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# Mathematical Principles in Visual Computing: Rigid Transformations

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SS 2021

# Learning goals

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- Understand the problems of dealing with rotations
- Understand how to represent rotations
- Understand the terms  $SO(3)$  etc.
- Understand the use of the tangent space
- Understand Euler angles, Axis-Angle, and quaternions
- Understand how to interpolate, filter and optimize rotations

# Outline

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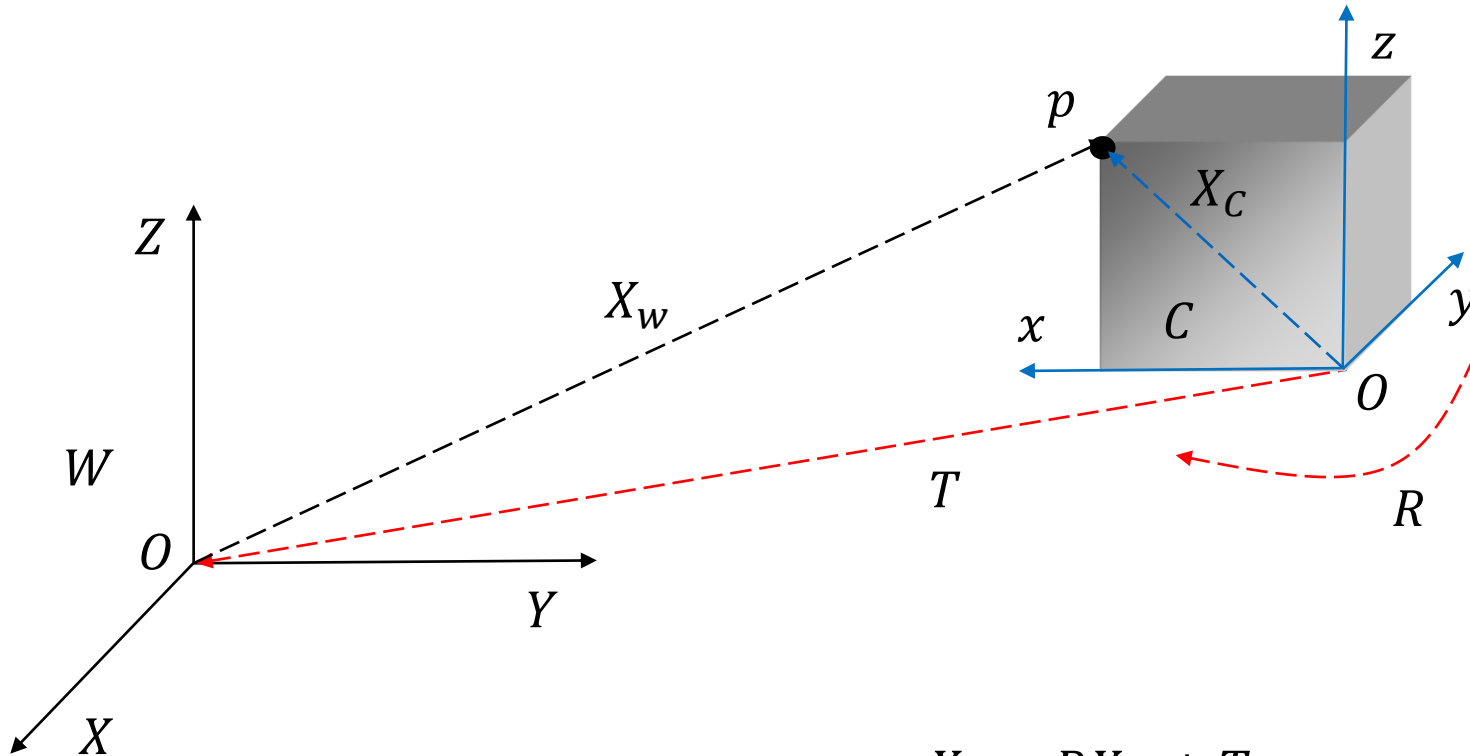
- Rigid transformations
- Problems with rotation matrices
- Properties of rotation matrices
- Matrix groups  $SO(3)$ ,  $SE(3)$
- Manifolds
- Tangent space
- Skew-symmetric matrices
- Exponential map
- Euler angles, angle-axis, quaternions
- Interpolation
- Filtering
- Optimization

# Motivation: 3D Viewer

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# Rigid transformations



- Coordinates are related by:

$$X_c = RX_w + T$$

$$\begin{bmatrix} X_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ 1 \end{bmatrix}$$

$$x \in \mathbb{R}^n$$

$$T \in \mathbb{R}^n$$

$$R \in \mathbb{R}^{n \times n}$$

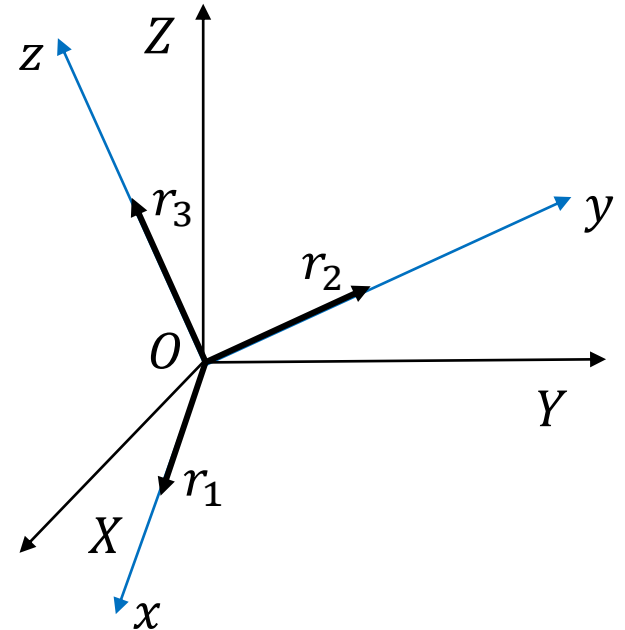
- Rigid transformation belong to the matrix group SE(3)
- What does this mean?

