Numerical simulation of fluid flow with fast Boundary-Domain Integral Method

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GAMM2022, Aachen 15.8.-19.8.2022

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Introduction

- Numerical investigation of fluids and solids (particle flow, mass transfer, heat transfer, stresses etc.)
- We have different numerical methods in order to solve a system of equations FEM, FVM and BEM
- The difference is in the accuracy of the numerical methods and computational cost
- The optimal numerical method has to have a high accuracy and low computational cost
- Most examples in practical engineering are non-linear

Boundary-Domain Integral Method

- The Boundary-Domain Integral Method is a numerical method which is used to solve Partial Differential Equations (PDE)
- The method is based on the Green's second identity
- The computational cost of the BDIM scales of $\mathcal{O}(nm)$, thus the application is limited to a small number of examples
- In the past different approximation methods were introduced, to reduce the computational demand to the order of $\mathcal{O}(m \log m)$ or $\mathcal{O}(m)$
- We use the $\mathcal{H}$-matrix or the $\mathcal{H}^2$-matrix form to approximate the full matrices
- To the $\mathcal{H}$-matrix an approximation is employed: SVD(Singular Value Decomposition), ACA(Adaptive Cross Approximation)
- Here we employ the $\mathcal{H}^2$-matrix form in combination with the ACA.

Governing equations

- The velocity-vorticity formulation of the Navier-Stokes equations

$$\vec{\nabla}_x \times \vec{\omega} + \nabla_x^2 \vec{v} = 0,$$

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla}_x)\vec{\omega} = (\vec{\omega} \cdot \vec{\nabla}_x)\vec{v} + \frac{1}{Re}\nabla_x^2 \vec{\omega} - \frac{Ra}{PrRe^2}\vec{\nabla}_x \times \vec{g}\theta$$

- For lid-driven cavity

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla}_x)\vec{\omega} = (\vec{\omega} \cdot \vec{\nabla}_x)\vec{v} + \frac{1}{Re}\nabla_x^2 \vec{\omega}$$

- $Re = \frac{v_0 L}{\vartheta_0}$ is the Reynolds number, where v_0 is the characteristic velocity, L the dimension and ϑ_0 kinematic viscosity

Modified Helmholtz equation

- The modified Helmhotz kinematic equation has this form

$$\frac{\vec{v}(\vec{x},t_n)-\vec{v}(\vec{x},t_{n-1})}{\Delta t}=\nabla_x^2\vec{v}(\vec{x},t_n)+\vec{\nabla}_x\times\vec{\omega}(\vec{x},t_n)$$

- Rearrangement and the abbreviations $\mu^2=\frac{1}{\Delta t}$, $b(\vec{x},t_n)=\frac{1}{\Delta t}\vec{v}(\vec{x},t_{n-1})+\vec{\nabla}_x\times\vec{\omega}(\vec{x},t_n)$ results in the Yukawa kinematic equation

$$\left(\nabla_x^2-\mu^2\right)\vec{v}(\vec{x},t_n)+b(\vec{x},t_n)=0$$

- Integral form of the modified Helmholtz equation

$$c(\vec{y})\vec{v}(\vec{y},t_n)+\int_\Gamma \vec{v}(\vec{x},t_n)q^*(\vec{y},\vec{x})d\Gamma=\int_\Gamma \vec{v}(\vec{x},t_n)\times[\vec{n}\times\vec{\nabla}]u^*(\vec{y},\vec{x})d\Gamma$$
$$+\int_\Omega \vec{\omega}(\vec{x},t_n)\times\vec{\nabla}u^*(\vec{y},\vec{x})d\Omega+\int_\Omega u^*(\vec{y},\vec{x})\mu^2\vec{v}(\vec{x},t_{n-1})d\Omega\ \ \forall\vec{y}\in\Gamma$$

- For the case that $\mu\rightarrow 0$ the equations reforms back to the kinematic equation form

Discretiztaion

- The respective shape functions are

$$\vec{v}(\vec{x}) \approx \sum_{a=1}^{9} \vec{v}_a \varphi_a(\vec{x}), \qquad \vec{q}(\vec{x},t) \approx \sum_{b=1}^{4} \vec{q}_b \psi_b(\vec{x}), \qquad \vec{\omega}(\vec{x}) \approx \sum_{c=1}^{27} \vec{\omega}_c \Phi_c(\vec{x})$$

