Polynomial Systems in Computer Vision

Many Computer Vision problems can be solved by finding the roots of a polynomial system:

- camera pose estimation from point correspondences;
- camera relative motion estimation from point correspondences;
- image distortion calibration;
- point triangulation;
- ...
Solving Polynomial Systems

- no general method;
Solving Polynomial Systems

- no general method;
- several mathematical tools exist. For a given problem, a tool can be more adapted than the others.
Gröbner Bases

- introduced in 1965 by Bruno Buchberger (now at the Johannes Kepler University in Linz) in his Ph.D. thesis (named after his advisor Wolfgang Gröbner) to study sets of polynomials
Let consider the following polynomial system:

\[ \begin{align*}
L_1 & \quad 2x^2 + y^2 - 2z + 3z^2 + 5 = 0 \\
L_2 & \quad x^2 + z + z^2 = 0 \\
L_3 & \quad x^2y^2 + y^2z^2 - 2 = 0
\end{align*} \]
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\end{align*}
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Hint: try to remove \( x \) from the first equation

Replace \( L_1 \) by \( L_1 - 2L_2 \):

\[
\begin{align*}
  L_1' & \quad y^2 - 4z + z^2 + 5 &= 0 \\
  L_2 & \quad x^2 + z + z^2 &= 0 \\
  L_3 & \quad x^2y^2 + y^2z^2 - 2 &= 0
\end{align*}
\]
A Real Polynomial System (continued)

\[ L_1' \quad \begin{cases} 
  y^2 - 4z + z^2 + 5 & = 0 \\
  x^2 + z + z^2 & = 0 \\
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\end{cases} \]
A Real Polynomial System (continued)

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\end{cases}
\end{align*}
\]

Hint: try to remove \( x \) from the second equation:
A Real Polynomial System (continued)

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\begin{align*}
L_1' & \quad \left\{ \begin{array}{l} 
y^2 - 4z + z^2 + 5 = 0 \\
x^2 + z + z^2 = 0 \\
x^2y^2 + y^2z^2 - 2 = 0 \\
y^2z + 2 = 0 \\
\end{array} \right. \\
L_2 & \quad x^2 + z + z^2 = 0 \\
L_3 & \quad x^2y^2 + y^2z^2 - 2 = 0 \\
L_4 & \quad y^2z + 2 = 0 \\
\end{align*}
\]

Hint: try to remove \(x\) from the second equation:

Adding \(y^2 L_2 - L_3\):

\[
\begin{align*}
L_1' & \quad \left\{ \begin{array}{l} 
y^2 - 4z + z^2 + 5 = 0 \\
x^2 + z + z^2 = 0 \\
x^2y^2 + y^2z^2 - 2 = 0 \\
y^2z + 2 = 0 \\
\end{array} \right. \\
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Hint: try to remove \( y \) from the first equation
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\begin{align*}
L'_{1} & \left\{ \begin{array}{ccc}
y^2 - 4z + z^2 + 5 & = & 0 \\
x^2 + z + z^2 & = & 0 \\
x^2y^2 + y^2z^2 - 2 & = & 0 \\
y^2z + 2 & = & 0 \\
z^2 - 4z^2 + z^3 - 2 & = & 0
\end{array} \right.
\end{align*}
\]

Hint: try to remove \( y \) from the first equation

Add \( zL'_{1} - L_{4} \):

\[
\begin{align*}
L'_{1} & \left\{ \begin{array}{ccc}
y^2 - 4z + z^2 + 5 & = & 0 \\
x^2 + z + z^2 & = & 0 \\
x^2y^2 + y^2z^2 - 2 & = & 0 \\
y^2z + 2 & = & 0 \\
5z - 4z^2 + z^3 - 2 & = & 0
\end{array} \right.
\end{align*}
\]
A Real Polynomial System (continued)

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\begin{align*}
L_1' & \quad y^2 - 4z + z^2 + 5 = 0 \\
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L_4 & \quad y^2 z + 2 = 0 \\
L_5 & \quad 5z - 4z^2 + z^3 - 2 = 0
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Hint: \( L_5 \) is a polynomial in \( z \) only
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Hint: \( L_5 \) is a polynomial in \( z \) only

\[ 5z - 4z^2 + z^3 - 2 = (z - 1)^2(z - 2) \]

Each possible value for \( z \) gives a new polynomial system in \( x \) and \( y \) only.
Solving a Univariate Polynomial

