Mathematical Principles in Vision and Graphics:
Projective Geometry – Part 2

Ass.Prof. Friedrich Fraundorfer

SS 2019

Outline

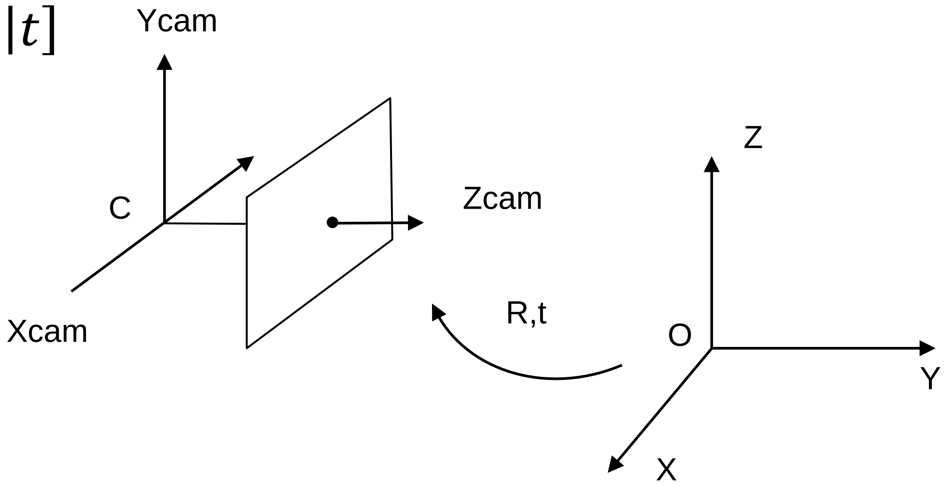
- Axioms of geometry
- Differences between Euclidean and projective geometry
- 2D projective geometry
  - Homogeneous coordinates
  - Points, Lines
  - Duality
- 3D projective geometry
  - Points, Lines, Planes
  - Duality
  - Plane at infinity
  - Image formation
  - Parameterizing a camera matrix
  - Projection of points and lines
- Geometric relationships
  - Epipolar constraint derivation
  - Stereo normal case
  - Triangulation
  - Camera pose estimation (algebraic, from points and lines)

Learning goals

- Understand the geometric interpretation of the camera matrix
- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation

Camera matrix (calibrated camera)

$$P = KR[I|-C] = K[R|-RC] = K[R|t]$$
$$t = -RC$$



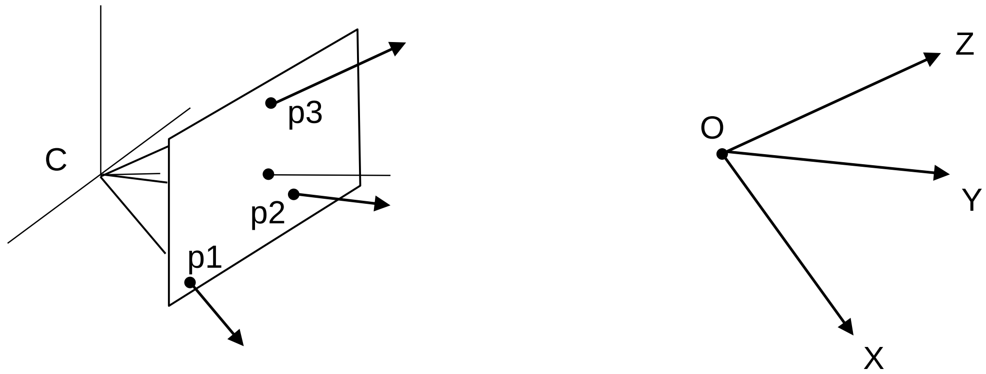
- Camera matrix is a coordinate transformation and then a projection
- C … 3x1 coordinate of the camera center in world coordinate
- R … 3x3 rotation matrix representing the orientation of the camera coordinate frame
- K … 3x3 calibration matrix

Geometric interpretation of camera matrix entries

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = [p_1 \quad p_2 \quad p_3 \quad p_4] = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}$$

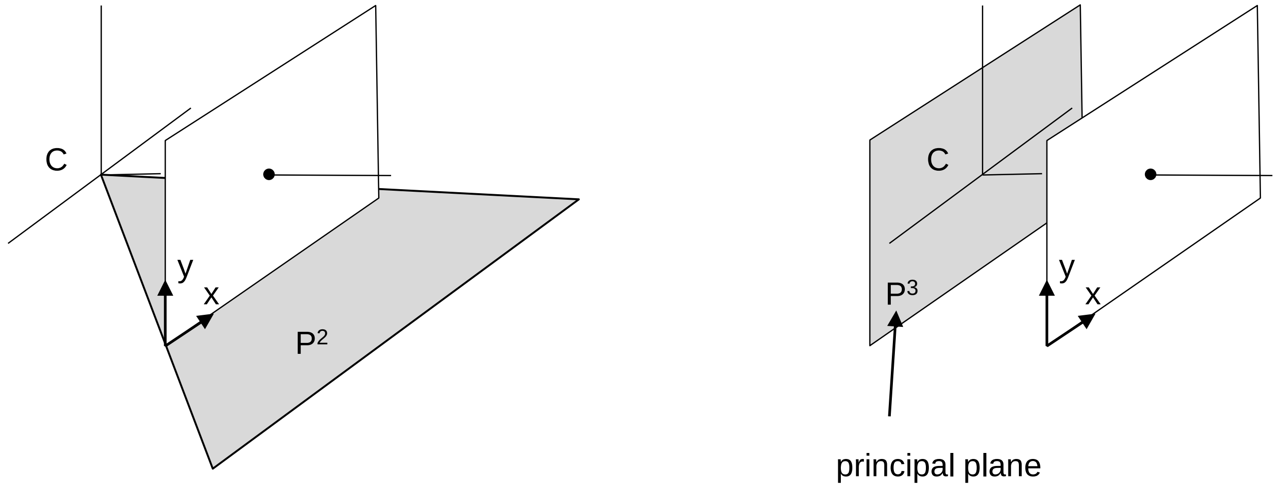
- Columns (3x1 vectors) are unit vectors of the coordinate system
- Rows transposed (1x4 vectors) are planes spanned by coordinate system axis

