Camera Drones
Lecture – Sensor Fusion

Prof. Friedrich Fraundorfer

WS 2023

Outline

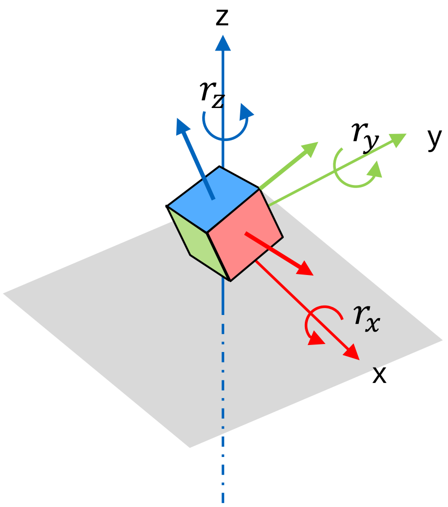
- Mathematical model for an IMU
- Sensor fusion methods
- Sensor fusion with a Kalman filter

Mathematical model for an IMU

- IMU measures attitude with respect to Earth's gravitational field (6DOF-IMU)
- IMU is a combination of an accelerometer (3-axis) and a gyroscope (3-axis)
- Attitude W will be represented through Euler angles in a world coordinate system (roll, pitch, yaw).

$$\boldsymbol{r} = \left[r_x, r_y, r_z\right]^T \quad [rad]$$

$$W = euler(r_x, r_y, r_z)$$



Mathematical model for an IMU

- Gyroscope:

$$\boldsymbol{\omega} = \hat{\boldsymbol{\omega}} + \boldsymbol{b}_g + \boldsymbol{n}_g$$

$$\boldsymbol{\omega} = [\boldsymbol{\omega}_x, \boldsymbol{\omega}_y, \boldsymbol{\omega}_z]^T \left[\frac{rad}{s}\right]$$

$\boldsymbol{\omega}$ … measurement of rotational velocity (angular rate)

$\hat{\boldsymbol{\omega}}$… true rotational velocity

$\boldsymbol{b}_g$ … bias of angular measurement

$\boldsymbol{n}_g$ … noise of the measurement of the rotational velocity

- Accelerometer:

$$\boldsymbol{a} = W^T(\hat{\boldsymbol{a}} - \boldsymbol{g}) + \boldsymbol{b}_a + \boldsymbol{n}_a$$

$$\boldsymbol{a} = [\boldsymbol{a}_x, \boldsymbol{a}_y, \boldsymbol{a}_z]^T \left[\frac{m}{s^2}\right]$$

$\boldsymbol{a}$ … measurement of acceleration

$\hat{\boldsymbol{a}}$… true acceleration

$\boldsymbol{b}_a$ … bias of acceleration measurement

$\boldsymbol{n}_a$ … noise of acceleration measurement

g …. acceleration due to gravity

W…. orientation of sensor in world frame

Measuring attitude by integrating rotational velocities

- Attitude update

$$W_{t+1} = euler(\triangle\, \boldsymbol{t}\, \boldsymbol{\omega}_t) W_t$$

- Initial value W_0 must be defined
- This integration leads to drift. Due to bias and noise the attitude deviation will increase continuously.
- Solution: Fusion with measurements of the accelerometer

Measuring attitude by integrating rotational velocities

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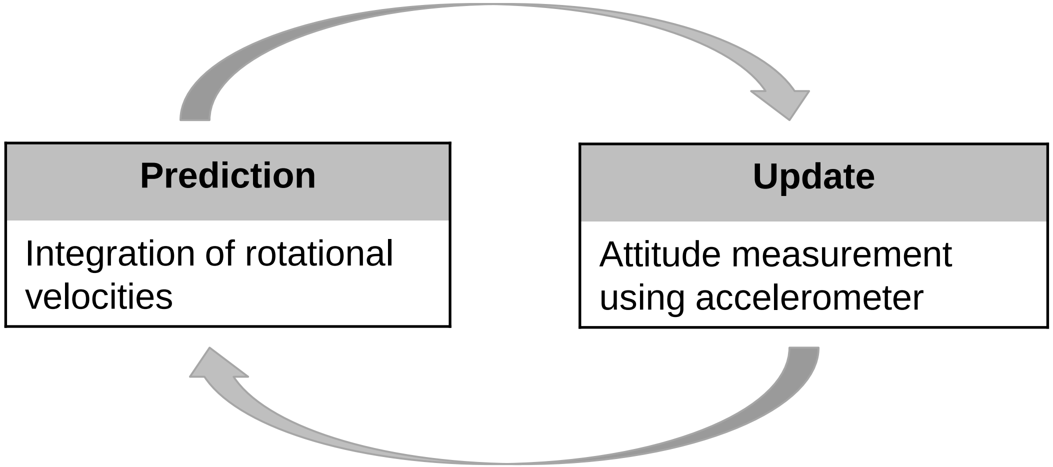
Sensor fusion principle:

1. **Attitude estimation through integration of rotational velocities**
2. **Correction of attitude estimate through attitude measurements of accelerometer**

Sensor fusion methods

- Complementary filter
- Madgwick filter
- Mahoney filter
- Kalman filter

Sensor fusion with a Kalman filter



- State vector $x_t = \begin{bmatrix} r_{x,t} \\ r_{y,t} \\ r_{z,t} \end{bmatrix}$

Kalman filter: Prediction

- Prediction of estimated mean and co-variance of state vector by integration of gyroscope measurements
- Constant attitude model

$$\hat{\mu}_{t+1} = A_{t+1}\hat{\mu}_t + B_{t+1}u_{t+1}$$

$$\hat{\Sigma}_{t+1} = A_{t+1}\hat{\Sigma}_t {A_{t+1}}^T + Q_{t+1}$$

$$\boldsymbol{A}_{t+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q…. covariance matrix of dynamic noise

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- Gyroscope measurements

$$\boldsymbol{u}_{t+1} = \boldsymbol{\omega}_t$$

- Integration of gyroscope measurements

$$\boldsymbol{B}_{t+1} = \begin{bmatrix} \triangle t & 0 & 0 \\ 0 & \triangle t & 0 \\ 0 & 0 & \triangle t \end{bmatrix}$$

Kalman filter: Update

- Update estimated mean and co-variance of state vector by comparison with attitude measurement of accelerometer

$$K_{t+1} = \hat{\Sigma}_{t+1} C^T \left(C \hat{\Sigma}_{t+1} C^T + R \right)^{-1}$$

$$\mu_{t+1} = \hat{\mu}_{t+1} + K_{t+1} (z_{t+1} - C \hat{\mu}_{t+1})$$

$$\Sigma_{t+1} = \hat{\Sigma}_{t+1} - K_{t+1} C \hat{\Sigma}_{t+1}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

R…. co−variance matrix of measurement noise

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R…. co−variance matrix of measurement noise

- Attitude measurement of accelerometer

$$z_{t+1} = \begin{bmatrix} r_{x,t} \\ r_{y,t} \\ 0 \end{bmatrix}$$

$$r_{x,t} = \operatorname{atan}\left(\frac{a_y}{\sqrt{{a_x}^2 + {a_z}^2}}\right)$$

$$r_{y,t} = \operatorname{atan}\left(\frac{a_x}{\sqrt{{a_y}^2 + {a_z}^2}}\right)$$

Learning goals

- What is measured by an IMU and how is it built?
- What is the mathematical model of an IMU?
- How does the integration of rotational velocities work?
- How is sensor fusion using a Kalman filter done?