- closed form up to degree 4;
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- closed form up to degree 4;
- for higher degrees:
  - the companion matrix method: The companion matrix of 
    \( p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \) is

\[
C = \begin{bmatrix}
0 & 0 & \ldots & 0 & -a_0 \\
1 & 0 & \ldots & 0 & -a_1 \\
0 & 1 & \ldots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -a_{n-1}
\end{bmatrix}
\]
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    C = \begin{bmatrix}
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    1 & 0 & & -a_1 \\
    & 1 & 0 & -a_2 \\
    & & \ddots & \ddots \\
    & & & 1 & -a_{n-1}
    \end{bmatrix}.
    \]
    Its eigenvalues are the roots of \( p(z) \) (because \( p(z) \) is the characteristic polynomial \( \det(zI - C) \) of \( C \)).
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\]

Its eigenvalues are the roots of \( p(z) \) (because \( p(z) \) is the characteristic polynomial \( \det(zI - C) \) of \( C \)).
- Sturm’s bracketing method (slightly less stable but much faster).
Two Gröbner bases

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\{ y^2 - 4z + z^2 + 5, x^2 + z + z^2, x^2y^2 + y^2z^2 - 2, y^2z + 2, 5z - 4z^2 + z^3 - 2 \}
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\{y^2 - 4z + z^2 + 5, x^2 + z + z^2, 5z - 4z^2 + z^3 - 2\}
\]

is also a Gröbner basis.
A Gröbner basis is a set of polynomials \( \{g_1, \ldots, g_t\} \), such that the system

\[
\begin{align*}
  g_1(x_1, \ldots, x_n) &= 0 \\
  \vdots \\
  g_t(x_1, \ldots, x_n) &= 0
\end{align*}
\]

has the same solutions as the original one, but with some specific properties that make the new system easier to solve than the original one, OR AT LEAST USEFUL to solve the original one.
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Tools

We can create new equations from:

- linear combinations of existing equations.
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2x^2 + xy + y^2 + 1 &= 0 \\
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in matrix form:

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\begin{bmatrix}
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\end{bmatrix}
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\end{bmatrix} = 0.
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xy \\
y^2 \\
1
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\]

After Gauss-Jordan elimination:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
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\end{bmatrix}
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xy \\
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- *algebraic* combinations of existing equations.
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- *algebraic* combinations of existing equations.
- the remainder of polynomial divisions (used by Buchberger’s algorithm).
Monomials

**Definition.** A monomial in $x_1, \ldots, x_n$ is a product of the form:

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where all the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers, sometimes noted $x^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Examples: $x, x^2, x^2y, x^2yz^3$
Polynomials

**Definition.** A polynomial $f$ in $x_1, \ldots, x_n$ with coefficients in a field $k$ is a finite linear combination with coefficients in $k$ of monomials. A polynomial is written in the form

$$f = \sum_{\alpha} a_{\alpha} x^\alpha, \quad a_{\alpha} \in k$$

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- $a_{\alpha}$ the **coefficient** of the monomial $x^\alpha$. 
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with

- $a_\alpha$ the coefficient of the monomial $x^\alpha$.
- If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a term of $f$. 
Notations: \( k[x_1, \ldots, x_n] \)

**Notation.** The set of all polynomials in \( x_1, \ldots, x_n \) with coefficients in \( k \) is denoted \( k[x_1, \ldots, x_n] \).
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\( k[x] \) is the set of polynomials in one variable: \( x^2 - x \in k[x], \)
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$k[x]$ is the set of polynomials in one variable: $x^2 - x \in k[x]$, $x^3 + 4x \in k[x]$.

$k[x, y]$ is the set of polynomials in two variables: $x^2 - y \in k[x, y]$, $x^3 + 2xy + y^2 \in k[x, y]$. 
Definition. Given a nonzero polynomial \( f \in k[x] \), let

\[
f = a_0x^m + a_1x^{m-1} + \ldots + a_m,
\]

where \( a_i \in k \) and \( a_0 \neq 0 \).
Definition - Leading Term $\text{LT}(f)$

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We will write \( \text{LT}(f) = a_0 x^m \).
Dividing Multivariate Polynomials?

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The answer is yes, but we need to decide which term of a polynomial is the leading term.
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For example, what is the leading term of \( x^2 + xy + y^2 \)?
Dividing Multivariate Polynomials?

Is there a division for polynomials in several variables?

The answer is yes, but we need to decide which term of a polynomial is the leading term.

For example, what is the leading term of $x^2 + xy + y^2$?

To decide, we will define a *monomial order*. 
Monomial Order

A monomial order is any relation on the set of monomials $x^\alpha$ in $k[x_1,\ldots,x_n]$ satisfying:

1. $\succ$ is a total (linear) ordering relation: there is only one possible to order in increasing order under $\succ$.
2. $\succ$ is compatible with multiplication: if $x^\alpha \succ x^\beta$ and $x^\gamma$ is any monomial, then $x^\alpha x^\gamma = x^\alpha + \gamma \succ x^\beta x^\gamma = x^\beta + \gamma$.
3. $\succ$ is a well-ordering: every nonempty set of monomials has a smallest element under $\succ$. 
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Monomial Order on $k[x]$

The only monomial order on $k[x]$ is the degree order, given by:

$$\ldots > x^{n+1} > x^n > \ldots > x^2 > x > 1.$$
Monomial Orders on $k[x_1, \ldots, x_n]$

For polynomials in several variables, there are many choices of monomial orders.
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Let’s first define an order on the variables: $x_1 > x_2 > \ldots > x_n$ (this is not a monomial order), and $x > y > z$. 

