Mathematical Principles in Vision and Graphics: Solving Polynomial Systems

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Polynomial Systems in Computer Vision

Many Computer Vision problems can be solved by finding the roots of a polynomial system:

- camera pose estimation from point correspondences;
- camera relative motion estimation from point correspondences;
- image distortion calibration;
- point triangulation;
- ...

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Solving Polynomial Systems

- no general method;
- several mathematical tools exist. For a given problem, a tool can be more adapted than the others.

Gröbner Bases

- introduced in 1965 by Bruno Buchberger (now at the Johannes Kepler University in Linz) in his Ph.D. thesis (named after his advisor Wolfgang Gröbner) to study sets of polynomials

A Polynomial System

Let consider the following polynomial system:

$$\begin{array}{l} L_1 \\ L_2 \\ L_3 \end{array} \left\{ \begin{array}{rcl} 2x^2 + y^2 - 2z + 3z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \end{array} \right.$$

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Replace L_1 by $L_1 - 2L_2$:

$$\begin{array}{c} L_1' \\ L_2 \\ L_3 \end{array} \begin{cases} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \end{cases}$$

A Real Polynomial System (continued)

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L_2 \\
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Hint: try to remove x from the second equation:

Adding $y^2L_2 - L_3$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \end{array} \begin{cases} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \end{cases}$$

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Add $zL_1' - L_4$:

$$\begin{array}{l} L_1' \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \begin{cases} y^2 - 4z + z^2 + 5 & = & 0 \\ x^2 + z + z^2 & = & 0 \\ x^2y^2 + y^2z^2 - 2 & = & 0 \\ y^2z + 2 & = & 0 \\ 5z - 4z^2 + z^3 - 2 & = & 0 \end{cases}$$

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L_3 \\
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\end{array}
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$5z - 4z^2 + z^3 - 2 = (z-1)^2(z-2)$

Each possible value for z gives a new polynomial system in x and y only.

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$$\mathbf{C} = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & 0 & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{n-1} \end{bmatrix}.$$

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    Its eigenvalues are the roots of $p(z)$ (because $p(z)$ is the characteristic polynomial $\det(z\mathbf{I} - \mathbf{C})$ of $\mathbf{C}$).
  - Sturm's bracketing method (slightly less stable but much faster).

Two Gröbner bases

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is also a Gröbner basis.

A Gröbner basis is a set of polynomials $\{g_1, \ldots, g_t\}$, such that the system

$$\left\{ \begin{array}{rcl} g_1(x_1, \ldots, x_n) & = & 0 \\ & \cdots & \\ g_t(x_1, \ldots, x_n) & = & 0 \end{array} \right.$$

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has the same solutions as the original one,

but with some specific properties that make the new system easier to solve than the original one, OR AT LEAST USEFUL to solve the original one.

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  After Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \\ 1 \end{bmatrix} = 0 \,.$$

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- the remainder of polynomial divisions (used by Buchberger's algorithm).

Monomials

Definition. A **monomial** in $x_1, \ldots, x_n$ is a product of the form:

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots \cdot x_n^{\alpha_n},$$

where all the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers, sometimes noted $\mathbf{x}^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Examples: x, x^2, x^2y, x^2yz^3

Polynomials

Definition. A **polynomial** f in $x_1, \ldots, x_n$ with coefficients in a field k is a finite linear combination with coefficients in k of monomials. A polynomial is written in the form

$$f = \sum_{\alpha} a_\alpha \mathbf{x}^\alpha, \quad a_\alpha \in k$$

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- a_α the **coefficient** of the monomial $\mathbf{x}^\alpha$.
- If $a_\alpha \neq 0$, then we call $a_\alpha \mathbf{x}^\alpha$ a **term** of f.

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$k[x, y]$ is the set of polynomials in two variables: $x^2 - y \in k[x, y]$, $x^3 + 2xy + y^2 \in k[x, y]$.

Definition - Leading Term $\mathrm{LT}(f)$

Definition. Given a nonzero polynomial $f \in k[x]$, let

$$f = a_0 x^m + a_1 x^{m-1} + \ldots + a_m \,,$$

where $a_i \in k$ and $a_0 \neq 0$.

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We will write $\mathrm{LT}(f) = a_0x^m$.

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To decide, we will define a *monomial order*.

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Monomial Order on $k[x]$

The only monomial order on $k[x]$ is the degree order, given by:

$$\ldots > x^{n+1} > x^n > \ldots > x^2 > x > 1.$$

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Let's first define an order on the variables: $x_1 > x_2 > \ldots > x_n$ (this is not a monomial order), and $x > y > z$.

Monomial Orders on $k[x_1, \ldots, x_n]$ - the Lexicographic Order $>_{lex}$

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$\rightarrow x^2yz^3 >_{lex} x^2z^4$ because $(2,1,3) - (2,0,4) = (0, \mathbf{1}, -1)$

Monomial Orders on $k[x_1, \ldots, x_n]$ - the Graded Reverse Lexicographic Order $>_{grevlex}$

Let x^α and x^β be monomials in $k[x_1, \ldots, x_n]$. $x^\alpha >_{grevlex} x^\beta$ if:

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$$xy^2 >_{grevlex} x^2 >_{grevlex} xy >_{grevlex} x >_{grevlex} y$$

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$\rightarrow xy^4z >_{grevlex} x^2y^2z^2$ because $1+4+1 = 2+2+2$ and

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- $\sum_i^n \alpha_i = \sum_i^n \beta_i$ and in the difference $\alpha - \beta$, the *rightmost* nonzero entry is *negative*.

