Mathematical Principles in Visual Computing: Multi-Camera-Systems

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Outline

- Multi-Camera-Systems
- The generalized camera
- Plücker coordinates
- Generalized Epipolar Constraint
- Intra- and Inter-camera correspondences
- Generalized PnP
Learning goals

- Understand Plücker-line coordinates
- Understand the use of Plücker-line coordinates to describe multi-camera systems
- Understand the properties of multi-camera systems
- Understand the concept of a generalized camera
- Understand the generalized epipolar concept
- Understand the P3P extension to the generalized camera P3P
PtGrey Ladybug

- 6 cameras
- 360 field of view
- Panorama images
Omnitour
Automotive around view system
The generalized camera

pinhole camera

generalized camera
The generalized camera

generalized camera

mixed configuration
Multi camera systems

general non-axial camera (locally-central)

axial camera

axial camera
Plücker coordinates

- Plücker line matrix (4x4)

\[ L = AB^T - BA^T \]

\[
\begin{bmatrix}
A_4 B_1 - A_1 B_4 \\
A_4 B_2 - A_2 B_4 \\
A_4 B_3 - A_3 B_4 \\
A_3 B_2 - A_2 B_3 \\
A_1 B_3 - A_3 B_1 \\
A_2 B_1 - A_1 B_2 
\end{bmatrix}
\]

- Plücker coordinates (6-vector)
Plücker coordinates

- 6-vector L consists of 2 parts

\[ a^T = (L_1 \ L_2 \ L_3) \quad b^T = (L_4 \ L_5 \ L_6) \]

\[ a^T b = 0 \]

- Rigid transformation

  - Point:

    \[ X' = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} X \]

  - Plücker coordinate

    \[ [a'] = \begin{bmatrix} R \\ -[t]_x R \\ 0 \end{bmatrix} [b] \]
Plücker coordinates

- Line intersection of $L_1$ and $L_2$

$$L_2^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} L_1 = a_2^T b_1 + b_2^T a_1 = 0$$
Plücker coordinates: Geometric interpretation

\[ L = (q, q') \]
\[ q' = P \times q \]
\[ q^T q' = 0 \]
All points on the line $L$ are expressed by the two vectors $q \times q'$ and $\alpha q$, where $\alpha$ is a scalar.
Plücker coordinates: Pinhole camera

- Camera center is point C
- Plücker coordinate is just standard homogeneous coordinate in this case

\[
L = (q, q') \\
q' = C \times q \\
P = C = (0, 0, 0) \\
L = (q, (0 0 0)')
\]
Plücker coordinates: Generalized camera

- Camera centers are not at the origin (0,0,0)

\[ L_1 = (R_1 x_1, C_1 \times R_1 x_1) \]

- \( R_1, C_1 \) ... camera center and orientation in a common coordinate system
- \( x_1 \) ... homogeneous (normalized) image coordinate
Relative motion

- Relative rotations are the same for all cameras
- Relative translations are all different due to the lever arm (direction as well as length).
- Motion is defined as full 6DOF as compared to single camera case with 5DOF (length of translation is defined!)

\[ R_{11'} = R_{22'} \]
\[ t_{11'} \neq t_{22'} \]
Relative motion, scale estimation

\[
\begin{align*}
&C_1'' \quad \lambda c_{1_{1'}} \\
&C_1' \\
&C_1 \\
&C_2' \\
&C_2'' \\
&C_2
\end{align*}
\]
Generalized epipolar constraint (GEC)

- Line correspondence $L\leftrightarrow L'$

\[
L = (q_1^T, q'_1^T)^T \\
L' = (q_2^T, q'_2^T)^T
\]

- Relative motion transforms Plücker coordinates (light ray)

\[
L' = \begin{pmatrix} Rq_1 \\ (Rq'_1 + t \times (Rq_1)) \end{pmatrix}
\]

- Generalized epipolar constraint (GEC) defines intersection of two light rays $([R,T]L,L')$

\[
q_2^T q'_1 + q'_2^T q_1 = 0 \\
q_1 \rightarrow Rq_1 \\
q'_1 \rightarrow Rq'_1 + t \times (Rq_1) \\
q_2^T \left( Rq'_1 + t \times (Rq_1) \right) + q'_2^T (Rq_1) = 0 \\
q_2^T Rq'_1 + q_2^T [t]_x Rq_1 + q'_2^T Rq_1 = 0
\]
Generalized epipolar constraint (GEC)

- Matrix form

\[ q_2^T R q'_1 + q_2^T [t]_x R q_1 + q'_2^T R q_1 = 0 \]

\[ L_2^T GL_1 = \begin{pmatrix} q_2 \\ q'_2 \end{pmatrix}^T \begin{bmatrix} [t]_x & R \\ R & 0 \end{bmatrix} \begin{pmatrix} q_1 \\ q'_1 \end{pmatrix} = 0 \]

\[ G = \begin{bmatrix} [t]_x & R \\ R & 0 \end{bmatrix} \]