# Properties of rotation matrices

Rotation matrix:

$$R = [r_1, r_2, r_3] \in \mathbb{R}^{3 \times 3}$$

$$R^T R = I, \det(R) = +1$$



Coordinates are related by:  $X_c = R X_w$

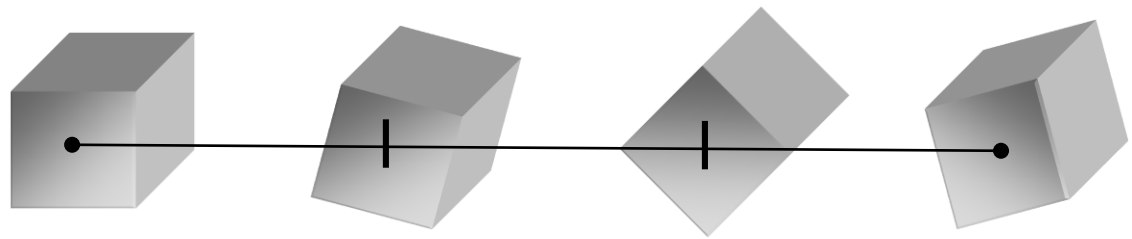
- Rotation matrices belong to the matrix group  $SO(3)$
- What does this mean?

# Problems with rotation matrices

- Optimization of rotations (bundle adjustment)

- Newton's method  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

- Linear interpolation



- Filtering and averaging

- E.g. averaging rotation from IMU or camera pose tracker for AR/VR glasses

# Matrix groups

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- The set of all the  $n \times n$  invertible matrices is a group w.r.t. the matrix multiplication:

$$GL(n) = (\{M \in \mathbb{R}^{n \times n} \mid \det(M) \neq 0\}, \times)$$

General linear group

- Reminder: A group is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element.
- The operation satisfies four conditions called the group axioms, namely closure, associativity, identity and invertibility.
- Closure means for all  $a, b$  in  $G$ , the result of the operation,  $a \cdot b$ , is also in  $G$ .



# Matrix groups

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- The set of all the  $n \times n$  orthogonal matrices is a group w.r.t. the matrix multiplication:

$$O(n) = (\{A \in GL(n) \mid A^{-1} = A^T\}, \times) \quad \text{Orthogonal group}$$

$$A \in O(n) \Rightarrow \det(A) = \pm 1$$

# Matrix groups

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- The set of all the  $n \times n$  orthogonal matrices with determinant equal to +1 is a group w.r.t. the matrix multiplication:

$$SO(n) = (\{A \in O(n) \mid \det(A) = +1\}, \times)$$

Special orthogonal group

- $SO(3)$  ... group of orthogonal  $3 \times 3$  matrices with  $\det = +1$  .... “rotation matrices”
- $R_3 = R_1 * R_2$  ...  $R_3$  is still an  $SO(3)$  element
- $R_3 = R_1 + R_2$  ...  $R_3$  is NOT an  $SO(3)$  element. Not a rotation matrix anymore.

# Matrix groups

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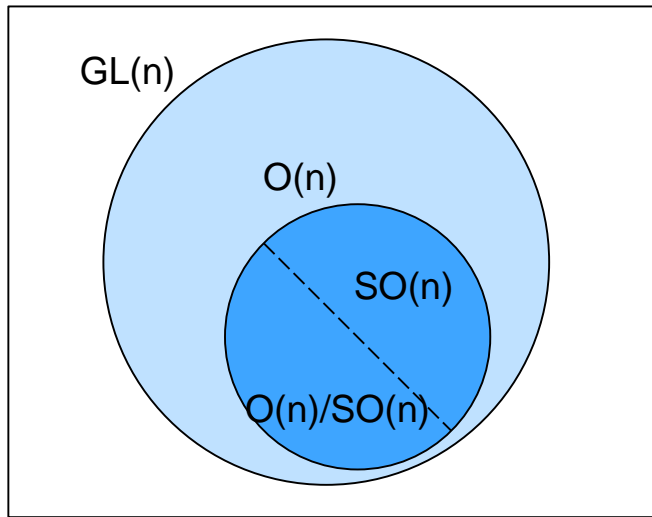
- The set of all the rigid transformations in  $\mathbb{R}^n$  is a group (not commutative) with the composition operation

$$\left( \left\{ F: \mathbb{R}^n \rightarrow \mathbb{R}^n \mid F \text{ rigid} \right\}, \circ \right)$$

- The set is isomorphic to the special Euclidean group  $SE(n)$
- The mathematical properties of a “rigid transformation” are specified by the special Euclidean group  $SE(n)$

$$RT = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

# Matrix groups: Summary



$\mathbb{R}^{n \times n}$

Vector space of all the  $n \times n$  matrices

$$GL(n) = (\{M \in \mathbb{R}^{n \times n} \mid \det(M) \neq 0\}, \times)$$

General linear group

$$O(n) = (\{A \in GL(n) \mid A^{-1} = A^T\}, \times)$$

Orthogonal group

$$SO(n) = (\{A \in O(n) \mid \det(A) = +1\}, \times)$$

Special orthogonal group

$$O(n)/SO(n) = (\{A \in O(n) \mid \det(A) = -1\})$$

Set of orthogonal matrices which do not preserve orientation (not a group)

$GL(n)$ ,  $O(n)$ ,  $SO(n)$  and  $SE(n)$  are all smooth manifolds  
(e.g. surfaces, curves, solids immersed in some big vector space)

# Special Euclidean group

- The special Euclidean group is constructed by the Cartesian product (a composition operation) from  $SO(n) \times \mathbb{R}^n$ .