Geometric interpretation of camera matrix entries



- p1,p2,p3 are the images of the axis directions (vanishing points)
- x-axis … D=(1,0,0,0) is imaged as p1=PD

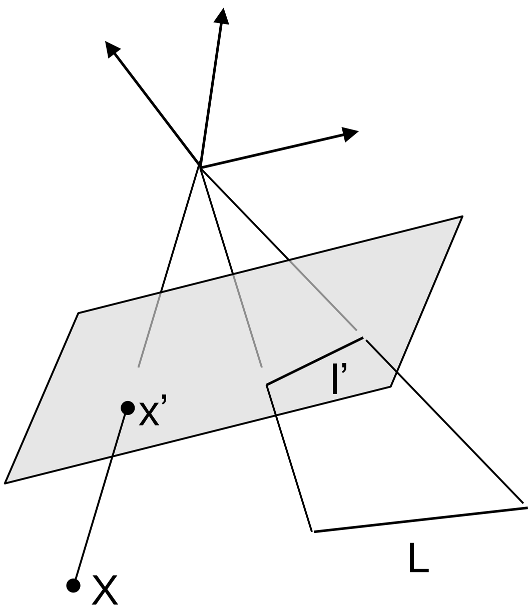
Geometric interpretation of camera matrix entries



- P^{3T} is the principal plane, the plane parallel to the image plane through the camera center
- $PX = (x, y, 0)$ all points on the principal plane have the following coordinates
- Plane condition, $P^{3T}X = 0$ (point X is on the plane if it fulfils the condition)
- Points on P2, $P^{2T}X = 0$, all points have coordinates $PX = (x, 0, z)$
- Plane P2 is defined by the line y=0 and the camera center

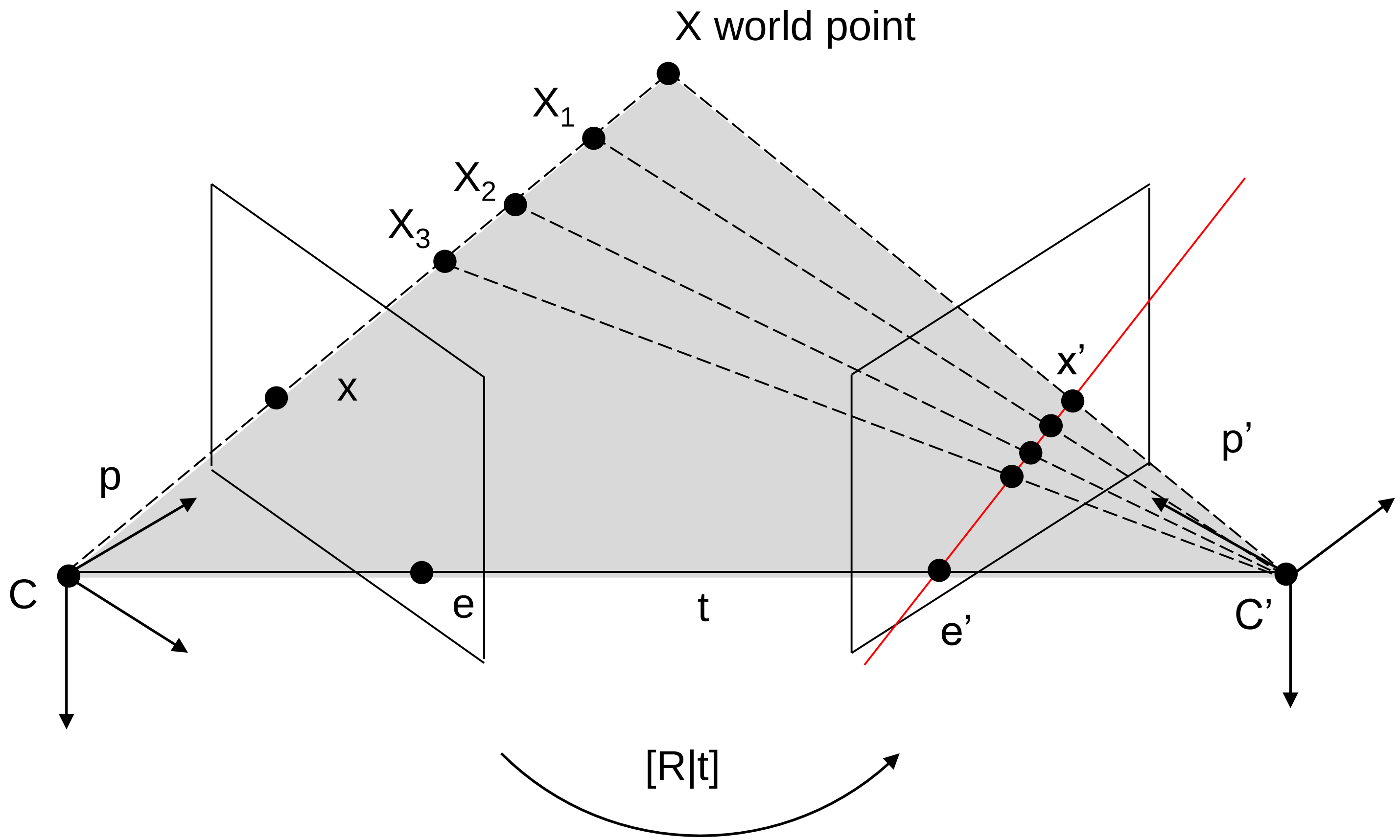
Point and line projection

- Point projection x = PX
- Line projection is more involved (line l is a 4x4 matrix)
- Therefore indirect projection:
  $l' = x' \times y' = PX \times PY$
  $L = \overline{X\,Y}$