Monomial Orders on $k[x_1, \ldots, x_n]$ - the Lexicographic Order $>_{lex}$

**Definition.** The lexicographic order: analogous to the ordering of words in a dictionary.
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For example, under this order $>_{lex}$:

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Formal definition: $x^\alpha >_{lex} x^\beta$ if in the difference $\alpha - \beta$ (which belongs to $\mathbb{Z}^n$), the leftmost nonzero entry is positive.
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$$x^2yz^3 >_{lex} x^2z^4 \quad \text{or} \quad x^2z^4 >_{lex} x^2yz^3$$
Definition. The lexicographic order: analogous to the ordering of words in a dictionary.

For example, under this order $\geq_{\text{lex}}$:

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Formal definition: $x^\alpha >_{\text{lex}} x^\beta$ if in the difference $\alpha - \beta$ (which belongs to $\mathbb{Z}^n$), the leftmost nonzero entry is positive.

$x^2yz^3 >_{\text{lex}} x^2z^4$ or $x^2z^4 >_{\text{lex}} x^2yz^3$?

$\rightarrow x^2yz^3 >_{\text{lex}} x^2z^4$ because $(2,1,3) - (2,0,4) = (0,1,-1)$
Monomial Orders on $k[x_1, \ldots, x_n]$ - the Graded Reverse Lexicographic Order $>_{\text{grevlex}}$

Let $x^\alpha$ and $x^\beta$ be monomials in $k[x_1, \ldots, x_n]$. $x^\alpha >_{\text{grevlex}} x^\beta$ if:

1. $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$, or
2. $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and in the difference $\alpha - \beta$, the rightmost nonzero entry is negative.

For example:

- $xy^2 >_{\text{grevlex}} x^2 >_{\text{grevlex}} xy >_{\text{grevlex}} x >_{\text{grevlex}} y$
- $x^2y^2z^2 >_{\text{grevlex}} xy^4z$ or $xy^4z >_{\text{grevlex}} x^2y^2z^2$ because $1+4+1 = 2+2+2$ and $(1, 4, 1) - (2, 2, 2) = (-1, 2, -1)$.
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Monomial Orders on $k[x_1, \ldots, x_n]$ - the Graded Reverse Lexicographic Order $>_{\text{grevlex}}$

Let $x^\alpha$ and $x^\beta$ be monomials in $k[x_1, \ldots, x_n]$. \( x^\alpha >_{\text{grevlex}} x^\beta \) if:

- $\sum_i^n \alpha_i > \sum_i^n \beta_i$, or if
- $\sum_i^n \alpha_i = \sum_i^n \beta_i$ and in the difference $\alpha - \beta$, the rightmost nonzero entry is negative.

Under this order $>_{\text{grevlex}}$:

\[ xy^2 >_{\text{grevlex}} x^2 >_{\text{grevlex}} xy >_{\text{grevlex}} x >_{\text{grevlex}} y \]

\[ x^2 y^2 z^2 >_{\text{grevlex}} xy^4 z \quad \text{or} \quad xy^4 z >_{\text{grevlex}} x^2 y^2 z^2 \]
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Under this order \( >_{\text{grevlex}} \):

\[
x y^2 >_{\text{grevlex}} x^2 \quad >_{\text{grevlex}} x y \quad >_{\text{grevlex}} x \quad >_{\text{grevlex}} y
\]

\[
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\]

\[
\rightarrow x y^4 z >_{\text{grevlex}} x^2 y^2 z^2 \quad \text{because} \quad 1 + 4 + 1 = 2 + 2 + 2 \quad \text{and}
\]
Monomial Orders on $k[x_1, \ldots, x_n]$ - the Graded Reverse Lexicographic Order $>_\text{grevlex}$

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Under this order $>_\text{grevlex}$:

- $xy^2 >_\text{grevlex} x^2 >_\text{grevlex} xy >_\text{grevlex} x >_\text{grevlex} y$

- $x^2y^2z^2 >_\text{grevlex} xy^4z$ or $xy^4z >_\text{grevlex} x^2y^2z^2$

$\rightarrow xy^4z >_\text{grevlex} x^2y^2z^2$ because $1 + 4 + 1 = 2 + 2 + 2$ and $(1,4,1) - (2,2,2) = (-1,2,-1)$
Monomial Orders

\[ x^3 y^2 z >_{\text{lex}} x^2 y^6 z^8 \]
\[ x^2 y^6 z^8 >_{\text{grevlex}} x^3 y^2 z \]

\[ x^2 y^2 z^2 >_{\text{lex}} xy^4 z \]
\[ xy^4 z >_{\text{grevlex}} x^2 y^2 z^2 \]
Why Several Orders?