- The matrix form of the modified Helmhotz equation for the solution of the domain velocity

$$[H]\{v_i\} = \left[H_k^t\right]\{v_j\} - \left[H_j^t\right]\{v_k\} + [D_k]\{\omega_j\} - [D_j]\{\omega_k\} + [B]\left\{v_i^{n-1}\right\}$$

- The matrix form for the solution of the boundary vorticity

$$\begin{aligned} &([n_i][D_i] + [n_j][D_j] + [n_k][D_k])\{\omega_i\}_\Gamma = \\ &([n_j][H_j^t] + [n_k][H_k^t])\{v_i\}_\Gamma + ([n_j][H] - [n_k][H_i^t])\{v_k\}_\Gamma \\ &-([n_i][H] + [n_j][H_i^t])\{v_j\}_\Gamma + [n_k][D_i]\{\omega_k\}_\Gamma + [n_j][D_i]\{\omega_j\}_\Gamma \\ &+[n_i][D_i]\{\omega_i\}_\Gamma - ([n_j][D_j]_{\Omega/\Gamma} + [n_k][D_k]_{\Omega/\Gamma})\{\omega_i\}_{\Omega/\Gamma} + [n_j][D_i]_{\Omega/\Gamma}\{\omega_j\}_{\Omega/\Gamma} \\ &+[n_k][D_i]_{\Omega/\Gamma}\{\omega_k\}_{\Omega/\Gamma} - [n_j][B]\left\{v_k^{n-1}\right\} + [n_k][B]\left\{v_j^{n-1}\right\}, \; i = 1,\,2,\,3 \end{aligned}$$

- There are 8 matrices of size $n \times m$ and $n \times m$

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([n_j][H_j^t] + [n_k][H_k^t])\{v_i\}_\Gamma + ([n_j][H] - [n_k][H_i^t])\{v_k\}_\Gamma \\
-([n_i][H] + [n_j][H_i^t])\{v_j\}_\Gamma + [n_k][D_i]\{\omega_k\}_\Gamma + [n_j][D_i]\{\omega_j\}_\Gamma \\
+[n_i][D_i]\{\omega_i\}_\Gamma - ([n_j][D_j]_{\Omega/\Gamma} + [n_k][D_k]_{\Omega/\Gamma})\{\omega_i\}_{\Omega/\Gamma} + [n_j][D_i]_{\Omega/\Gamma}\{\omega_j\}_{\Omega/\Gamma} \\
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\end{aligned}$$

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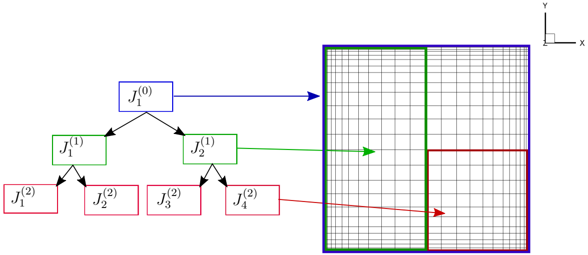
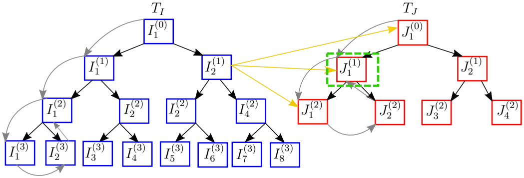
$\mathcal{H}$-structure

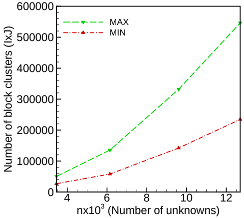
- Let us consider a matrix $[B]$ that is of the form $n \times m$
- We split the matrix into smaller matrices $[B]_{\hat{n}\times\hat{m}}$
- In order to form the $\mathcal{H}^2$-matrix and $\mathcal{H}$-matrix we build cluster trees with the bottom-up approach
- The cluster tree that is built from the boundary elements is T_J and the cluster tree that is built from the domain cells is T_I