Epipolar constraint

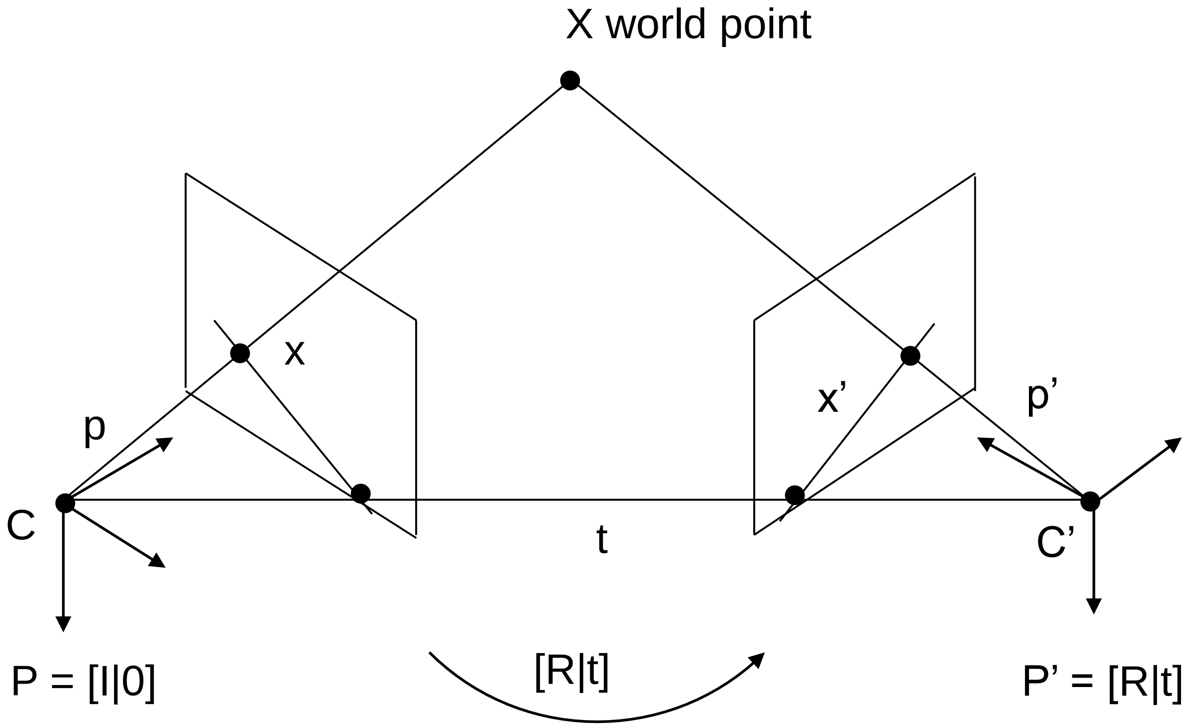
- An image point x lies on its corresponding epipolar line in the corresponding image.



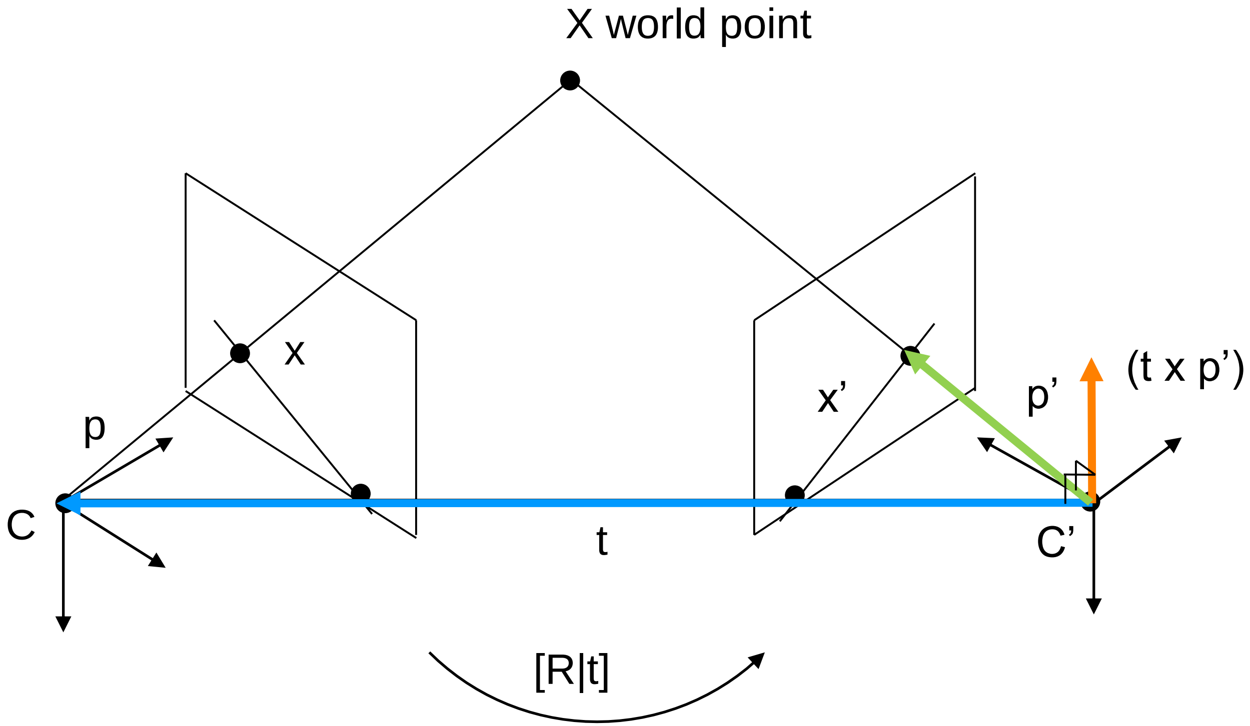
Epipolar constraint – derivation by coplanarity condition

- Vector p and t define a plane
- Vector p’ and t define also a plane
- Both planes must have the same normal
- What we seek is a relation between p and p’

