The Numerical Assembly Technique for arbitrary planar beam systems based on an improved homogeneous solution

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Content

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Overview

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- General formulation of the boundary and interface conditions for arbitrary planar beam structures
- Study of the homogeneous solution of the Timoshenko beam
- Generating the system matrix by means of the Numerical Assembly Technique
- Determining the eigenvalues and eigenvectors of the system matrix

Aim and scope

- Natural Frequencies of any continuous planar beam structure considering the Timoshenko beam theory
- Analytical solutions of the differential equation*s
- Improvement of the computational performance by use of an improved homogeneous solution

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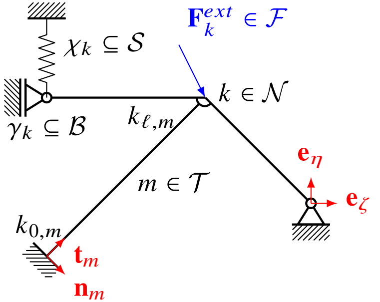
Content

Definition of the frame structure

- $\mathcal{N}$ ... set of nodes
- $\mathcal{S}$ ... set of springs
- $\mathcal{T}$ ... set of beams
- $\mathcal{F}$ ... set of external loads
- $\mathcal{B}$ ... set of bearings



- Each beam $m \in \mathcal{T}$ has
  - two nodes $k_{0,m}, k_{\ell,m} \in \mathcal{N}$
  - parameters ρ_m, E_m, I_m, A_m
  - rigid or hinged at k_0, k_ℓ
  - $\mathbf{t}_m = \frac{k_{\ell,m} - k_{0,m}}{||k_{\ell,m} - k_{0,m}||}$
  - $\mathbf{n}_m \perp \mathbf{t}_m$
- Each node $k \in \mathcal{N}$ has
  - set of beams $\beta_k \subseteq \mathcal{T}$, $|\beta_k| \geq 1$
  - set of springs $\chi_k \subseteq \mathcal{S}$, $|\chi_k| \in \{0, 3\}$
  - external loads $\mathbf{F}_k^{ext} \in \mathcal{F}$
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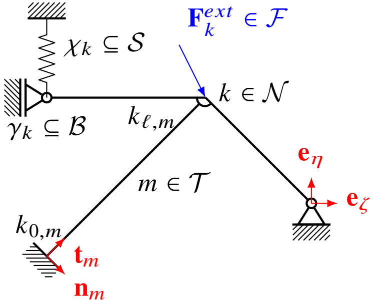

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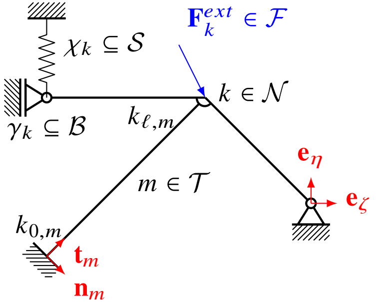

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Preliminaries

- The rotation of the beam axis ψ_m
- The resulting deformation

$$\mathbf{U}_m(\xi) = U_m(\xi) \cdot \mathbf{t}_m + W_m(\xi) \cdot \mathbf{n}_m$$

- Internal bending moment M_m
- The resulting internal force

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Boundary conditions

- Kinetics

$$\mathbf{F}_k = \sum_{m\in\beta_k} \mathbf{S}_{m|k} + \mathbf{B}_k^L + \mathbf{F}_k^{ext} + \mathbf{F}_k^L = \mathbf{0} \qquad \text{free vib.} \longrightarrow \mathbf{F}_k^{ext} = \mathbf{0}$$

$$M_k = \sum_{m\in\beta_k} M_{m|k} + B_k^R + M_k^{ext} + M_k^R = 0 \qquad \text{free vib.} \longrightarrow M_k^{ext} = 0$$

- Springs

$$\mathbf{F}_k^L = -c_L(\mathbf{U}_k \cdot \zeta) \cdot \zeta \qquad c_L \text{ ...longitudinal spring constant}$$

$$M_k^R = -c_\psi \psi_k \qquad c_\psi \text{ ...rotational spring constant}$$

- Bearings

$\mathbf{B}_k^L$ ...unknown resulting force

B_k^R ...unknown rotational moment

linked to the blocked degrees of freedom due to bearings!

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Bearings

We define the subsets $\beta_k^r \subset \beta_k$ and $\beta_k^h \subset \beta_k$

- Pinned

$$\mathbf{U}_{m|k} = 0 \qquad m \in \beta_k$$

- Roller

$$\mathbf{U}_{m|k} \cdot \mathbf{e}_\eta = 0 \qquad m \in \beta_k$$

- Clamped bearing

$$\begin{aligned} \mathbf{U}_{m|k} &= 0 \qquad m \in \beta_k \\ \psi_{m|k} &= 0 \qquad m \in \beta_k^r \end{aligned}$$

- Parallel guide

$$\begin{aligned} \mathbf{U}_{m|k} \cdot \mathbf{e}_\eta &= 0 \qquad m \in \beta_k \\ \psi_{m|k} &= 0 \qquad m \in \beta_k^r \end{aligned}$$

Coupling conditions

At node k we define a fixed beam

- $m_f \in \beta_k$

$$\mathbf{U}_{m_f|k} = \mathbf{U}_{m|k} \qquad \forall\, m \in \beta_k \backslash m_f$$