Under this order $>_{grevlex}$:

$$xy^2 >_{grevlex} x^2 >_{grevlex} xy >_{grevlex} x >_{grevlex} y$$

$x^2y^2z^2 >_{grevlex} xy^4z$ or $xy^4z >_{grevlex} x^2y^2z^2$?

$\rightarrow xy^4z >_{grevlex} x^2y^2z^2$ because $1+4+1 = 2+2+2$ and $(1,4,1) - (2,2,2) = (-1, 2, -\mathbf{1})$

Monomial Orders

$$x^3y^2z >_{lex} x^2y^6z^8$$

$$x^2y^6z^8 >_{grevlex} x^3y^2z$$

$$x^2y^2z^2 >_{lex} xy^4z$$

$$xy^4z >_{grevlex} x^2y^2z^2$$

Why Several Orders?

Computing Gröbner bases with $>_{grevlex}$ is usually more efficient.

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Computing Gröbner bases with $>_{grevlex}$ is usually more efficient.

Computing Gröbner bases with $>_{lex}$ yields a polynomial system that can be easily solved.

Cool

If

- we use the monomial order $>_{lex}$ to compute a Gröbner basis and
- the solution set is finite,

then a univariate polynomial (in the last variable) is in the basis.

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For example, the Gröbner basis for $\langle x^2 - y^2 + 1,\ xy - 1 \rangle$ is $\langle y^4 - y^2 - 1,\ x - y^3 + y \rangle$.

The system

$$\begin{cases} x^2 - y^2 + 1 & = & 0 \\ xy - 1 & = & 0 \end{cases}$$

has the same solutions as the system:

$$\begin{cases} y^4 - y^2 + -1 & = & 0 \\ x - y^3 + y & = & 0 \end{cases}$$

but the latter is much simpler to solve.

A More Ugly Example

A Gröbner basis for

$$\left\{\begin{array}{rcl} x^2-2xz+5 & = & 0 \\ xy^2+yz+1 & = & 0 \\ 3y^2-8xz & = & 0 \end{array}\right.$$

under $>_{lex}$ is

A More Ugly Example

A Gröbner basis for

$$\begin{cases} x^2 - 2xz + 5 & = & 0 \\ xy^2 + yz + 1 & = & 0 \\ 3y^2 - 8xz & = & 0 \end{cases}$$

under $>_{lex}$ is

$$\{-81 + 4320z - 86400z^2 + 766272z^3 - 2513488z^4 - 295680z^5 - 242496z^6 + 61440z^8, -2472389942760 + 1450790919y + 987224793696600z - 1312504296363936z^2 + 575639999170068z^3 + 711670127441280z^4 + 549519027506496z^5 - 103266809856000z^6 - 139421921341440z^7, 6503592729600 + 1450790919x - 257416379643438z + 340063949002032z^2 - 148570799195514800z^3 - 183578218716480z^4 - 141847372728576z^5 + 26347944960000z^6 + 359882180198400z^7\}$$

Algorithms to Compute a Gröbner basis

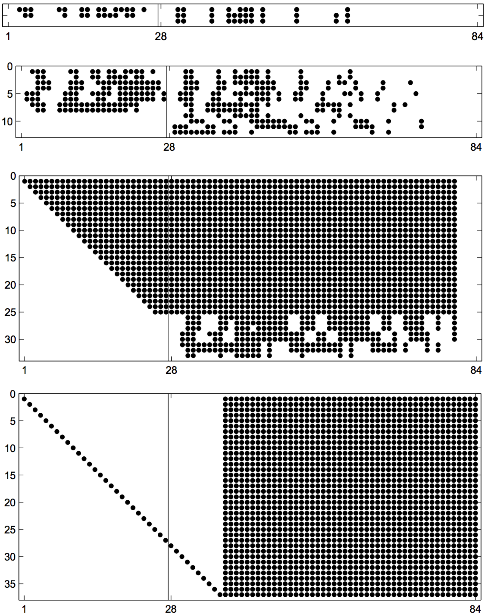
First algorithm to compute a Gröbner basis: the Buchberger algorithm.

More recent algorithms are more efficient ($F4$ and $F5$ algorithms by Faugère).

In (Kukelova, 2008):

1. Start with $d \leftarrow 1$;
2. Multiply each equation of the current system by every possible monomial of degree d;
3. Simplify the system with Gauss-Jordan elimination;
4. If not a Gröbner basis, set $d \leftarrow d + 1$, and iterate from 1.

Computation Steps for (Stewenius, 2005)



Albeit

Unfortunately, computation of Gröbner bases under the lexicographic ordering ($>_{lex}$) is often intractable for real problems.

Using the graded reverse lexicographical ordering ($>_{grevlex}$) usually yields more tractable computations.

Unfortunately, the resulting polynomial system is not necessarily easy to solve.

Fortunately, other properties of Gröbner bases can be used to find the solutions.

$>_{lex}$ versus $>_{grevlex}$: Example

Computing a Gröbner basis for

$$\left\{\begin{array}{rcl}
d_1^2 + Ad_1d_2 + d_2^2 - F^2 & = & 0\\
d_1^2 + Bd_1d_3 + d_3^2 - F^2 & = & 0\\
d_2^2 + Cd_2d_3 + d_3^2 - G^2 & = & 0\\
d_2^2 + Dd_2d_4 + d_4^2 - F^2 & = & 0\\
d_3^2 + Ed_3d_4 + d_4^2 - F^2 & = & 0
\end{array}\right.$$

under $>_{grevlex}$: less than a second (but 130 polynomials in a 96Kb text file).

under $>_{lex}$: more than a week

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