Mathematical Principles in Visual Computing: Sylvester Resultant Prof. Friedrich Fraundorfer SS2021

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Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown x:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$
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$\det(\mathrm{Syl}(f,g))$ is called the Sylvester Resultant of f and g.

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Let's consider y as a constant, and write these two polynomials as polynomials in x:

$$p_{1,y}(x) = 6x^2 + (3y - y^2)x + (y+1)$$
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→ First solve for y (we will see later how it can be done).