- G … generalized essential matrix (6x6)
- G contains essential matrix! (E = [t]_x R)
Linear algorithm for G

\[
L^T_2 G L_1 = \begin{pmatrix} q_2 \\ q_2' \end{pmatrix}^T \begin{bmatrix} [t]_x & R & R \\ R & 0 \end{bmatrix} \begin{pmatrix} q_1 \\ q_1' \end{pmatrix} = 0
\]

\[
L'^T G L = \begin{pmatrix} x' \\ v' \times x' \end{pmatrix}^T \begin{bmatrix} E & R \\ R & 0 \end{bmatrix} \begin{pmatrix} x \\ v \times x \end{pmatrix} = 0
\]

- E is 3x3, R is 3x3, i.e. in total 18 unknowns
- Each line correspondence gives 1 equation like

\[
x'^T E x + (v' \times x')^T R x + x'^T R (v \times x) = 0
\]

- Linear solution needs 17 point correspondences to compute G
Linear algorithm for G

\[
A_i^T y = \begin{bmatrix}
x_1'x_1 \\
x_1'x_2 \\
x_1'x_3 \\
x_2'x_1 \\
x_2'x_2 \\
x_2'x_3 \\
x_3'x_1 \\
x_3'x_2 \\
x_3'x_3 \\
(v_2'x_3' - v_3'x_2')x_1 + x_1'(v_2 x_3 - v_3 x_2) \\
(v_2'x_3' - v_3'x_2')x_2 + x_1'(v_3 x_1 - v_1 x_3) \\
(v_2'x_3' - v_3'x_2')x_3 + x_1'(v_1 x_2 - v_2 x_1) \\
(v_3'x_1' - v_1'x_3')x_1 + x_2'(v_2 x_3 - v_3 x_2) \\
(v_3'x_1' - v_1'x_3')x_2 + x_2'(v_3 x_1 - v_1 x_3) \\
(v_3'x_1' - v_1'x_3')x_3 + x_2'(v_1 x_2 - v_2 x_1) \\
(v_1'x_2' - v_2'x_1')x_1 + x_3'(v_2 x_3 - v_3 x_2) \\
(v_1'x_2' - v_2'x_1')x_2 + x_3'(v_3 x_1 - v_1 x_3) \\
(v_1'x_2' - v_2'x_1')x_3 + x_3'(v_1 x_2 - v_2 x_1)
\end{bmatrix}^T
\begin{bmatrix}
E_{11} \\
E_{12} \\
E_{13} \\
E_{21} \\
E_{22} \\
E_{23} \\
E_{31} \\
E_{32} \\
E_{33} \\
R_{11} \\
R_{12} \\
R_{13} \\
R_{21} \\
R_{22} \\
R_{23} \\
R_{31} \\
R_{32} \\
R_{33}
\end{bmatrix}
\]
Two types of correspondences

- Intra-camera correspondences: Correspondences from the same camera
- Inter-camera correspondences: Correspondences from different cameras
Two types of correspondences

- Problem when rotation is identity and there are only intra-camera correspondences. \( R = I \) \( t'_c = t_c \)

\[
x'^TEx + (t'_c \times x')^TRx + x'^TR(t_c \times x) = 0
\]

\[
x'^TEx = 0
\]

- Scale not observable anymore
Generalized PnP

- Very similar to perspective PnP
- Use Plücker coordinates
- But: How many points needed? How many solutions?
- We will show that 3 points are needed and this yields 8 solutions.
Generalized PnP

Three 2D-3D correspondences

Express rays as Plücker lines

Find unknown depths

Absolute orientation

$R, t$
Generalized PnP

\[ l_i = (u_i, u_i') \]

\[ X_i^V = u_i \times u_i' + \lambda_i u_i \]

\[ d_{ij} = \| X_i - X_j \|^2 = \| X_i^V - X_j^V \|^2 \] depths

\[ d_{ij} = \| X_i - X_j \|^2 = \| (u_i \times u_i' + \lambda_i u_i) - (u_j \times u_j' + \lambda_j u_j) \|^2 \]
Generalized PnP

- 3 distances lead to 3 equations with unknown depths (lambda)

\[ k_{11}\lambda_1^2 + (k_{12}\lambda_2 + k_{13})\lambda_1 + (k_{14}\lambda_2^2 + k_{15}\lambda_2 + k_{16}) = 0 \]
\[ k_{21}\lambda_2^2 + (k_{22}\lambda_3 + k_{23})\lambda_2 + (k_{24}\lambda_3^2 + k_{25}\lambda_3 + k_{26}) = 0 \]
\[ k_{31}\lambda_3^2 + (k_{32}\lambda_3 + k_{33})\lambda_3 + (k_{34}\lambda_3^2 + k_{35}\lambda_3 + k_{36}) = 0 \]

- Eliminating variables using resultants (determinant of Sylvester matrix) leads to an 8 degree polynomial

\[ A\lambda_3^8 + B\lambda_3^7 + C\lambda_3^6 + D\lambda_3^5 + E\lambda_3^4 + F\lambda_3^3 + G\lambda_3^2 + H\lambda_3 + I = 0 \]

- No closed form solution possible (Root solving with companion matrix or Sturm bracket method)
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