$\mathcal{H}$-structure

- The block cluster tree is a combination of cluster tree T_I and T_J
- After the block clusters are formed, each one is tested for admissibility:
  - $min\{dim(I_j^{(i)}), dim(J_k^{(i)})\} \leq \eta\ dist(I_j^{(i)}, J_k^{(i)})$
  - $max\{dim(I_j^{(i)}), dim(J_k^{(i)})\} \leq \eta\ dist(I_j^{(i)}, J_k^{(i)})$
- The matrix is eliminated if Frobenius norm $||\hat{B}_{\hat{m}\times\hat{n}}|| \leq 10^{-15}$
- The integral kernel is approximated if $dist(I_j^{(i)}, J_k^{(i)}) \geq dist_m$





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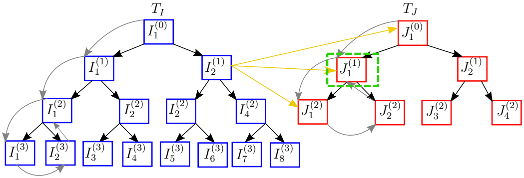
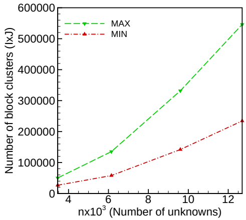
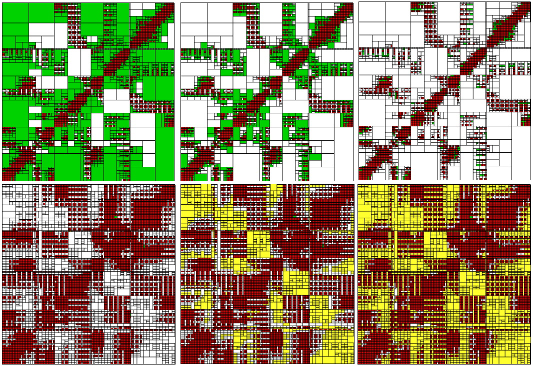




$\mathcal{H}$-structure

- Matrix $[B]_{n\times m}$ for value of $\mu = 20$ and $\mu = 50$
- Yelow matrices have the Frobenius norm $||\hat{B}_{\hat{n}\times\hat{m}}|| \leq 10^{-15}$, for the green matrices the integral kernel is not approximated and white matrices the integral kernel is approximated



Approximation of the kernel

- Approximation of the fundamental solution and its derivative with the Lagrange interpolation function:

$$u^{*}\left(\vec{y},\vec{x}\right)\approx\sum_{\iota=1}^{\alpha^3}\sum_{\kappa=1}^{\beta^3}\mathcal{L}_{\iota}\left(\vec{y}\right)u^{*}\left(\vec{y}_{\iota},\vec{x}_{\kappa}\right)\mathcal{L}_{\kappa}\left(\vec{x}\right)$$

$$\mathcal{L}_{\iota}(\vec{y})=\prod_{\iota_1\neq\ell}\frac{\xi^1-\xi^1_{\ell}}{\xi^1_{\iota_1}-\xi^1_{\ell}}\times\prod_{\iota_2\neq\ell}\frac{\xi^2-\xi^2_{\ell}}{\xi^2_{\iota_2}-\xi^2_{\ell}}\times\prod_{\iota_3\neq\ell}\frac{\xi^3-\xi^3_{\ell}}{\xi^3_{\iota_3}-\xi^3_{\ell}}\quad \iota_1,\iota_2,\iota_3,\ell=1,\dots,\alpha^3$$

$$q^{*}\left(\vec{y},\vec{x}\right)\approx\sum_{\iota=1}^{\alpha^3}\sum_{\kappa=1}^{\beta^3}\mathcal{L}_{\iota}\left(\vec{y}\right)u^{*}\left(\vec{y}_{\iota},\vec{x}_{\kappa}\right)n_i\frac{\partial\mathcal{L}_{\kappa}\left(\vec{x}\right)}{\partial x_i}$$