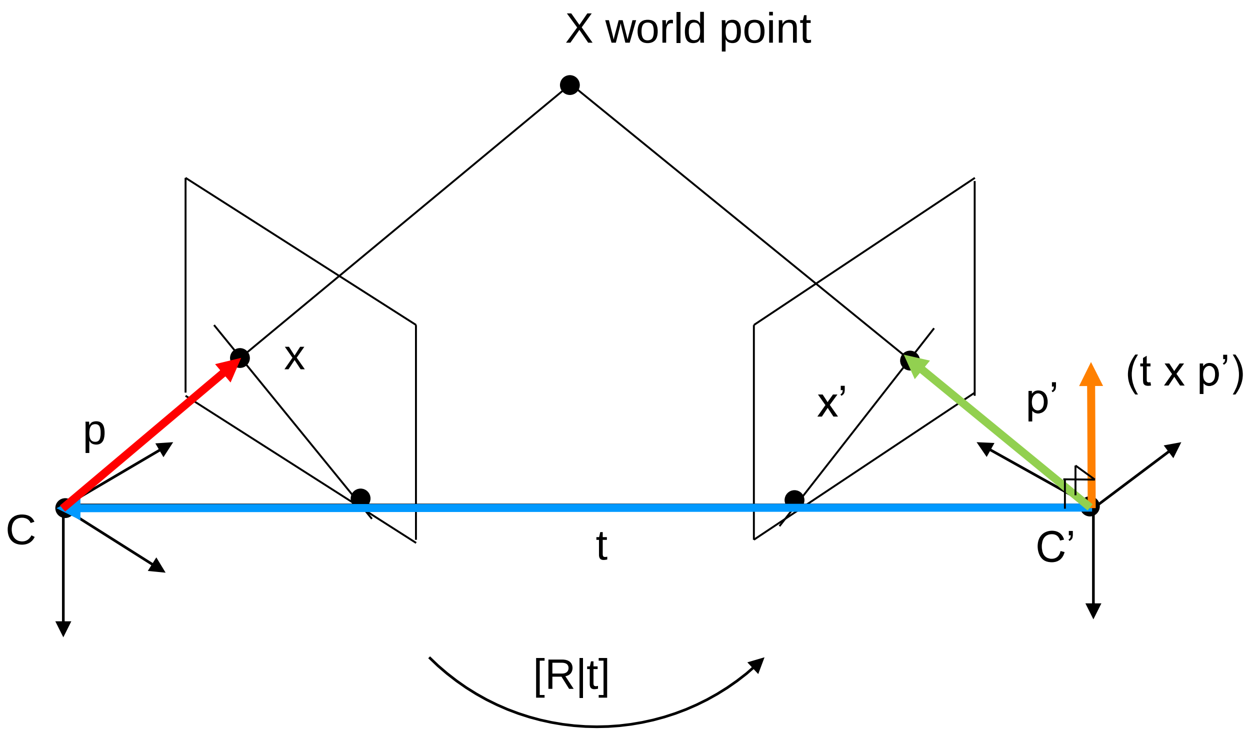
Epipolar constraint – derivation by coplanarity condition



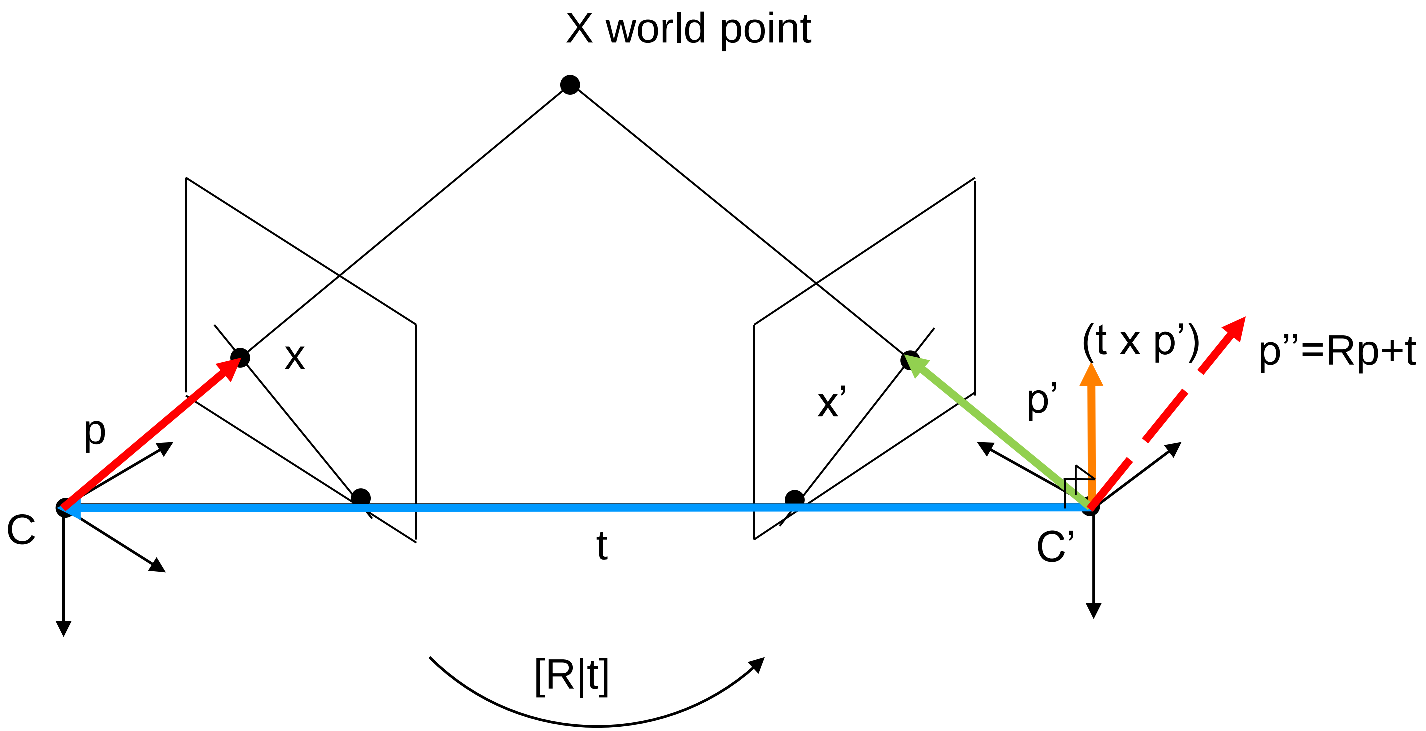
Epipolar constraint – derivation by coplanarity condition



