Fast formation and assembly for spline-based fictitious domain methods

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Outline

1. Fast formation and assembly
2. Application to fictitious domains
3. Numerical results

Fast formation and assembly – key ingredients

- Sum factorization (2D tensor product basis) (P. Antolin et al., CMAME 2015)

$$m_{\boldsymbol{i},\boldsymbol{j}} = \int_{\Omega} B_{\boldsymbol{i}}(\boldsymbol{\xi}) B_{\boldsymbol{j}}(\boldsymbol{\xi}) c(\boldsymbol{\xi}) d\boldsymbol{\xi}$$

$$= \int_{\Omega_1} B_{i_1}(\xi_1) B_{j_1}(\xi_1) \times \left[\int_{\Omega_2} B_{i_2}(\xi_2) B_{j_2}(\xi_2) c(\xi_1, \xi_2) d\xi_2 \right] d\xi_1$$

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- Weighted quadrature $\mathbb{Q}_i$ (for smooth splines) (F. Calabrò et al., CMAME 2017)

$$\mathbb{Q} = \sum_k B_i(x_k) B_j(x_k) w_k := \int_{\Omega} B_i(\xi) B_j(\xi) d\xi$$

$$\Rightarrow \mathbb{Q}_i = \sum_k B_j(x_k) w_{k,i} := \int_{\Omega} B_j(\xi) \left(B_i(\xi) d\xi \right)$$

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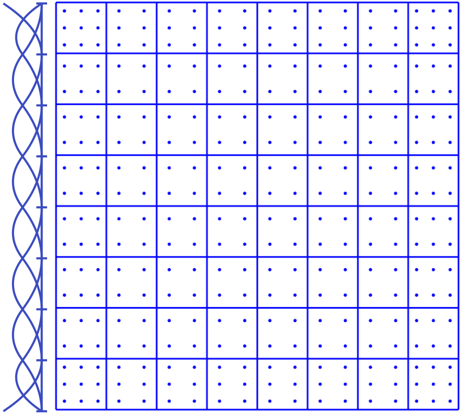
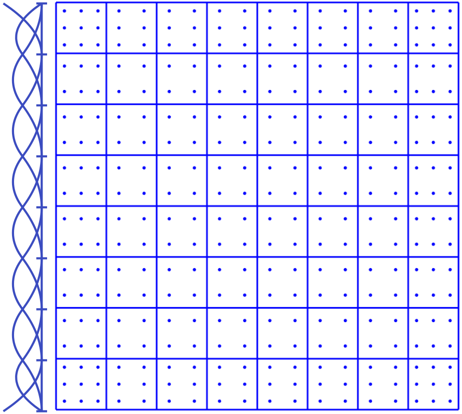
- Row/column-based assembly

Weighted quadrature point distribution



Bi-degree $p = 2$, C^{p-1} B-spline basis

Weighted quadrature point distribution



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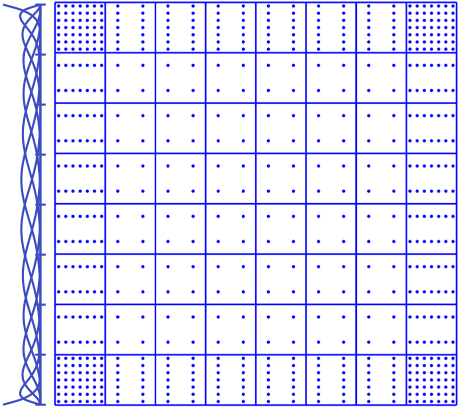
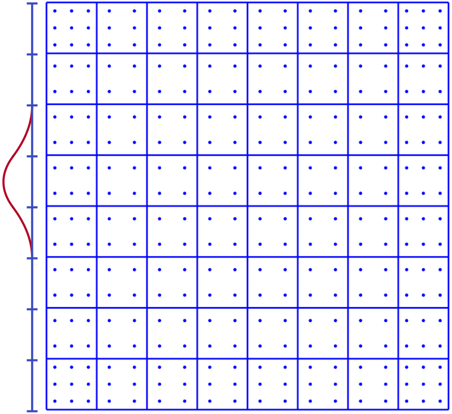