Approximation of the kernel

- The Lagrange interpolation of the fundamental solution gives this:

$$\hat{h}_{fa} = \sum_{\iota=1}^{\alpha^3} \sum_{\kappa=1}^{\beta^3} \mathcal{L}_\iota(\vec{y}_f)\, u^*(\vec{y}_\iota, \vec{x}_\kappa) \int_{supp(\varphi_a)} \varphi_a(\vec{x})\, n_i \frac{\partial \mathcal{L}_\kappa(\vec{x})}{\partial x_i} d\Gamma_a$$

$$\hat{h}^t_{ifa} = \sum_{\iota=1}^{\alpha^3} \sum_{\kappa=1}^{\beta^3} \mathcal{L}_\iota(\vec{y}_f)\, u^*(\vec{y}_\iota, \vec{x}_\kappa) \int_{supp(\varphi_a)} \varphi_a(\vec{x})[n_j \frac{\partial \mathcal{L}_\kappa(\vec{x})}{\partial x_k} - n_k \frac{\partial \mathcal{L}_\kappa(\vec{x})}{\partial x_j}] d\Gamma_a$$

$$\hat{d}^t_{ifc} = \sum_{\iota=1}^{\alpha^3} \sum_{\kappa=1}^{\beta^3} \mathcal{L}_\iota(\vec{y}_f)\, u^*(\vec{y}_\iota, \vec{x}_\kappa) \int_{supp(\Phi_c)} \Phi_c(\vec{x}) \frac{\partial \mathcal{L}_\kappa(\vec{x})}{\partial x_i} d\Omega_c$$

$$\hat{b}_{fc} = \sum_{\iota=1}^{\alpha^3} \sum_{\kappa=1}^{\beta^3} \mathcal{L}_\iota(\vec{y}_f)\, u^*(\vec{y}_\iota, \vec{x}_\kappa) \int_{supp(\Phi_c)} \Phi_c(\vec{x})\, \mathcal{L}_\kappa(\vec{x})\, d\Omega_c$$

Approximation of the kernel

- The matrix formulation of the admissible block cluster has this form after the approximation:

$$[\hat{H}] = [\hat{U}][\hat{S}][\hat{V}_H] \qquad [\hat{H}_i^t] = [\hat{U}][\hat{S}][\hat{V}_{Hi}^t]$$
$$[\hat{D}_i] = [\hat{U}][\hat{S}][\hat{V}_{Di}] \qquad [\hat{B}] = [\hat{U}][\hat{S}][\hat{V}_B]$$

- The matrices $[V_x]$ are compressed by employing the nested cluster basis:

$$\mathcal{L}_\iota(\vec{x}) = \sum_{\lambda=1}^{\gamma^3} \mathcal{L}_\iota(\vec{x}_\lambda)\mathcal{L}'_\lambda(\vec{x}), \qquad \vec{n}(\vec{x}) \cdot \vec{\nabla}_\eta \mathcal{L}_\kappa(\vec{x}) = \sum_{\lambda=1}^{\gamma^3} \mathcal{L}_\kappa(\vec{x}_\lambda)(\vec{n}(\vec{x}) \cdot \vec{\nabla}_x \mathcal{L}'_\lambda(\vec{x}))$$

- This gives the matrices $[T_x]$:

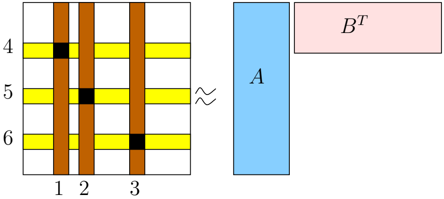
$$T_\lambda = \mathcal{L}'_\lambda(\vec{x}), \quad (T_H)_{a\lambda} = \int_{supp(\varphi_a)} \varphi_a(\vec{x}) n_i \frac{\partial \mathcal{L}'_\lambda(\vec{x})}{\partial x_i} d\Gamma_a$$

$$(T_{Hi}^t)_{a\lambda} = \int_{supp(\psi_i)} \varphi_a(\vec{x})[n_j \frac{\partial \mathcal{L}'_\lambda(\vec{x})}{\partial x_k} - n_k \frac{\partial \mathcal{L}'_\lambda(\vec{x})}{\partial x_j}] d\Gamma_a$$

$$(T_{Di})_{c\lambda} = \int \Phi_c(\vec{x}) \frac{\partial \mathcal{L}'_\lambda(\vec{x})}{\partial x_i} d\Omega_c \quad (T_B)_{c\lambda} = \int \Phi_c(\vec{x}) \mathcal{L}'_\lambda(\vec{x}) d\Omega_c$$

