
r

r =
2*x*z - 5
```

Action matrix example

$$\overline{x}^G = x$$
$$\overline{x^2}^G = -5 + 2xz$$
$$\overline{xy}^G = xy$$
$$\overline{xz}^G = xz$$
$$\overline{x^2y}^G = -5y + \frac{3}{20}xy - \frac{2}{10}yz - z^2$$
$$\overline{x^2z}^G = -\frac{3}{8}yz - \frac{3}{8}$$
$$\overline{xyz}^G = -z^2 - \frac{3}{20}yz + \frac{3}{40}xy$$
$$\overline{xz^2}^G = \frac{5}{2}z - \frac{3}{16}yz - \frac{3}{16}$$

The basis monomials are $\{[1], [x], [y], [z], [xy], [xz], [yz], [z^2]\}$

Action matrix example

$$
\mathbf{M}_x = \begin{array}{c}
\begin{matrix} x & x^2 & xy & xz & x^2y & x^2z & xyz & xz^2 \end{matrix} \\
\begin{bmatrix}
0 & -5 & 0 & 0 & 0 & -\frac{3}{8} & 0 & -\frac{3}{16} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \\
0 & 0 & 1 & 0 & \frac{3}{20} & 0 & \frac{3}{40} & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{10} & -\frac{3}{8} & -\frac{3}{20} & -\frac{3}{16} \\
0 & 0 & 0 & 0 & -2 & 0 & -1 & 0
\end{bmatrix}
\end{array}
\begin{matrix} 1 \\ x \\ y \\ z \\ xy \\ xz \\ yz \\ z^2 \end{matrix}
$$

The real eigenvalues for $\mathbf{M}_x$ are $-1.10137..$ and $0.9660..$, which are the possible values for x.

Action matrix example

The real eigenvalues for $\mathbf{M}_x$ are $-1.10137..$ and $0.9660..$, which are the possible values for x.

We still have to find the corresponding values for y and z.

Possible strategies:

1. do the same with $\mathbf{M}_y$ and $\mathbf{M}_z$, and check for every possible combination (x, y, z) if it is a valid solution.
2. for each possible value for x, plug it in the system and solve the resulting system (which is now only in y and z).

The first option is more stable numerically.

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