Bi-degree $p = 6$, C^{p-1} B-spline basis

Weighted quadrature point distribution



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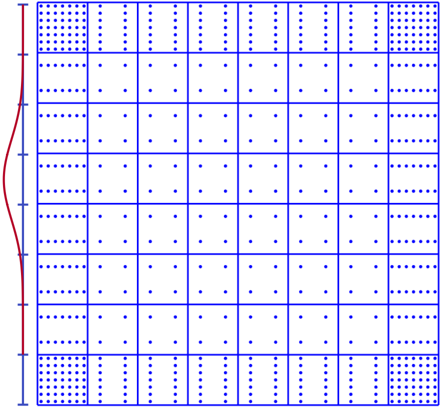

Bi-degree $p = 6$, C^{p-1} B-spline basis

Fast formation and assembly – impact

FLOPS for 3D mass matrix for C^{p-1} splines

(R. Hiemstra et al., CMAME 2019)

- $\mathcal{O}(p^3)$ quadrature points per element

Assembly \ Formation	Quadrature loop	Sum factorization
Element loop	$c \cdot p^9$	$c_1 \cdot p^5 + c_2 \cdot p^6 + c_3 \cdot p^7$
Row/column loop	$c \cdot p^9$	$c_1 \cdot p^7 + c_2 \cdot p^6 + c_3 \cdot p^5$

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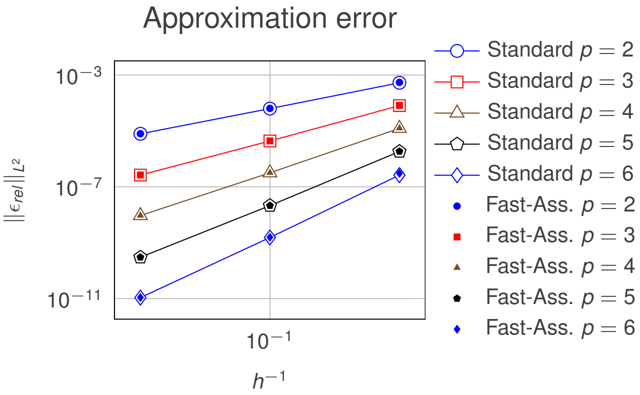
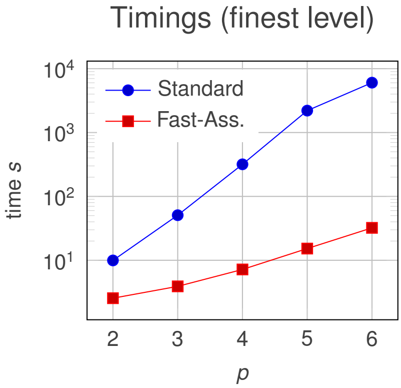
Row/column loop	$c \cdot p^6$	$c_1 \cdot p^4 + c_2 \cdot p^4 + c_3 \cdot p^4$

Fast formation and assembly – impact

Single threat formation of a 3D mass matrix





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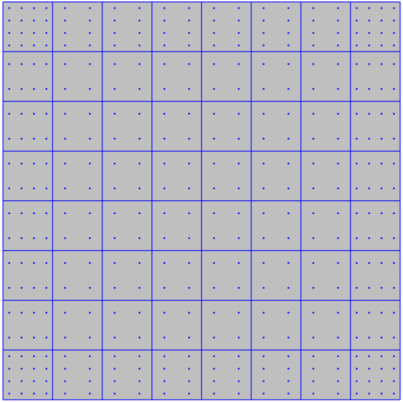
The challenge of fictitious domains

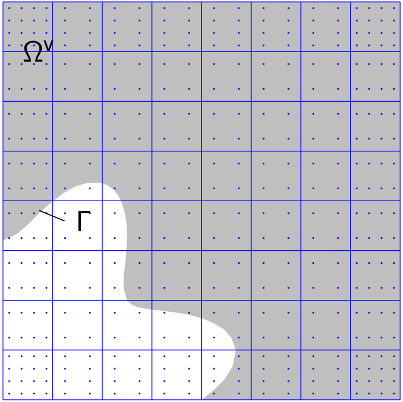
- Sum factorization and weighted quadrature take full advantage of the tensor product structure ...



Bi-cubic C^{p-1} background mesh

The challenge of fictitious domains

- Sum factorization and weighted quadrature take full advantage of the tensor product structure ...
- ... which is violated by an arbitrarily located interface Γ.

