Mathematical Principles in Vision and Graphics:
Projective Geometry – Part 2

Ass.Prof. Friedrich Fraundorfer

SS 2019
Outline

- Axioms of geometry
- Differences between Euclidean and projective geometry
- 2D projective geometry
  - Homogeneous coordinates
  - Points, Lines
  - Duality
- 3D projective geometry
  - Points, Lines, Planes
  - Duality
  - Plane at infinity
  - Image formation
    - Parameterizing a camera matrix
    - Projection of points and lines
- Geometric relationships
  - Epipolar constraint derivation
  - Stereo normal case
  - Triangulation
  - Camera pose estimation (algebraic, from points and lines)
Learning goals

- Understand the geometric interpretation of the camera matrix
- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation
Camera matrix (calibrated camera)

\[ t = -RC \]

- Camera matrix is a coordinate transformation and then a projection
- \( C \) … 3x1 coordinate of the camera center in world coordinate
- \( R \) … 3x3 rotation matrix representing the orientation of the camera coordinate frame
- \( K \) … 3x3 calibration matrix
Geometric interpretation of camera matrix entries

\[ P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = [p_1 \ p_2 \ p_3 \ p_4] = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix} \]

- Columns (3x1 vectors) are unit vectors of the coordinate system
- Rows transposed (1x4 vectors) are planes spanned by coordinate system axis
Geometric interpretation of camera matrix entries

- $p_1, p_2, p_3$ are the images of the axis directions (vanishing points)
- $x$-axis … $D=(1,0,0,0)$ is imaged as $p_1=PD$
Geometric interpretation of camera matrix entries

- $P^3T$ is the principal plane, the plane parallel to the image plane through the camera center
- $PX = (x, y, 0)$ all points on the principal plane have the following coordinates
- Plane condition, $P^3TX = 0$ (point X is on the plane if it fulfils the condition)
- Points on P2, $P^2TX = 0$, all points have coordinates $PX = (x, 0, z)$
- Plane P2 is defined by the line y=0 and the camera center
Point and line projection

- Point projection $x = PX$
- Line projection is more involved (line $l$ is a 4x4 matrix)
- Therefore indirect projection:
  \[ l' = x' \times y' = PX \times PY \]
  \[ L = \overline{XY} \]
Epipolar constraint

- An image point \( x \) lies on its corresponding epipolar line in the corresponding image.
Epipolar constraint – derivation by coplanarity condition

- Vector $p$ and $t$ define a plane
- Vector $p'$ and $t$ define also a plane
- Both planes must have the same normal
- What we seek is a relation between $p$ and $p'$
Epipolar constraint – derivation by coplanarity condition
Epipolar constraint – derivation by coplanarity condition

X world point

C

C'

p

p'

x

x'

(t x p')

[R|t]
Epipolar constraint – derivation by coplanarity condition
Epipolar constraint – derivation by coplanarity condition

\[ p'' = Rp + t \]

\( X \) world point

\( p \)

\( p' \)

\( t \)

\( C \)

\( C' \)

\( [R|t] \)
Epipolar constraint – derivation by coplanarity condition

\[ t \times p' = t \times p'' \]
\[ t \times p' = t \times (Rp + t) \]
\[ with \; p'' = Rp + t \]
\[ t \times p' = t \times Rp + t \times t \]
\[ p'^T (t \times p') = 0 \]
\[ p'^T (t \times Rp) = 0 \]

\[ p'^T [t] x Rp = 0 \]
\[ E \]
\[ p'^T Ep = 0 \]

E is called the Essential matrix
Fundamental matrix

- \( p,p' \) from the Essential matrix derivation are in normalized coordinates
- \( x,x' \) are image coordinates, \( x=Kp, x'=Kp \)
- By replacing \( p,p' \) with \( x,x' \) one gets the Fundamental matrix

\[
\begin{align*}
    p'^T E p &= 0 \\
    x'^T K^{-T} E K^{-1} x &= 0 \\
    x'^T F x &= 0 \\
    F &= K^{-T} E K^{-1}
\end{align*}
\]
Epipolar lines

- The corresponding line $l'$ to image coordinate $x$
- $l'$ is the line connecting the epipole $e'$ and the image coordinate $x'$
- Hypothesis: $l' = Fx$
- Point $x'$ must lie on $l'$, thus $x'^T l' = 0$
- Now $x'^T Fx = 0$
Stereo case

\[ R = I_{3 \times 3} \quad T = [T_x \ 0 \ 0]^T \]
Stereo case

\[ R = I_{3 \times 3} \]

\[ T = [T_x \ 0 \ 0]^T \]

\[ E = [T]_x R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_x \\ 0 & T_x & 0 \end{bmatrix} \]

\[
\begin{bmatrix}
    x' \\
    y' \\
    1
\end{bmatrix}
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & -T_x \\
    0 & T_x & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
    x' \\
    y' \\
    1
\end{bmatrix}
\begin{bmatrix}
    0 \\
    -T_x \\
    T_xy
\end{bmatrix} = 0
\]

\[-y'T_x + T_xy = 0\]
Triangulation

- Compute coordinates of world point $X$ given the measurements $x$, $x'$ and the camera projection matrices $P$ and $P'$
Triangulation

- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)
- $x \times (PX) = 0$ and $x' \times (P'X) = 0$
- Can be rewritten into equation system $AX = 0$ to solve for $X$
Triangulation

X world point

X

C

C'
Triangulation

\[ P = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix} \]

\[
x \times (PX) = 0 \text{ and } x' \times (P'X) = 0
\]

\[
x(P_3^TX) - (P_1^TX) = 0
\]

\[
y(P_3^TX) - (P_2^TX) = 0
\]

\[
x(P_2^TX) - y(P_1^TX) = 0
\]

\[
\begin{bmatrix}
xP_3^T - P_1^T \\
yP_3^T - P_2^T \\
x'P_3^T - P_1^T \\
y'P_3^T - P_2^T
\end{bmatrix} X = 0
\]
Camera pose estimation

- perspective-n-point problem
- Goal is to estimate camera matrix $P$ such that $m_1 = PM_1$
- $m_1, M_1, m_2, M_2, m_3, M_3$ are known
- Algebraic linear solution requires 6 3D-2D point correspondences, minimal nonlinear solution requires only 3
Camera pose estimation

- Derivation similar to Triangulation, but now entries of P are the unknowns instead of X
- Condition: Measurement vector x needs to have the same direction as projection of X (cross-product equals 0)
Camera pose estimation

- Derivation similar to Triangulation, but now entries of $P$ are the unknowns instead of $X$
- Condition: Measurement vector $x$ needs to have the same direction as projection of $X$ (cross-product equals 0)

\[ x \times (PX) = 0 \text{ for all pairs } x \leftrightarrow X \]
\[ y(P_3^TX) - w(P_2^TX) = 0 \]
\[ x(P_3^TX) - w(P_1^TX) = 0 \]
\[ x(P_2^TX) - y(P_1^TX) = 0 \]

\[
\begin{bmatrix}
0 & -wX^T & yX^T \\
-wX^T & 0 & xX^T \\
-yX^T & xX^T & 0
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
P_3
\end{bmatrix} = 0
\]
Recap - Learning goals

- Understand the geometric interpretation of the camera matrix
- Understand and derive the epipolar constraint
- Understand the geometric concept of the epipolar plane
- Understand triangulation
- Understand camera pose estimation