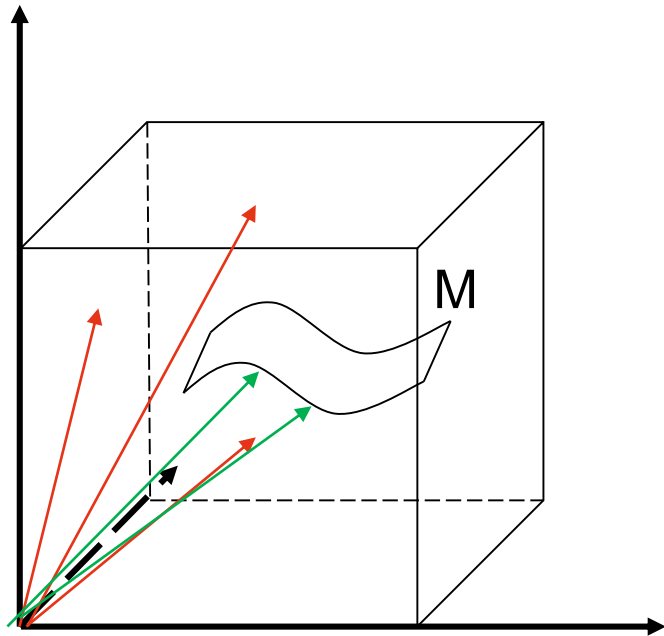
$$SE(n) = (SO(n) \times \mathbb{R}^n, \times)$$

$$(M, t) \times (S, q) = (MS, Mq + t)$$

- The Cartesian product defines where the values of  $SO(n)$  and  $\mathbb{R}^n$  (rotation and translation) go to form the transformation matrix

$$RT = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$
$$SE(3) = \begin{bmatrix} SO(3) & \mathbb{R}^3 \\ 0 & 1 \end{bmatrix}$$

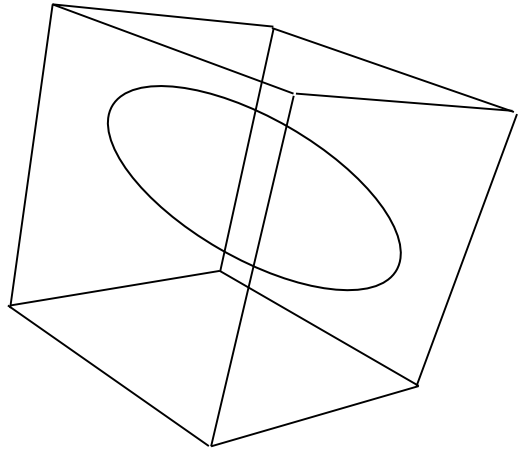
# Manifolds



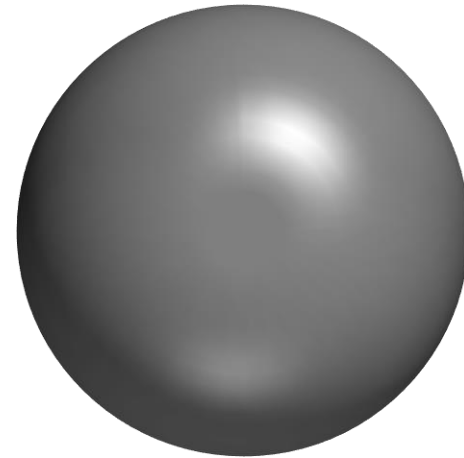
- Non-mathematical definition:  
Manifolds are shapes like curves, surfaces, and volumes in Euclidean space (i.e. a vector space)
- A sphere is an example of a manifold of dimension 2.
- The matrix groups  $SO(3)$ ,  $SE(3)$  are manifolds.

# Shapes of $SO(2)$ and $SO(3)$

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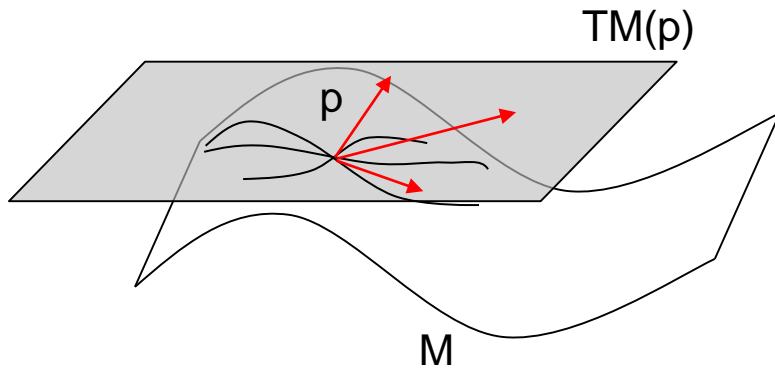


$SO(2)$  ... 1-manifold



$SO(3)$  ... 3-manifold (3-sphere)  
A solid ball in  $\mathbb{R}^3$

# Tangent space of a manifold



$V =$  vector space

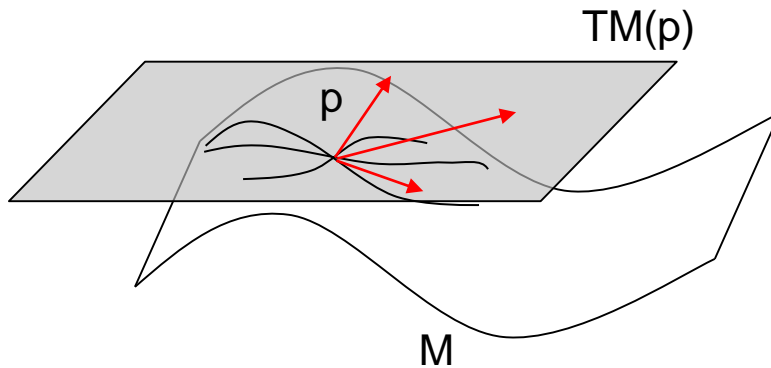
$M \dots k$ -manifold

The tangent space of the manifold  $M$  in  $p$  (every point  $p$  on the manifold has a different tangent space) is isomorphic to a subspace of  $V$ .