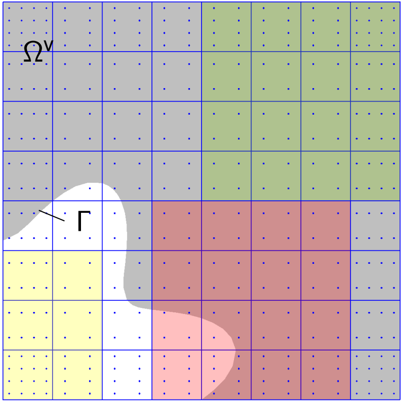


Cut bi-cubic C^{p-1} background mesh

Function types of cut background meshes

Classification by valid support size $\mathcal{S}_i^{\mathrm{v}} := \mathrm{supp}\{B_i\} \cap \overline{\Omega^{\mathrm{v}}}$

- **Exterior** if $\mathcal{S}_i^{\mathrm{v}} = \emptyset$
- **Interior** if $\mathcal{S}_i^{\mathrm{v}} = \mathrm{supp}\{B_i\}$
- **Cut** if $0 < |\mathcal{S}_i^{\mathrm{v}}| < |\mathrm{supp}\{B_i\}|$



Cut bi-cubic C^{p-1} background mesh

Integration of cut basis functions

Split of the basis function's valid support $\mathcal{S}_i^{\mathrm{v}}$ into...

$$\int_{\mathcal{S}_i^{\mathrm{v}}} B_i(\xi)d\xi = \int_{\mathcal{S}_i^{\mathrm{c}}} B_i(\xi)d\xi + \int_{\mathcal{S}_i^{\mathrm{r}}} B_i(\xi)d\xi$$

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- **Cut** part $\mathcal{S}_i^{\text{c}}$:
  Element-wise assembly using **higher-order accurate** quadrature
  (e.g., T.-P. Fries et al., CMAME 2017; R.I. Saye, SISC 2015; R.I. Saye, J. Comput. Phys. 2022)

Integration of cut basis functions

Split of the basis function's valid support $\mathcal{S}_{\boldsymbol{i}}^{\text{v}}$ into...

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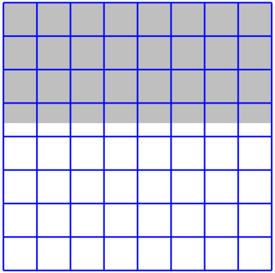
- **Regular** part $\mathcal{S}_{\boldsymbol{i}}^{\text{r}}$:
  Integration that **exploits the tensor product structure**

Integration of $\mathcal{S}_i^r$: weighted quadrature (WQ)

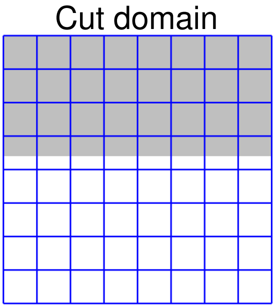
Cut domain



Mass matrix deviation

p	Error
1	3.330e-03
2	3.389e-04
3	3.038e-04
4	1.782e-04
5	1.599e-04
6	1.282e-04

Integration of $\mathcal{S}_i^r$: weighted quadrature (WQ)



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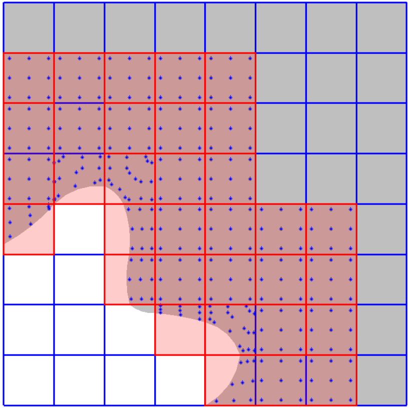
Straightforward use of weighted quadrature is **not** possible since a **discontinuity** is introduced in the integration domain.

Integration of $\mathcal{S}_i^r$: Gauss quadrature (GQ)

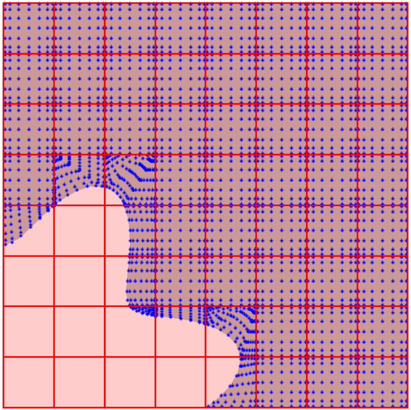
Gauss points for all cut basis functions