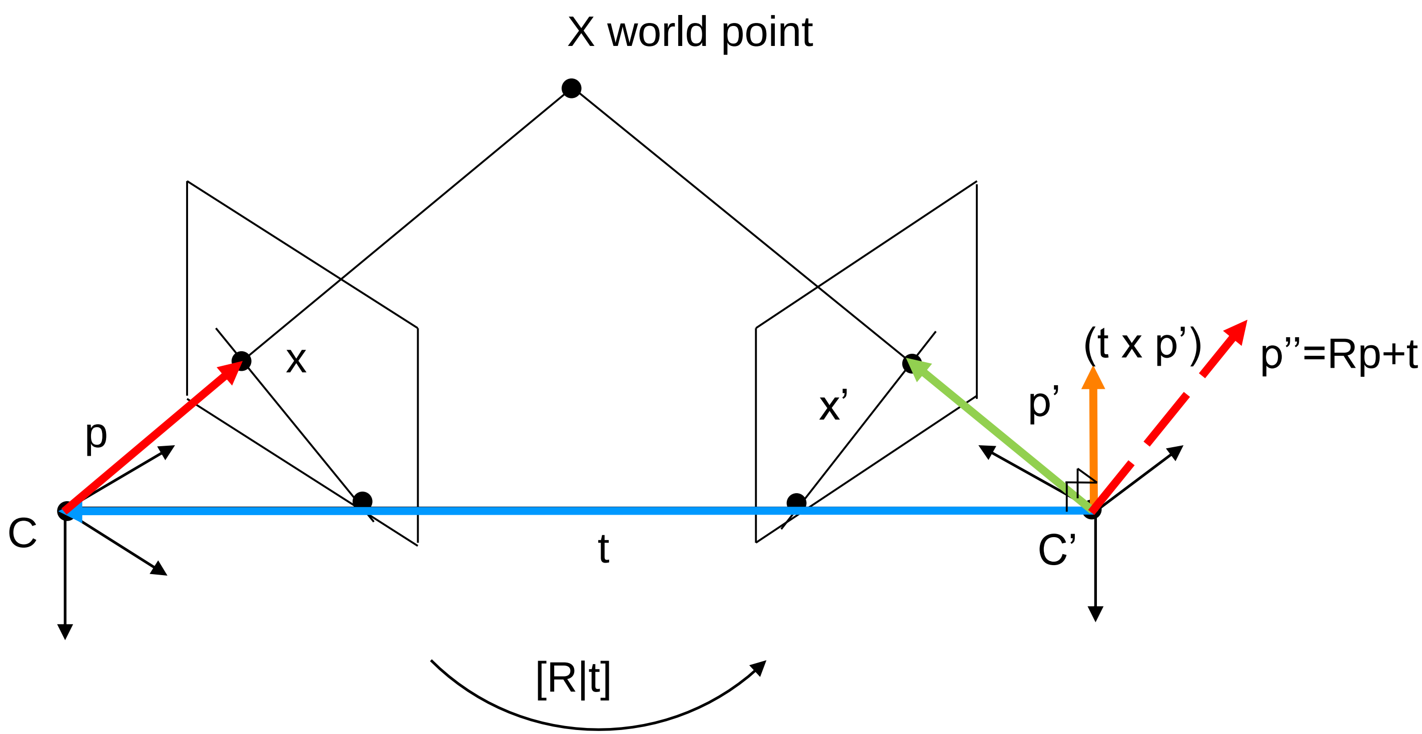
Epipolar constraint – derivation by coplanarity condition



Epipolar constraint – derivation by coplanarity condition



Epipolar constraint – derivation by coplanarity condition



$$t \times p' = t \times p''$$
$$t \times p' = t \times (Rp + t)$$
$$with\; p'' = Rp + t$$
$$t \times p' = t \times Rp + t \times t$$
$$p'^T (t \times p') = 0$$
$$p'^T (t \times Rp) = 0$$

$$p'^T \underbrace{[t]_x R}_{E} p = 0$$

$$\boxed{p'^T E p = 0}$$

E is called the Essential matrix

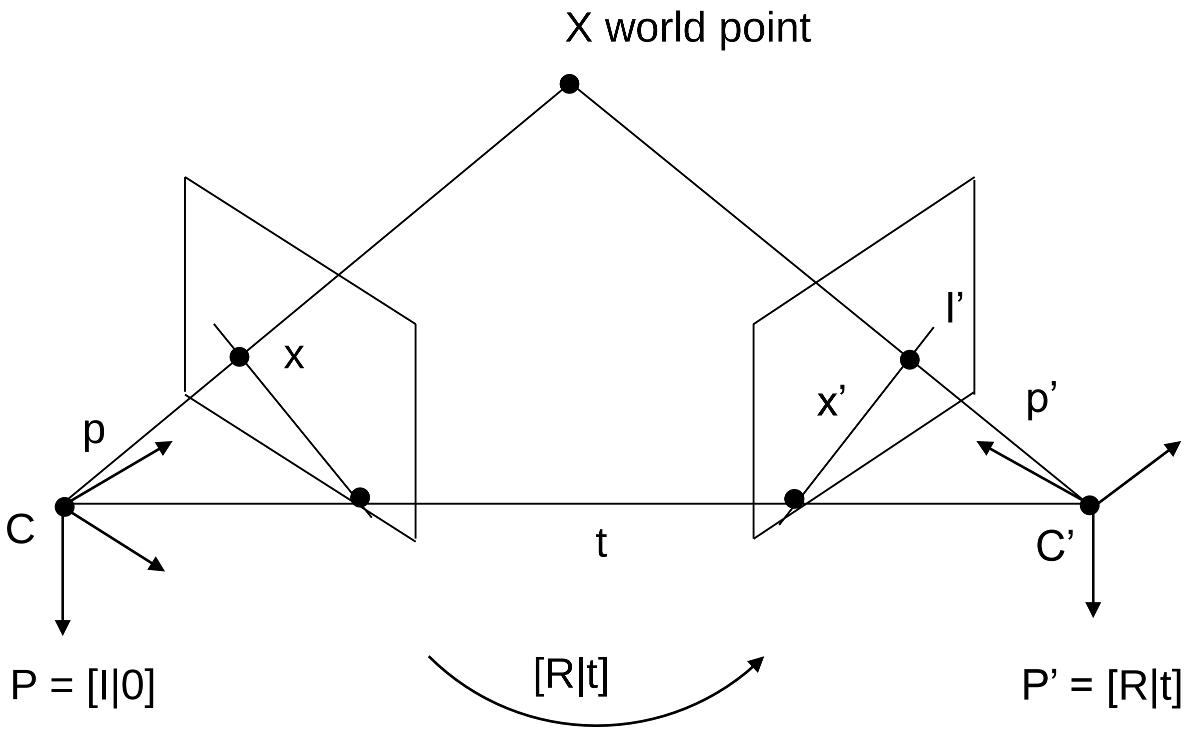
Fundamental matrix

- p,p’ from the Essential matrix derivation are in normalized coordinates
- x,x’ are image coordinates, x=Kp, x’=Kp
- By replacing p,p’ with x,x’ one gets the Fundamental matrix