Approximation of the kernel

- To further reduce the memory cost the matrix $[S]$ is approximated with the ACA. The algorithm is of this form:
  - Set $R^0 = [\hat{S}]$
  - For $\ell = 1, 2, \ldots, k$
    1. $(i^*, j^*)^\ell = ArgMax|R^{\ell-1}|$
    2. $\tau^\ell = (R^{\ell-1}_{i^*,j^*})^{-1}$
    3. $\vec{a}_\ell = \tau^\ell R^{\ell-1}_{:,j^*}, \ \vec{b}_\ell = (R^{\ell-1}_{i^*,:})^T$
    4. $R^\ell = R^{\ell-1} - \vec{a}_\ell \vec{b}_\ell \ , \hat{S}^\ell = \hat{S}^{\ell-1} + \vec{a}_\ell \vec{b}_\ell$
  - If $(\left\|R^\ell\right\|_F \leq \varepsilon \left\|\hat{S}^\ell\right\|_F \vee \ell = k)$ Stop
  - EndFor



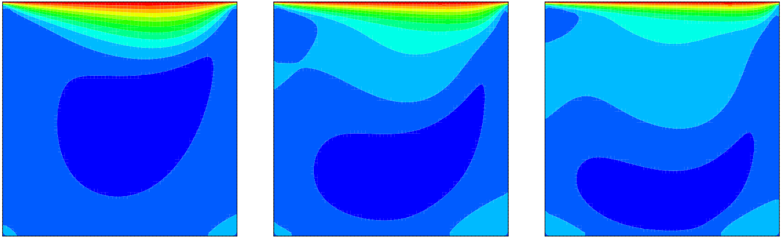
- Pleas note only the matrices $[T_x]$ have to be saved in memory. This reduces the complexity to linear $\mathcal{O}(m)$. However the CPU time for the matrix-vector product increases.

Numerical test

- Flow flow in a lid driven cavity was simulated
- The solved velocity and vorticity field was compared with the non-approximated and approximated version

$$RMS_\omega = \left(\frac{\sum_{i=1}^{n} (\omega_i - \tilde{\omega}_i)^2}{\sum_{i=1}^{n} (\omega_i)^2} \right)^{\frac{1}{2}}$$

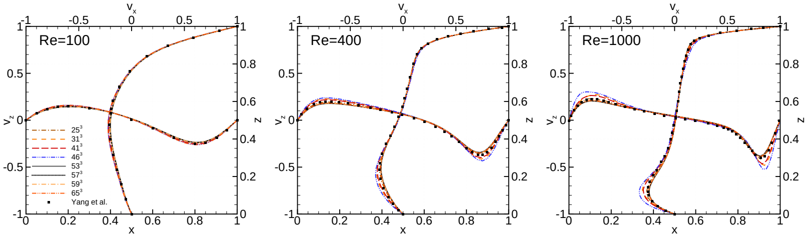
- The Reynolds number was changed from 100 to 400 and 1000



- The velocity v_x on plane x-z at y=0.5.

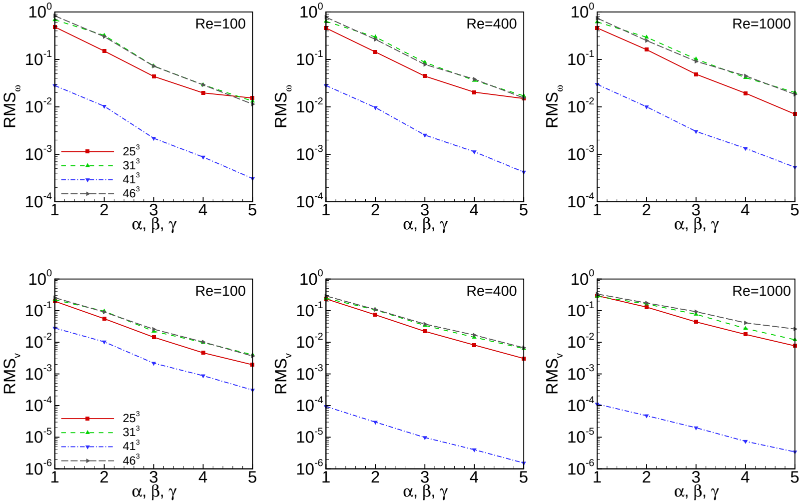
Results

- Velocity profile for the three Reynolds numbers and different mesh densities
- J. Yen Yang, S. Chang Yang, Y. Nan Chen, C. An Hsu, Implicit Weighted ENO Schemes for the Three-Dimensional Incompressible Navier – Stokes Equations, Journal of Computational Physics 487(1998) 464–487