Bi-degree $p = 2$

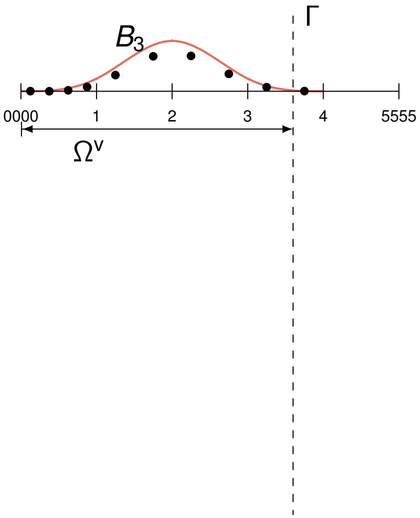
Integration of $\mathcal{S}_i^r$: Gauss quadrature (GQ)

Gauss points for all cut basis functions



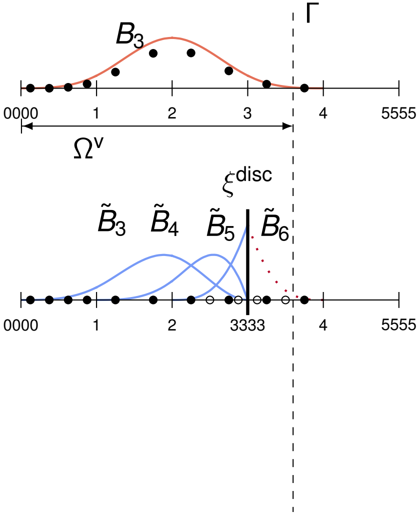
Bi-degree $p = 6$

Discontinuous weighted quadrature (DWQ)



Cut B-spline and
its WQ rule

Discontinuous weighted quadrature (DWQ)

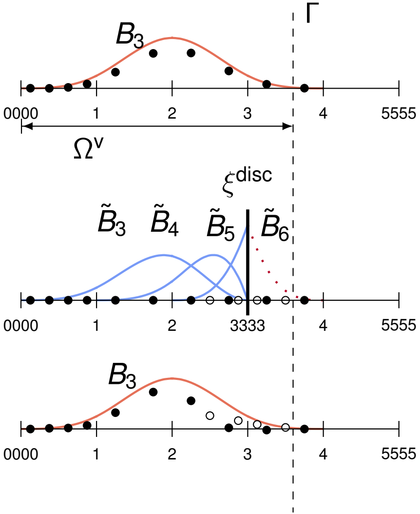


Cut B-spline and its WQ rule

Knot insertion at ξ^{disc} and subdivision matrix **S**

Disc. basis and additional nested quadrature points

Discontinuous weighted quadrature (DWQ)



Cut B-spline and its WQ rule

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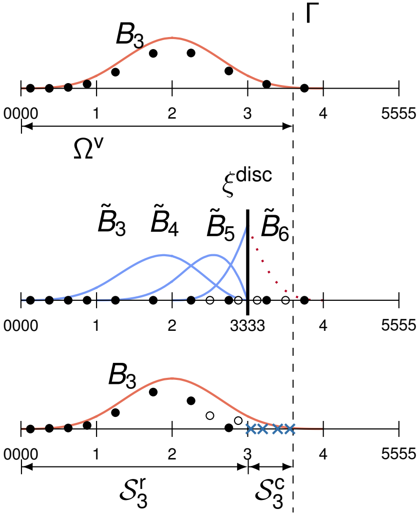
Disc. basis and additional nested quadrature points

Compute weights $\tilde{w}$ and multiply them by $\mathbf{S}^T$

Initial cut B-spline and its DWQ rule

Discontinuous weighted quadrature (DWQ)



Cut B-spline and its WQ rule

Knot insertion at ξ^{disc} and subdivision matrix $\mathbf{S}$

Disc. basis and additional nested quadrature points

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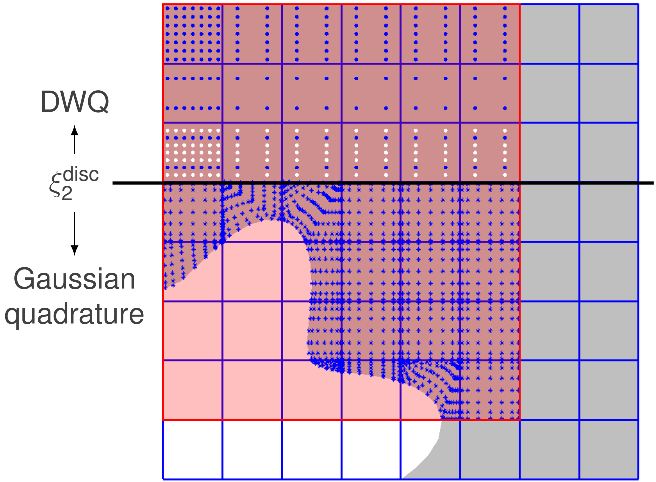
Initial cut B-spline and its DWQ rule