# Tangent space of a manifold

$V$  = vector space with dimension  $n$

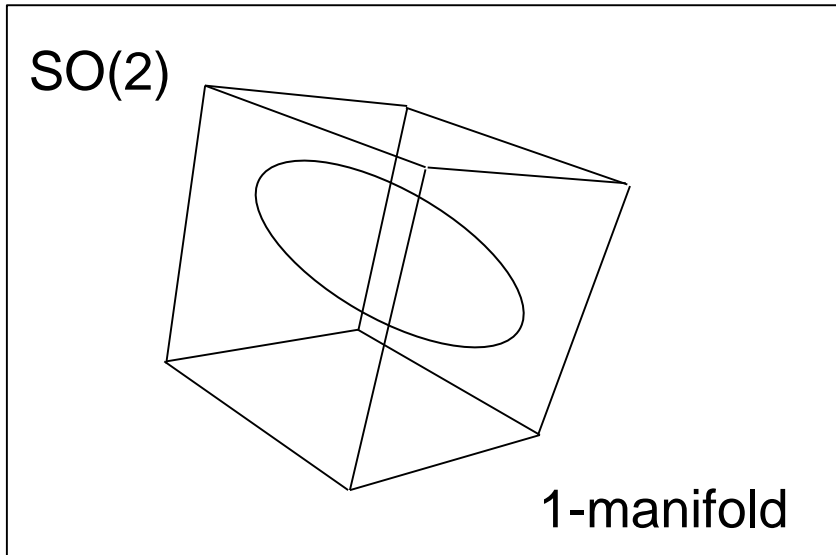


$M$  ...  $k$ -manifold

- $TM(p)$  is a vector space (subspace of  $V$ ) and has dimension  $k$ .
- 1-manifold (curves)  $\rightarrow$  1 dim TM (lines)
- 2-manifold (surface)  $\rightarrow$  2 dim TM (planes)
- 3-manifold (volumes, e.g. 3-sphere)  $\rightarrow$  3 dim TM (full volumes)

# Tangent space of SO(2) and SO(3)

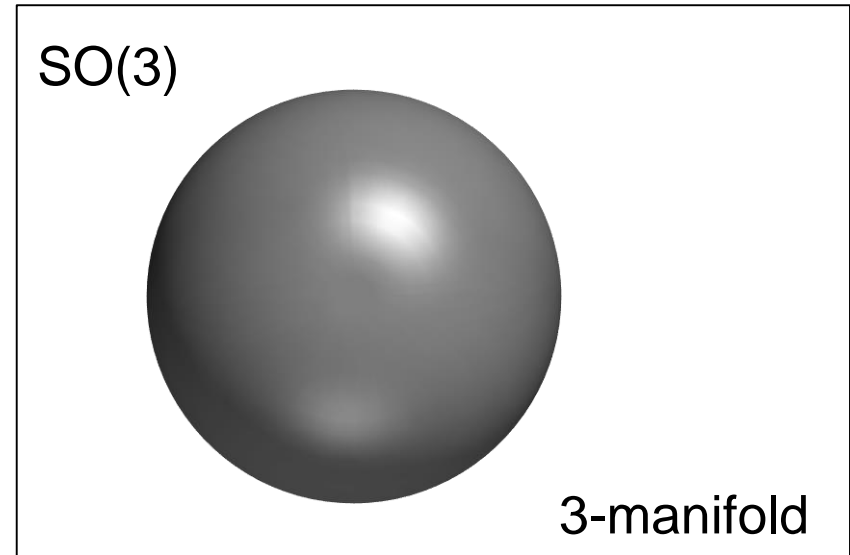
$\mathbb{R}^{2 \times 2}$



TSO(2) is a vector space  
with dimension 1  
subspace of  $\mathbb{R}^{2 \times 2}$

subspaces are defined by matrices

$\mathbb{R}^{3 \times 3}$



TSO(3) is a vector space  
with dimension 3  
subspace of  $\mathbb{R}^{3 \times 3}$

# Skew-symmetric matrices

- $M$  is skew-symmetric iff  $M^T = -M$

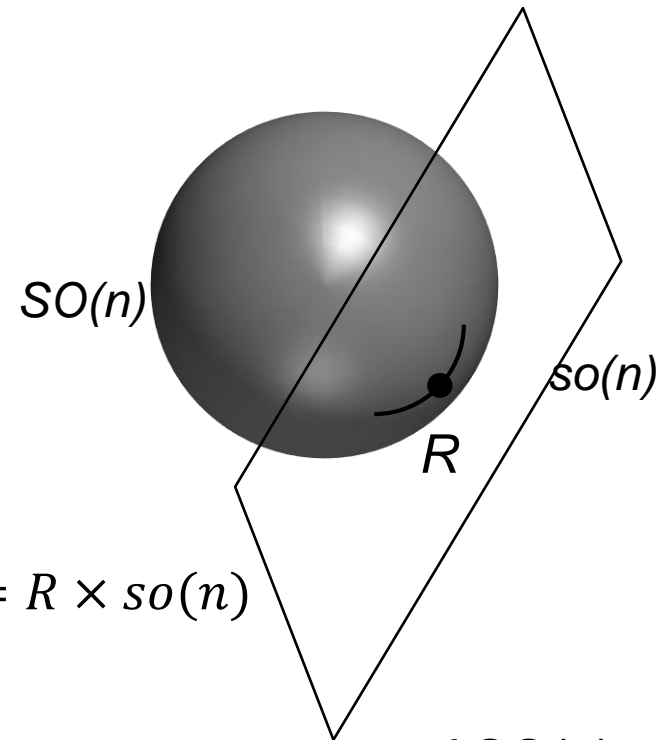
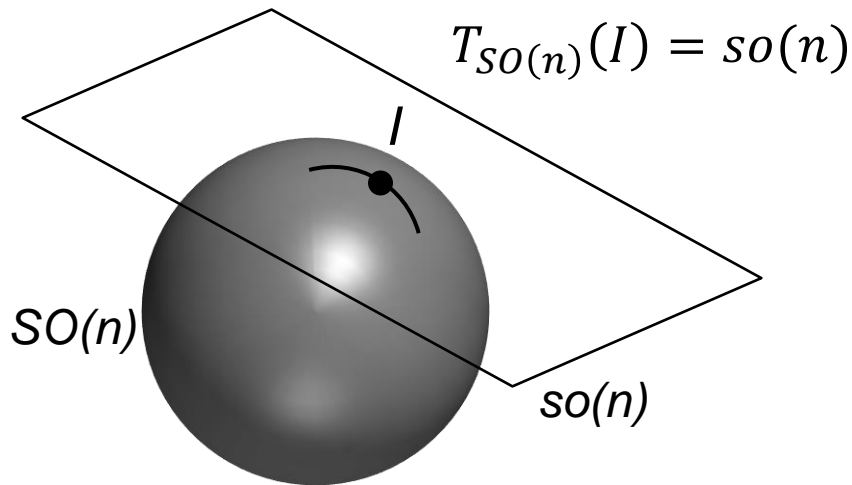
$$M^T = -M \quad \begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$so(n) = (\{M \in \mathbb{R}^{n \times n} \mid M^T = -M\}, +, [])$$