Computing Gröbner bases with $>_{\text{grevlex}}$ is usually more efficient.
Why Several Orders?

Computing Gröbner bases with $>_\text{grevlex}$ is usually more efficient. Computing Gröbner bases with $>_\text{lex}$ yields a polynomial system that can be easily solved.
If

- we use the monomial order $>_\text{lex}$ to compute a Gröbner basis
- the solution set is finite,

then a univariate polynomial (in the last variable) is in the basis.
Cool

If

- we use the monomial order $\succ_{lex}$ to compute a Gröbner basis
- the solution set is finite,

then a univariate polynomial (in the last variable) is in the basis.

For example, the Gröbner basis for $\langle x^2 - y^2 + 1, xy - 1 \rangle$ is $\langle y^4 - y^2 - 1, x - y^3 + y \rangle$. 
Cool

If

- we use the monomial order $>_\text{lex}$ to compute a Gröbner basis
- the solution set is finite,

then a univariate polynomial (in the last variable) is in the basis.

For example, the Gröbner basis for $\langle x^2 - y^2 + 1, \ xy - 1 \rangle$ is $\langle y^4 - y^2 - 1, \ x - y^3 + y \rangle$.

The system

\[
\begin{align*}
    x^2 - y^2 &+ 1 = 0 \\
    xy - 1 & = 0
\end{align*}
\]

has the same solutions as the system:

\[
\begin{align*}
    y^4 - y^2 &+ -1 = 0 \\
    x - y^3 + y & = 0
\end{align*}
\]

but the latter is much simpler to solve.
A More Ugly Example

A Gröbner basis for

\[
\begin{align*}
    x^2 - 2xz + 5 &= 0 \\
    xy^2 + yz + 1 &= 0 \\
    3y^2 - 8xz &= 0
\end{align*}
\]

under \( >_{\text{lex}} \) is
A More Ugly Example

A Gröbner basis for

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\begin{align*}
  x^2 - 2xz + 5 &= 0 \\
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\end{align*}
\]

under $>_\text{lex}$ is

\[
\{-81 + 4320z - 86400z^2 + 766272z^3 - 2513488z^4 - 295680z^5 - 242496z^6 + 61440z^8, -2472389942760 + 1450790919y + 98722479369600z - 1312504296363936z^2 + 5756399991700688z^3 + 711670127441280z^4 + 549519027506496z^5 - 10326680985600z^6 - 139421921341440z^7, 6503592729600 + 1450790919x - 257416379643438z + 3400639490020320z^2 - 14857079919551480z^3 - 1835782187164800z^4 - 1418473727285760z^5 + 26347944960000z^6 + 359882180198400z^7\}
\]
First algorithm to compute a Gröbner basis: the Buchberger algorithm.

More recent algorithms are more efficient (\(F4\) and \(F5\) algorithms by Faugère).

In (Kukelova, 2008):

1. Start with \(d \leftarrow 1\);
2. Multiply each equation of the current system by every possible monomial of degree \(d\);
3. Simplify the system with Gauss-Jordan elimination;
4. If not a Gröbner basis, set \(d \leftarrow d + 1\), and iterate from 1.
Computation Steps for (Stewenius, 2005)
Albeit

Unfortunately, computation of Gröbner bases under the lexicographic ordering ($>_{lex}$) is often intractable for real problems.

Using the graded reverse lexicographical ordering ($>_{grevlex}$) usually yields more tractable computations.

Unfortunately, the resulting polynomial system is not necessarily easy to solve.

Fortunately, other properties of Gröbner bases can be used to find the solutions.
>lex versus >grevlex: Example

Computing a Gröbner basis for

\[
\begin{align*}
    d_1^2 + Ad_1d_2 + d_2^2 - F^2 &= 0 \\
    d_1^2 + Bd_1d_3 + d_3^2 - F^2 &= 0 \\
    d_2^2 + Cd_2d_3 + d_3^2 - G^2 &= 0 \\
    d_2^2 + Dd_2d_4 + d_4^2 - F^2 &= 0 \\
    d_3^2 + Ed_3d_4 + d_4^2 - F^2 &= 0
\end{align*}
\]

under >grevlex: less than a second (but 130 polynomials in a 96Kb text file).

under >lex: more than a week
Further Reading and References I


Further Reading and References II