- • DWQ point
- × Gauss point

Discontinuous weighted quadrature (DWQ)



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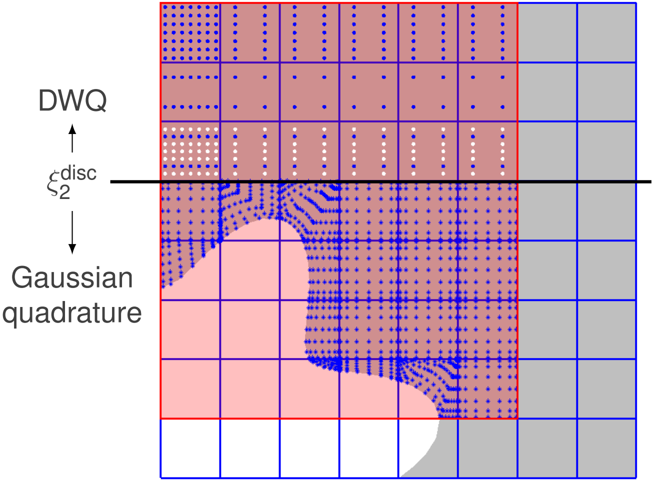

- Implementation remark: Use Gauss point locations for DWQ to reuse evaluations

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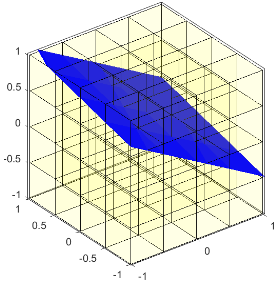
3D mass matrix

- L^2-projection of the target function

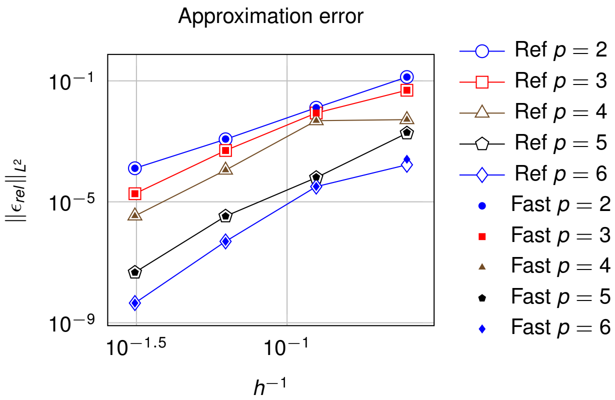
$$f = \sin(2xz)\cos(3yz)$$

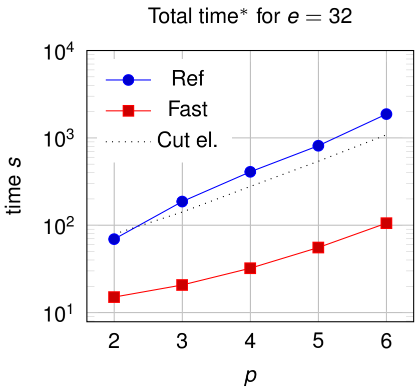
- Degrees $p = \{2, \ldots, 6\}$
- Elements per dimension $e = \{4, 8, 16, 32\}$
- Ill-conditioning due to cut elements is treated by extended B-splines (K. Höllig et al., SINUM 2002)



Domain of interest

3D mass matrix: Convergence and timings





* Timings without cut elements

Conclusions

Fast formation and assembly key ingredients:

- Spline-based background mesh
- Sum factorization (P. Antolin et al., CMAME 2015)
- Row/column assembly
- (Discontinuous[1]) weighted quadrature (F. Calabrò et al., CMAME 2017)

[1] BM, Fast formation and assembly of isogeometric Galerkin matrices for trimmed patches, INdAM Series 2021

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Impact

- Significant reduction of FLOPS for setting up system matrices
- Overall performance depends on degree, interface shape, and mesh size

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Impact

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Potential drawbacks

- Weighted quadrature does not preserve symmetry
- Code requires a completely new structure
- Many quantities have to be pre-computed for sum factorization

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