Special orthogonal Lie algebra

- The term “algebra” means that  $so(n)$  is a vector space (more specific than group)
- We have addition in the vector space now (in  $SO(n)$  it was only multiplication)

# Skew-symmetric matrices

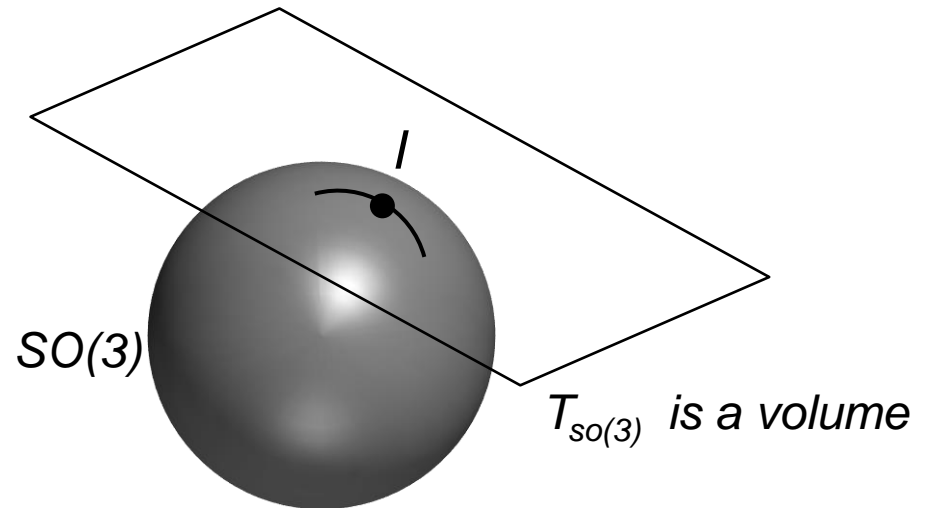
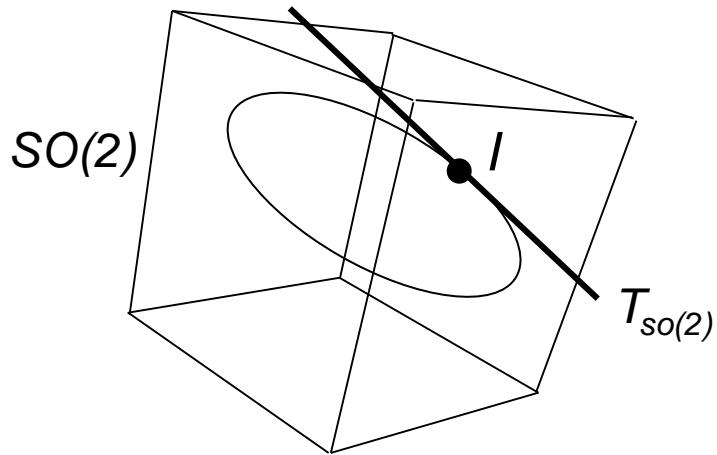


- The special orthogonal Lie algebra is the tangent space of  $SO(n)$  at identity.
- The tangent space of  $SO(n)$  in any other point  $R$  is a rotated version of  $so(n)$
- $T_{SO(n)}(R)$  is not a skew-symmetric matrix anymore, but a rotated one

# so(2) and so(3)

$$\begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -1 \\ -6 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

- so(3) is a vector space of dimension 3
- so(2) is a vector space of dimension 1



# The hat operator

- The hat operator is used to form skew-symmetric matrices
- for  $so(3)$ :

$$\hat{\cdot} : \mathbb{R}^3 \rightarrow so(3) \quad (\widehat{(x, y, z)}) \rightarrow \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

- for  $so(2)$ :

$$\hat{\cdot} : \mathbb{R} \rightarrow so(2) \quad \hat{x} \rightarrow \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$$

- Different notation

- $[t]_x = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$

# The hat operator

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- The hat operator is used to define the cross-product in matrix form

$$a \times b = \hat{a}b \quad \forall a, b \in \mathbb{R}^3$$

# Lie groups

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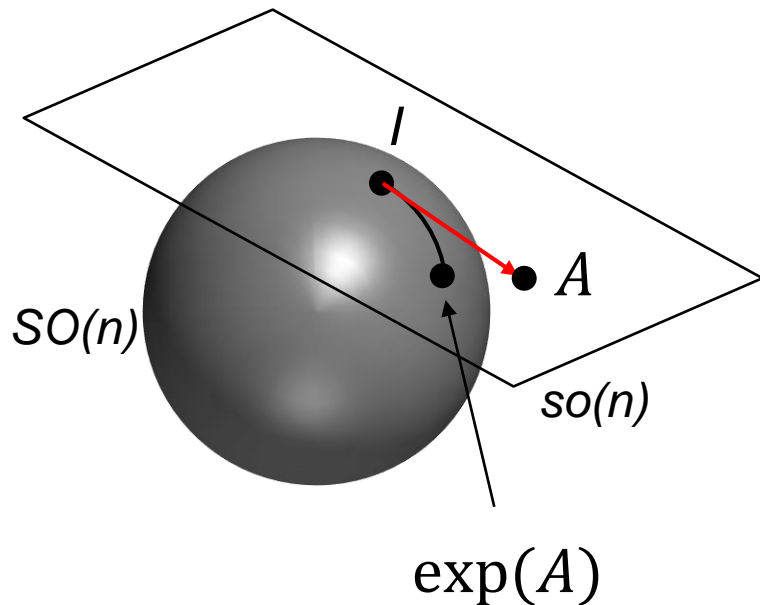
- $GL(n)$ ,  $O(n)$ ,  $SO(n)$  and  $SE(n)$  are all Lie groups (groups which are a smooth manifold where the operation is a differentiable function between manifolds)
- Also we have seen that the group of skew-symmetric matrices is called Lie algebra  $\mathfrak{so}(n)$  and is the tangent space of the special orthonormal group  $SO(n)$
- But how to compute an element of the tangent space  $\mathfrak{so}(n)$  from  $SO(n)$  or vice versa?
- The exponential map!