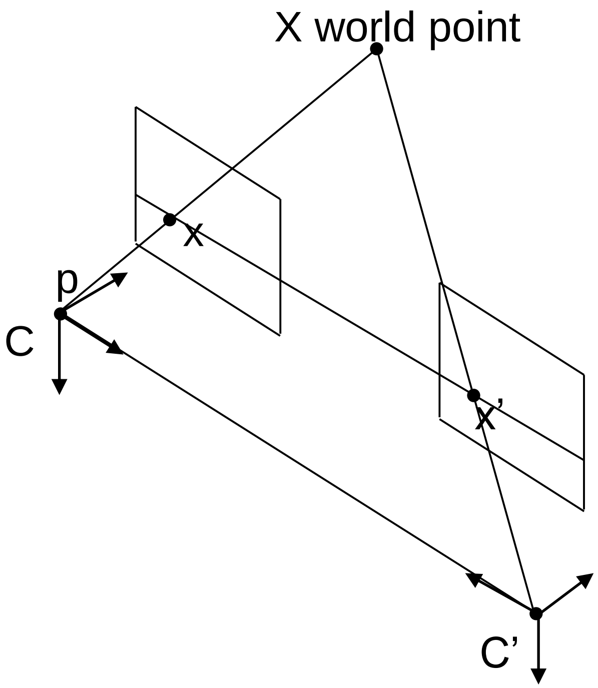
$$p'^T E p = 0$$
$$x'^T K^{-T} E K^{-1} x = 0$$
$$x'^T F x = 0$$
$$\mathrm{F} = K^{-T} E K^{-1}$$

Epipolar lines

- The corresponding line l’ to image coordinate x
- l’ is the line connecting the epipole e’ and the image coordinate x’
- Hypothesis: $l' = Fx$
- Point x’ must lie on l’, thus $x'^T l' = 0$
- Now $x'^T F x = 0$

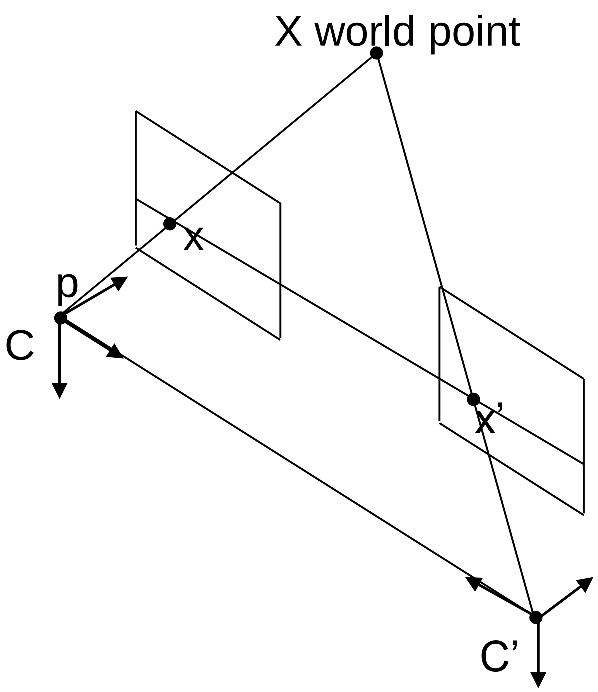


Stereo case



$$R = I_{3x3} \qquad T = [T_x \quad 0 \quad 0]^T$$

Stereo case



$$R = I_{3x3} \qquad T = [T_x \quad 0 \quad 0]^T$$

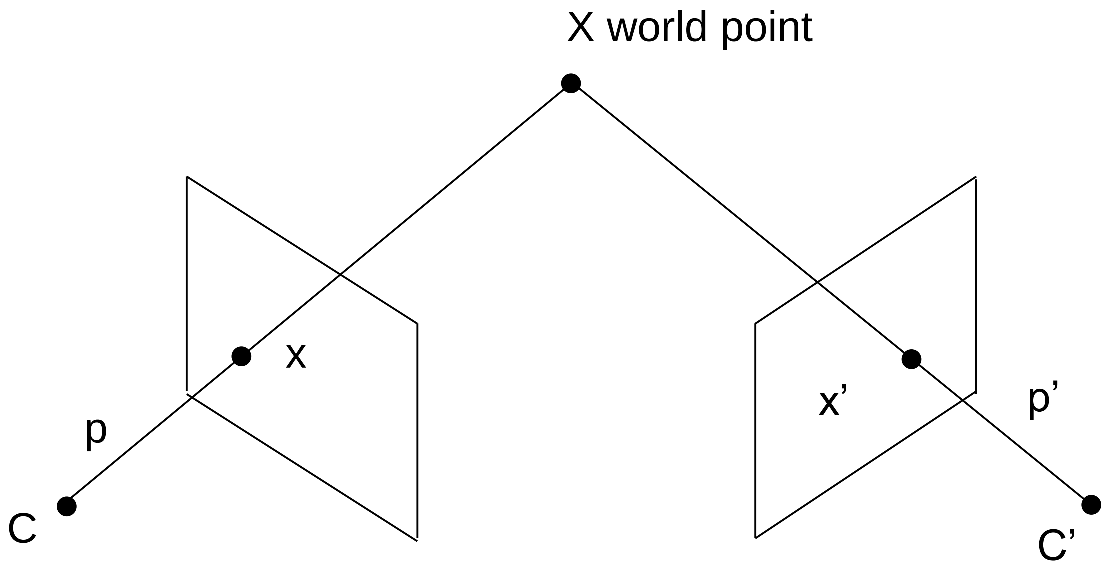
$$E = [T]_x R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix}$$

$$[x' \quad y' \quad 1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$[x' \quad y' \quad 1] \begin{bmatrix} 0 \\ -T_x \\ T_x y \end{bmatrix} = 0$$

