Mathematical Principles in Vision and Graphics: Sylvester Resultant

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Sylvester Matrix and Sylvester Resultant

Let's consider two polynomials of a *single* unknown x:

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$$

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Sylvester matrix of f and g:

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$\det(\mathrm{Syl}(f,g))$ is called the Sylvester Resultant of f and g.

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$$p_2(x,y) = x^2y + 5x + 4y - 1$$

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Let's consider y as a constant, and write these two polynomials as polynomials in x:

$$p_{1,y}(x) = 6x^2 + (3y - y^2)x + (y + 1)$$
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$\rightarrow$ First solve for y (we will see later how it can be done).