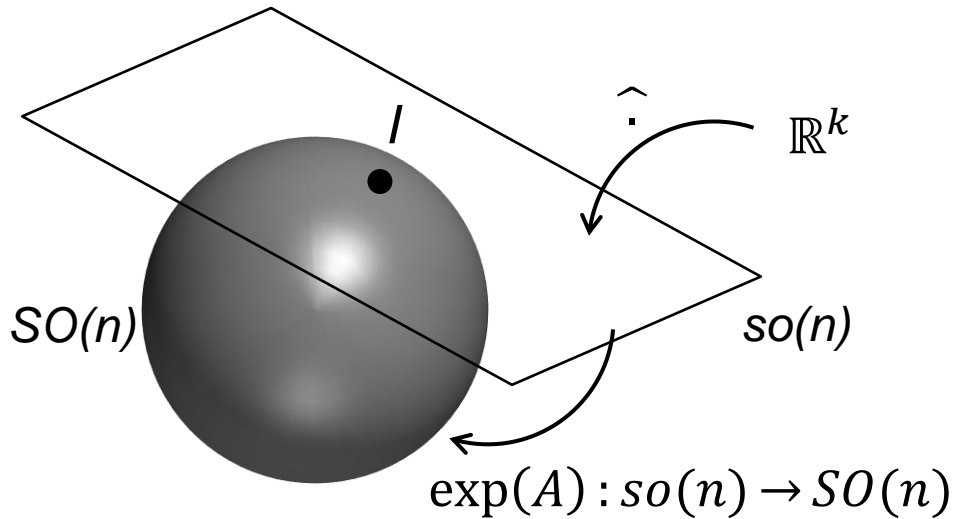
# Exponential map

- Given a Lie group  $G$ , with its related Lie algebra  $\mathfrak{g} = TG(I)$ , there always exists a smooth map from Lie algebra  $\mathfrak{g}$  to the Lie group  $G$  called exponential map

$$\exp: \mathfrak{g} \rightarrow G$$



# Exponential map



$$\omega \in \mathbb{R}^k$$

$$\hat{\omega} \in so(n)$$

$$\exp(\hat{\omega}) = \sum_{k=0}^{\infty} \frac{1}{k!} \hat{\omega}^k \in SO(n)$$

$$\omega \in \mathbb{R}^k \rightarrow \exp(\hat{\omega})$$

- Angle-axis representation for rotations:  
 $\omega \dots$  is the angle-axis representation ( $\mathbb{R}^3$ )  
 $\exp(\omega)$  is the 3x3 rotation matrix (element of  $SO(3)$ )

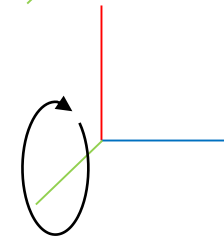
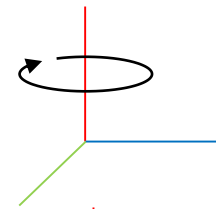
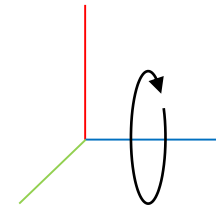
# Euler angles

- Euler's Theorem for rotations: Any element in  $SO(3)$  can be described as a sequence of three rotations around the canonical axes, where no successive rotations are about the same axis.

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



*For any  $R \in SO(3)$  there  $\exists \alpha, \beta, \gamma \mid R = R_x(\alpha)R_y(\beta) R_z(\gamma)$*

- $\alpha, \beta, \gamma$  are called Euler angles of  $R$  according to the XYZ representation (3 DOF/parameters)

# Euler angles

- Given  $M$  (element of  $SO(3)$ ) there are 12 possible ways to represent it

$$M \in SO(3) \text{ there } \exists \alpha, \beta, \gamma \mid M = R_x(\alpha)R_y(\beta) R_z(\gamma)$$

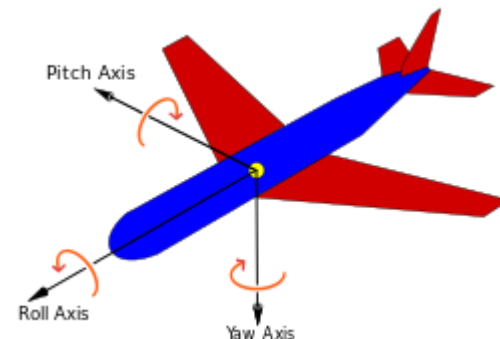
$$M \in SO(3) \text{ there } \exists \alpha, \beta, \gamma \mid M = R_x(\alpha)R_z(\gamma)R_y(\beta)$$

$$M \in SO(3) \text{ there } \exists \alpha, \beta, \gamma \mid M = R_x(\alpha)R_z(\gamma)R_x(\beta)$$

$$M \in SO(3) \text{ there } \exists \alpha, \beta, \gamma \mid M = R_z(\alpha)R_x(\gamma)R_z(\beta)$$