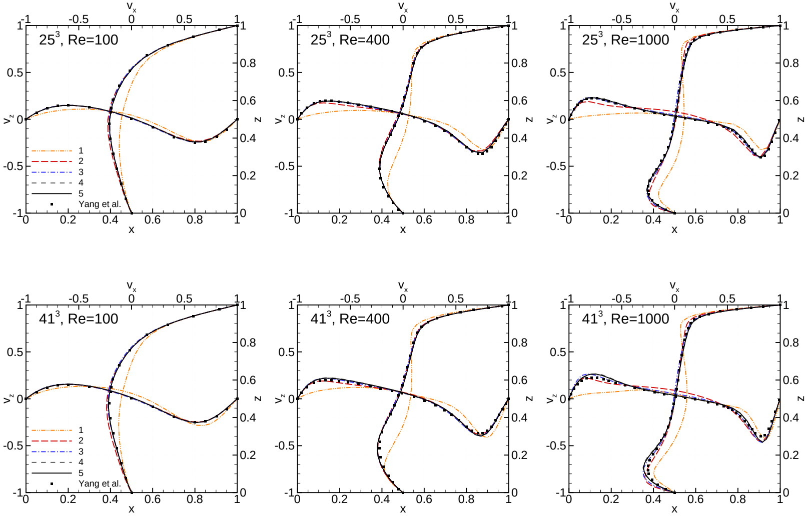
Results

- Influence of the $\mathcal{H}^2$-approximation without the ACA on the solution of the vorticity and velocity field:



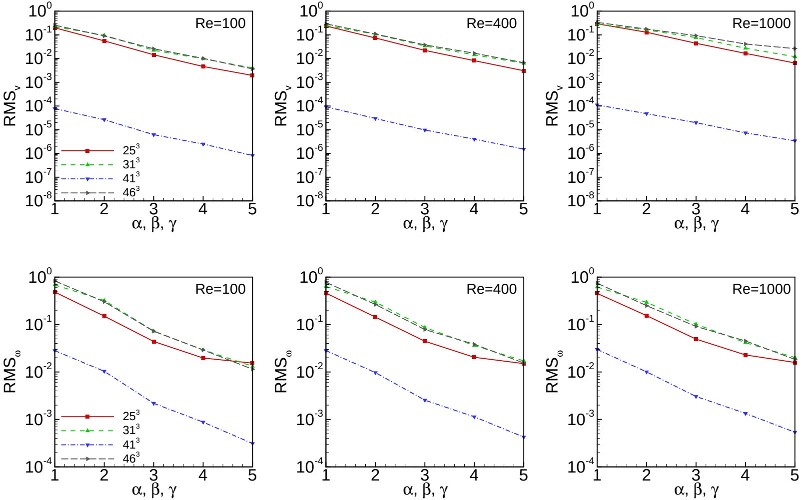
Results

- The velocity profile solved with $\mathcal{H}^2$-approximation without the ACA compression:



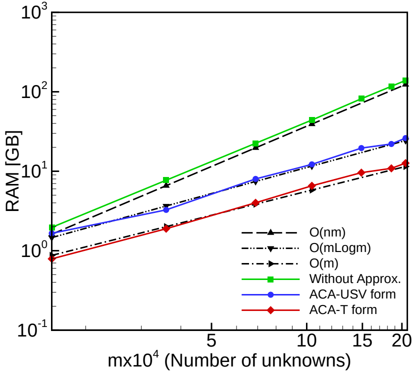
Results

- The velocity profile solved with $\mathcal{H}^2$-approximation with the ACA compression
- The ACA stopping condition is $\varepsilon = 10^{-8}$



Conclusion

- The presented method can reduce the complexity to linear $\mathcal{O}(m)$
- This allows to solve cases with a denser mesh
- The amount of computer memory needed to store the matrices depending on the mesh density:



- The accuracy of the solution depends interpolation accuracy and ACA stopping condition
- CPU-time to solve a case increases depending on the matrix approximation

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