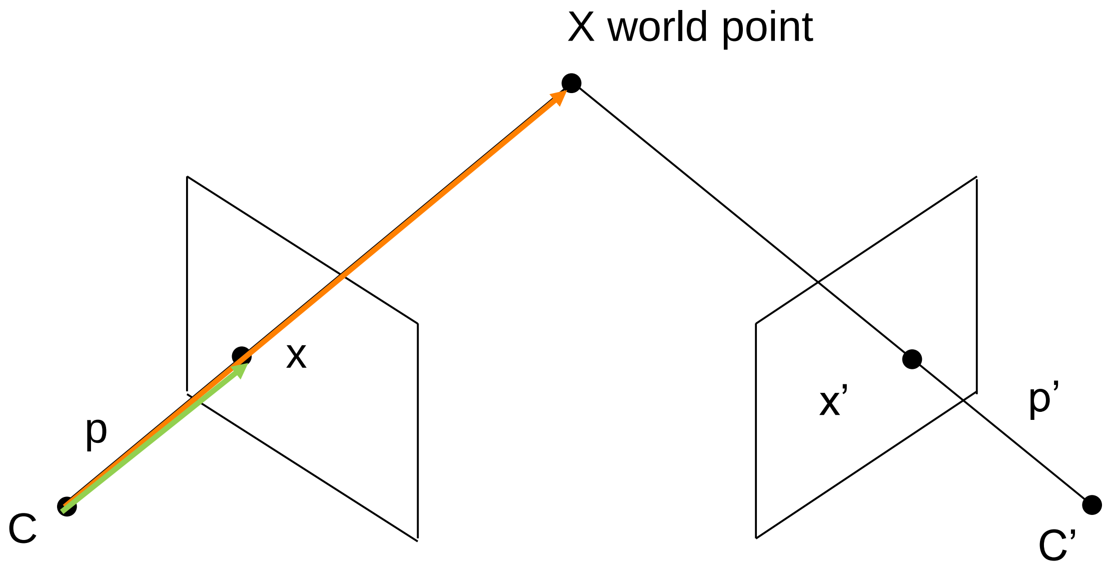
$$-y' T_x + T_x y = 0$$

Triangulation



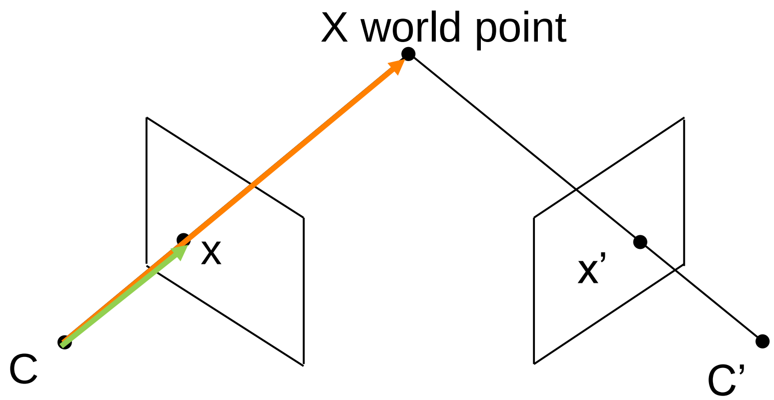
- Compute coordinates of world point X given the measurements x, x' and the camera projection matrices P and P'

Triangulation

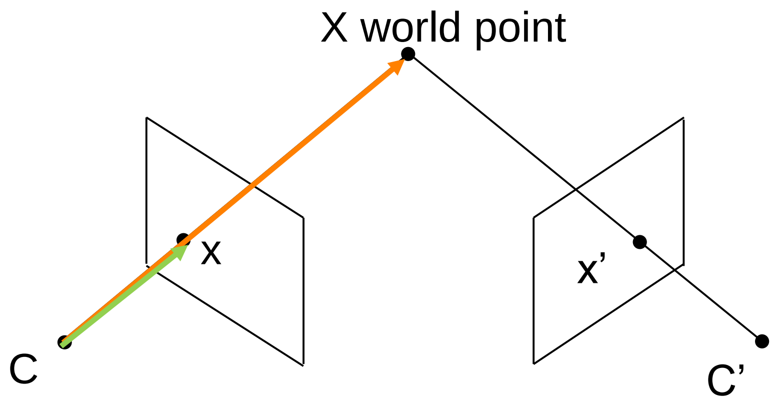


- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)
- $x \times (PX) = 0 \; and \; x' \times (P'X) = 0$
- Can be rewritten into equation system $AX = 0$ to solve for X

Triangulation



Triangulation

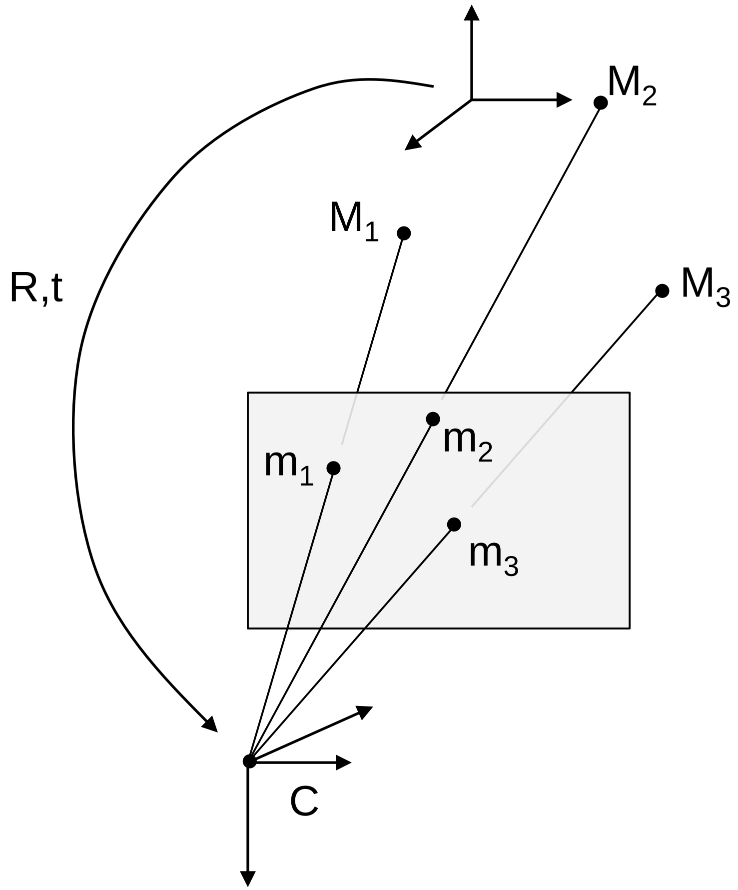


$$P = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix}$$

$$x \times (PX) = 0 \; and \; x' \times (P'X) = 0$$
$$x(P_3^T X) - (P_1^T X) = 0$$
$$y(P_3^T X) - (P_2^T X) = 0$$
$$x(P_2^T X) - y(P_1^T X) = 0$$