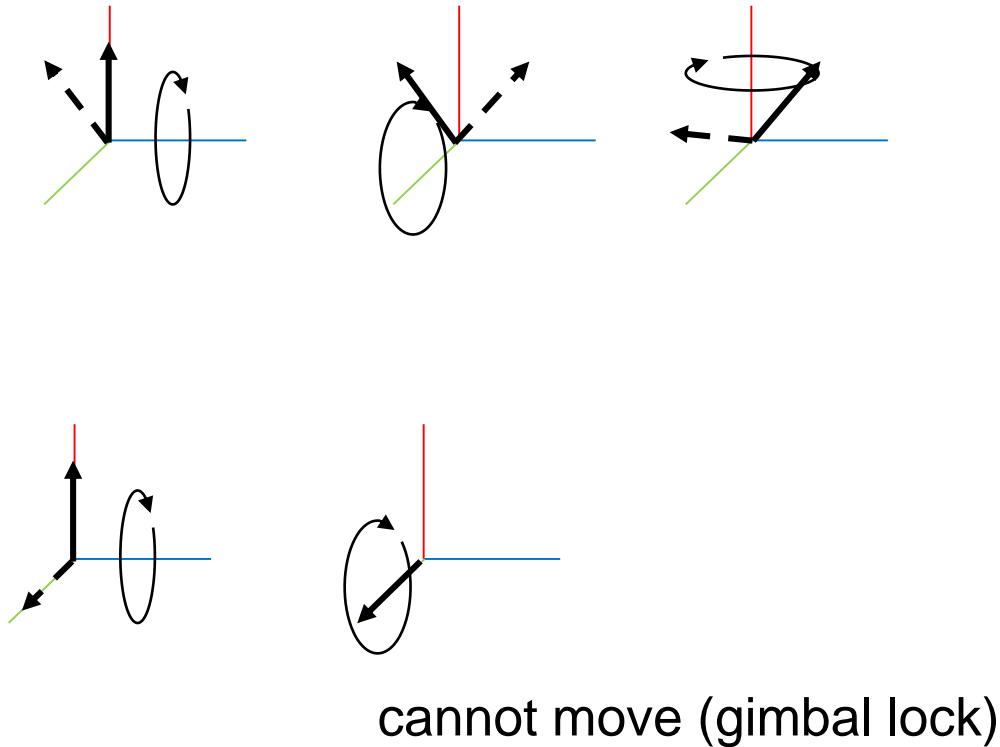
...

- A common convention is ZYX corresponding to a rotation first around the x-axis (roll), then the y-axis (pitch) and finally around the z-axis (yaw)



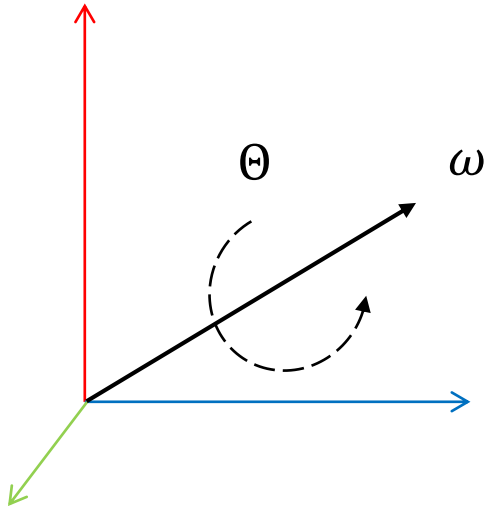
# Euler angles

- The parameterization has singularities, called gimbal lock
- A gimbal lock happens when after a rotation around an axis, two axes align, resulting in a loss of one degree of freedom



# Angle-Axis

- Euler's rotation theorem also states that any rotation can be expressed as a single rotation about some axis.



- The axis can be represented as a three-dimensional unit vector, and the angle by a scalar.
- 3 DOF/parameters
- Angle-axis defines a unique mapping and does not have gimbal lock

# Angle-Axis

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- The operation to compute the rotation matrix  $SO(3)$  from the angle-axis parameters is by using the exponential map!

$$\omega \in \mathbb{R}^k \rightarrow \exp(\hat{\omega})$$

- The exponential map can be computed in closed form using the Rodrigues formula

$$R = I + (\sin\Theta)K + (1 - \cos\Theta)K^2$$

$$K = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- There also exists the inverse

# Quaternions

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- Quaternions are extensions of complex numbers, with 3 imaginary values instead of 1:

$$a+ib+jc+kd$$

- Like the imaginary component of complex numbers, squaring the components gives:

$$i^2=j^2=k^2=-1$$

- One way to express a quaternion is as a pair consisting of the real value and the 3D vector consisting of the imaginary components:

$$q=(a,w) \text{ with } w=(b,c,d)$$

- It is basically a 4-vector



# Quaternions

- If  $q=a+ib+jc+kd$  is a unit quaternion ( $\|q\|=1$ ), then  $q$  corresponds to a rotation:

$$R(q) = \begin{bmatrix} 1 - 2c^2 - 2d^2 & 2bc - 2ad & 2bd + 2ac \\ 2bc + 2ad & 1 - 2b^2 - 2d^2 & 2cd - 2ab \\ 2bd - 2ac & 2cd + 2ab & 1 - 2b^2 - 2c^2 \end{bmatrix}$$

- Because  $q$  is a unit quaternion, we can write  $q$  as:

$$q = \left( \cos\left(\frac{a}{2}\right), \sin\left(\frac{a}{2}\right) w \right), \quad \|w\| = 1$$

- It turns out the  $q$  corresponds to the rotation whose:
  - Axis of rotation is  $w$  and
  - Angle of the rotation is  $a$

# Interpolation in $SO(3)$

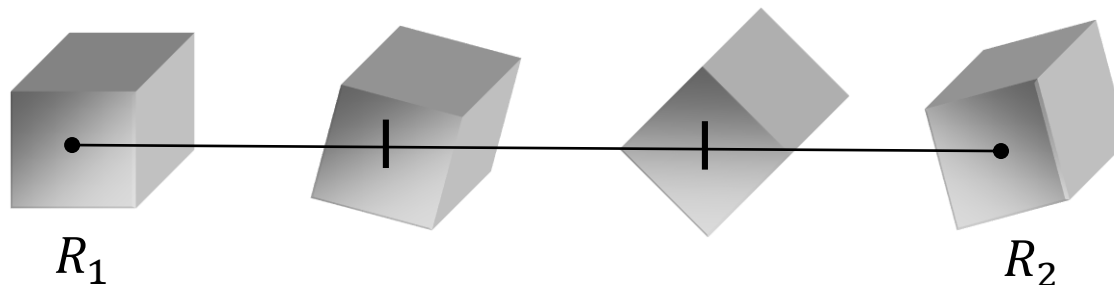
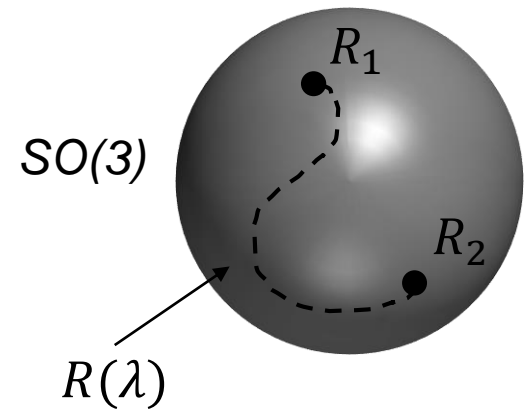
- Given two rotation matrices  $R_1, R_2$  one would like to find a smooth path in  $SO(3)$  connecting these two matrices

