Mathematical Principles in Vision and Graphics: Sylvester Resultant

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SS2020

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Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown x:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$$

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$\det(\mathrm{Syl}(f,g))$ is called the Sylvester Resultant of f and g.

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Let's consider y as a constant, and write these two polynomials as polynomials in x:

$$\begin{aligned} p_{1,y}(x) &= 6x^2 + (3y - y^2)x + (y+1) \\ p_{2,y}(x) &= yx^2 + 5x + (4y-1) \end{aligned}$$

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$\rightarrow$ First solve for y (we will see later how it can be done).