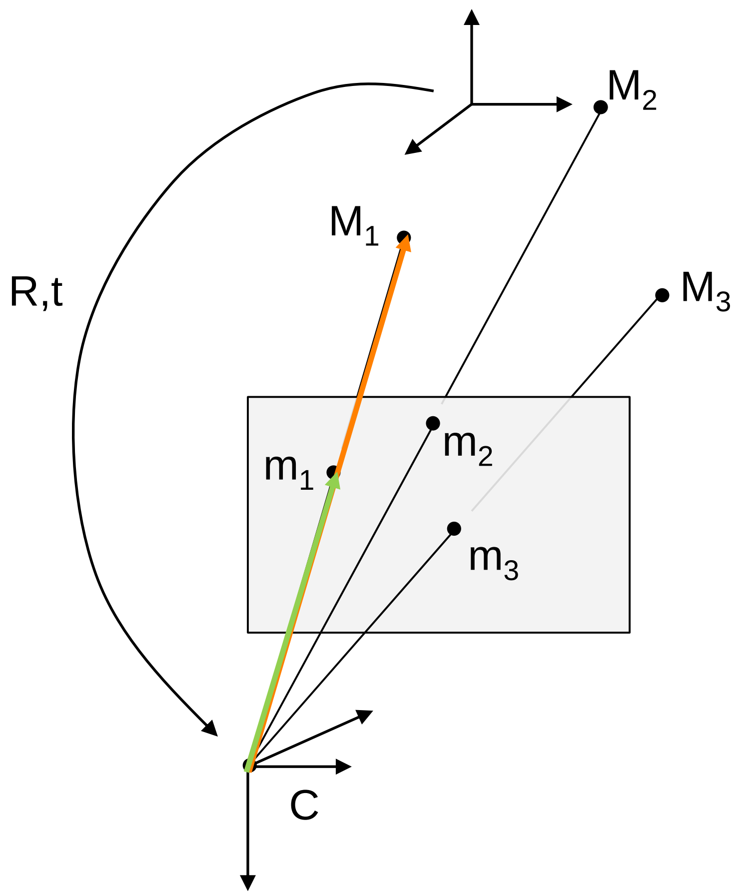
$$\begin{bmatrix} xP_3^T - P_1^T \\ yP_3^T - P_2^T \\ x'P'^T_3 - P'^T_1 \\ y'P'^T_3 - P'^T_2 \end{bmatrix} X = 0$$

Camera pose estimation



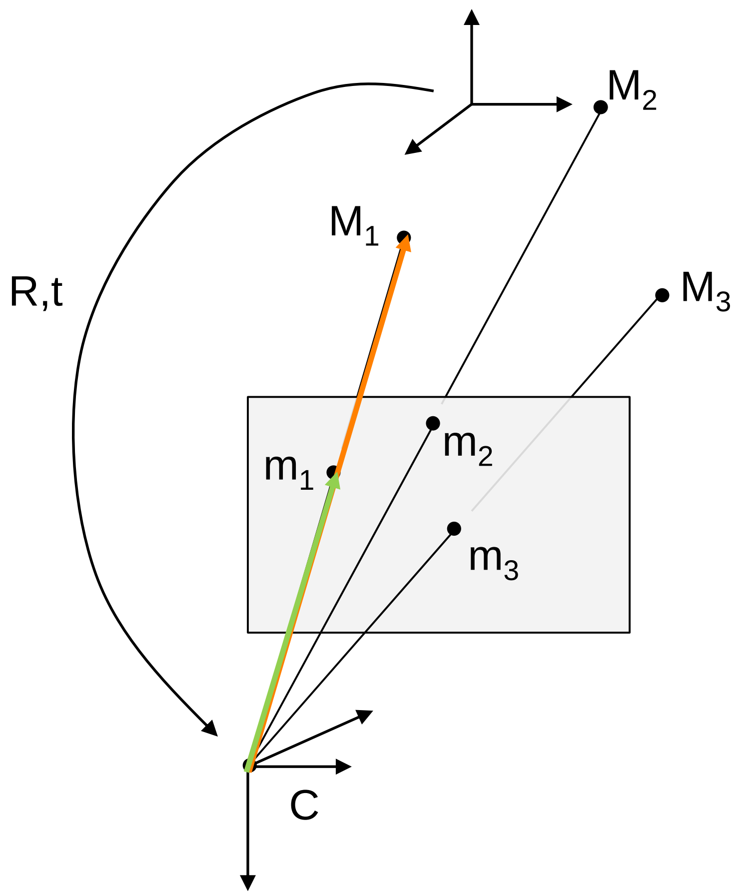
- perspective-n-point problem
- Goal is to estimate camera matrix P such that $m_1 = PM_1$
- $m_1, M_1, m_2, M_2, m_3, M_3$ are known
- Algebraic linear solution requires 6 3D-2D point correspondences, minimal nonlinear solution requires only 3

Camera pose estimation



- Derivation similar to Triangulation, but now entries of P are the unknowns instead of X
- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)

Camera pose estimation



- Derivation similar to Triangulation, bot now entries of P are the unknowns instead of X
- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)

$$x \times (PX) = 0 \; for \; all \; pairs \; x \leftrightarrow X$$
$$y(P_3^T X) - w(P_2^T X) = 0$$
$$x(P_3^T X) - w(P_1^T X) = 0$$
$$x(P_2^T X) - y(P_1^T X) = 0$$

$$\begin{bmatrix} 0 & -wX^T & yX^T \\ -wX^T & 0 & xX^T \\ -yX^T & xX^T & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$

Recap - Learning goals

- Understand the geometric interpretation of the camera matrix
- Understand and derive the epipolar constraint
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