$$R(\lambda) \in SO(3), \lambda \in [0,1]$$

$R(\lambda)$  smooth

$$R(0) = R_1$$

$$R(1) = R_2$$



# Interpolation in $SO(3)$

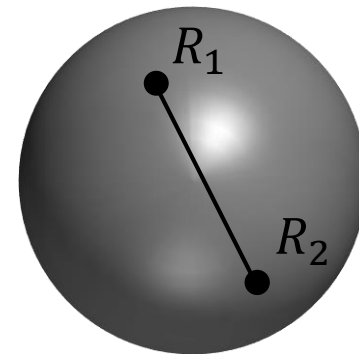
- Approach 1: Linearly interpolate  $R_1$  and  $R_2$  as matrices (naive approach)

$$R(\lambda) = \pi_{SO(3)}(\lambda R_1 + (1 - \lambda)R_2)$$

Projection onto sphere  
(not accurate)

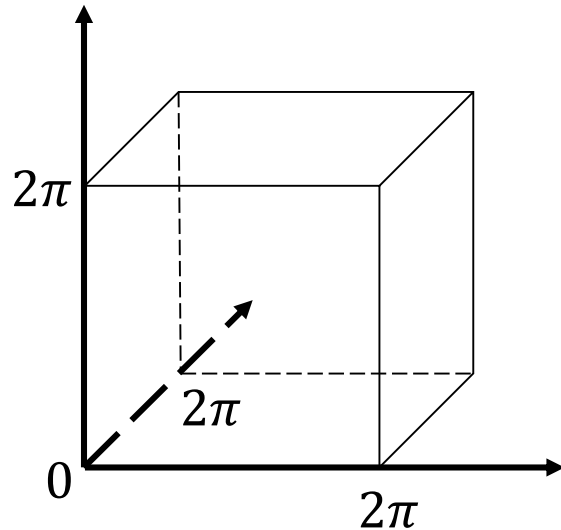
Not an element of  $SO(3)$ ,  
not a rotation matrix at all

$$\pi_{SO(3)}(M) = \arg \min_{R \in SO(3)} \|M - R\|_F^2$$



# Interpolation in $SO(3)$

- Approach 2: Linearly interpolate  $R_1$  and  $R_2$  using Euler angles

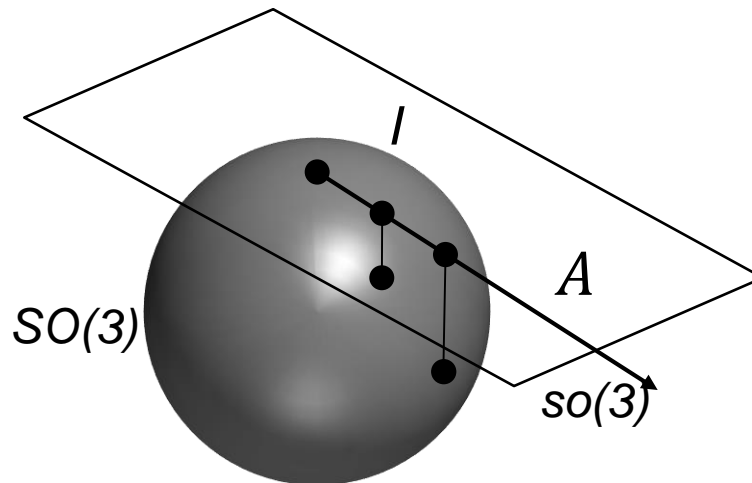


- Each axis is interpolated independently
- If  $R_1$  and  $R_2$  are too far apart  $\rightarrow$  not intuitive motion

# Interpolation in $SO(3)$

- Approach 3: Linearly interpolate  $R_1$  and  $R_2$  using angle-axis

$$\omega(\lambda) = \lambda\omega_1 + (1 - \lambda)\omega_2$$



- Interpolation happens in tangent space (vector space) and is then projected using the exponential map onto the manifold

## Filtering in $SO(3)$

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- Given  $n$  different noisy measurements for the rotation of an object

$$R_1, \dots, R_n$$

- What is the filtered average of it?

# Filtering in SO(3)

- Possible approaches:

- Average the rotation matrices  $R_i$

$$\frac{1}{n} \sum_{i=1}^n R_i \quad (\text{not rotation})$$

- Average the Euler angles of each  $R_i$

$$\left( \frac{1}{n} \sum_{i=1}^n \alpha_i, \frac{1}{n} \sum_{i=1}^n \beta_i, \frac{1}{n} \sum_{i=1}^n \gamma_i \right)$$

- Average the angle-axis of each  $R_i$

$$\frac{1}{n} \sum_{i=1}^n \omega_i$$

- Average the quaternions of each  $R_i$

$$\frac{1}{n} \sum_{i=1}^n q_i$$

(is rotation)

- All equally problematic and do not accurately respect the noise model

# Optimization in SO(3)

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- Newton-Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Naive approach:

- $X_n$  are elements of rotation matrix. Then the update step (addition) would not result in a rotation matrix.

- $X_n$  are Euler angles:

- To evaluate  $f(X_n)$  the rotation matrix has to be created from the Euler angles. Could lead to gimbal lock.
- Derivatives of Euler angle construction has to be computed.

- $X_n$  are elements of the tangent space  $\mathfrak{so}(3)$

- Represents angle-axis notation
- No gimbal lock
- Minimal representation of 3 parameters



# Learning goals - Recap

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- Understand the problems of dealing with rotations
- Understand how to represent rotations
- Understand the terms  $SO(3)$  etc.
- Understand the use of the tangent space
- Understand Euler angles, Axis-Angle, and quaternions
- Understand how to interpolate, filter and optimize rotations