Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown $x$:

\[ f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0 \]
\[ g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 \]
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Sylvester matrix of $f$ and $g$:

\[
\text{Syl}(f, g) = \begin{bmatrix}
    a_m & \cdots & \ldots & a_0 \\
    \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    b_n & \cdots & \ldots & b_0
\end{bmatrix}
\]

$\det(\text{Syl}(f, g))$ is called the Sylvester Resultant of $f$ and $g$. 

$f$ and $g$ have a common root if and only if $\det(\text{Syl}(f, g)) = 0$. 

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Using the Sylvester Resultant
to solve a polynomial system of two equations and TWO unknowns:

\[
\begin{pmatrix}
6 & (3y - y^2) & (y + 1) & 0 \\
0 & 6 & (3y - y^2) & (y + 1) \\
y & 5 & (4y - 1) & 0 \\
0 & y & 5 & (4y - 1)
\end{pmatrix}
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to solve a polynomial system of two equations and TWO unknowns:

\[ p_1(x, y) = 6x^2 + 3xy - xy^2 + y + 1 \]
\[ p_2(x, y) = x^2y + 5x + 4y - 1 \]

\[ \begin{cases} 
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Let’s consider \( y \) as a constant,
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p_1(x, y) = 6x^2 + 3xy - xy^2 + y + 1 \quad \left\{ \begin{array}{l} p_1(x, y) = 0 \\ p_2(x, y) = 0 \end{array} \right.
\]

\[
p_2(x, y) = x^2y + 5x + 4y - 1
\]

Let’s consider \( y \) as a constant, and write these two polynomials as polynomials in \( x \):

\[
p_{1,y}(x) = 6x^2 + (3y - y^2)x + (y + 1)
\]

\[
p_{2,y}(x) = yx^2 + 5x + (4y - 1)
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\[p_{1,y} \text{ and } p_{2,y} \text{ should have roots in common, and}
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\[\det(\text{Syl}(p_{1,y}, p_{2,y})) = 0:\]
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\[ \text{det}(\text{Syl}(p_{1, y}, p_{2, y})) \text{ is a polynomial in } y \text{ (only)!} \]
\[ \text{det}(\text{Syl}(p_{1, y}, p_{2, y})) = 4y^6 - 20y^5 + 153y^4 - 310y^3 + 781y^2 - 276y + 36 \]
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→ First solve for \( y \) (we will see later how it can be done).