- $m_r \in \beta_k^r$

$$\psi_{m_r|k} = \psi_{m|k} \qquad \forall\, m \in \beta_k^r \backslash m_r$$

- $m_h \in \beta_k^h$

$$M_{m|k} = 0 \qquad \forall\, m \in \beta_k^h \backslash m_h$$

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Differential equation

- The governing differential equation $W^{IV}(x) + dW''(x) + eW(x) = 0$
  - Timoshenko beam theory
  - undamped, free vibration
  - constant cross section
  - frequency domain
- with the abbreviations

$$d = a + b + c \qquad e = ab$$

$$a = \frac{\omega^2 \rho}{kG} \qquad b = \frac{\omega^2 \rho}{E} - c \qquad c = \frac{GkA}{EI}$$

- The characteristic polynomial

$$r^4 + dr^2 + e = 0$$

with the condition $\tilde{z} = r^2$ leads to

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Eigenvalues

- Roots

$$\tilde{z}_1 = \frac{1}{2}(-d + \sqrt{\Delta}) \qquad \tilde{z}_1 \in \mathbb{R} \quad \forall\omega$$

$$\tilde{z}_2 = \frac{1}{2}(-d - \sqrt{\Delta}) \qquad \tilde{z}_2 \in \mathbb{R}^{<0} \quad \forall\omega$$

where

$$\Delta = d^2 - 4e \qquad \Delta \wedge d \in \mathbb{R}^{>0} \quad \forall\omega$$

- Cut-off frequency

$$\tilde{z}_1 = 0 \rightarrow \omega_{co} = \sqrt{\frac{GkA}{\rho I}}$$

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General solutions

- First spectrum of natural frequencies

$$\tilde{z}_1 \in \mathbb{R}^{>0} \ni \omega < \omega_{co}$$

- General solution

$$W_m(\xi) = C_{m,1}e^{-i\sqrt{\tilde{z}_2}\xi} + C_{m,2}e^{i\sqrt{\tilde{z}_2}\xi} + C_{m,3}e^{-\sqrt{\tilde{z}_1}\xi} + C_{m,4}e^{\sqrt{\tilde{z}_1}\xi}$$

- Common homogeneous solution

$$W_m(\xi) = P_{m,1}\cos(\lambda_2\xi) + P_{m,2}\sin(\lambda_2\xi) + P_{m,3}\cosh(\lambda_1\xi) + P_{m,4}\sinh(\lambda_1\xi) \quad (1)$$

where

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Improved homogeneous solution

- Euler's formula

$$\cos(\xi) + sin(\xi) = \left(\frac{1}{2} + \frac{i}{2}\right) e^{-i\xi} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{i\xi}$$

- Implemented in the general solution

$$W_m(\xi) = P_{m,1}\cos(\lambda_2\xi) + P_{m,2}\sin(\lambda_2\xi) + C_{m,3}e^{-\lambda_1\xi} + C_{m,4}e^{\lambda_1\xi} \tag{2}$$

- Dividing last term by $e^{\lambda_1 L}$ so that

$$e^{\lambda_1\xi}e^{-\lambda_1 L} = e^{\lambda_1(\xi-L)}$$

- Improved homogeneous solution

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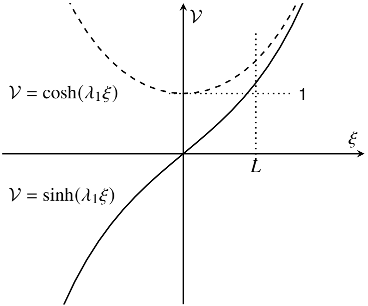
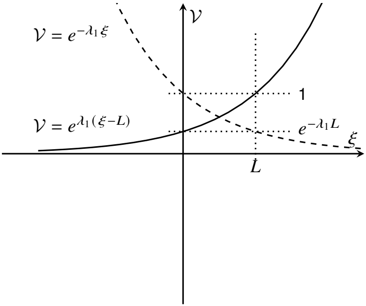
Study of function terms

- The value range of each term in eq. (3) is $\mathcal{V} \in \left[e^{-\lambda_1 L}, 1\right]$





Content

Numerical Assembly Technique

1. Homogeneous solutions

$$U_m(\xi) = Q_{m,1}\cos\left(c_m\xi\right) + Q_{m,2}\sin\left(c_m\xi\right)$$

$$W_m(\xi) = P_{m,1}\cos(\lambda_2\xi) + P_{m,2}\sin(\lambda_2\xi) + C_{m,3}e^{-\lambda_1\xi} + C^*_{m,4}e^{\lambda_1(\xi-L)}$$

2. Enforcement of boundary and coupling conditions
3. System of equations

$$\mathbf{A}(\omega)\mathbf{c} = \mathbf{0} \qquad \mathbf{A} \in \mathbb{R}^{n\times n} \quad n = |\mathcal{T}|\cdot 6 + |\mathcal{B}|$$

4. Non-trivial solutions for ω (Regula-Falsi Method)

$$f(\omega) = \det(\mathbf{A}) = 0$$

5. Solving for the integration constants, bearing forces and moments
6. Computing eigenvectors, mode shape

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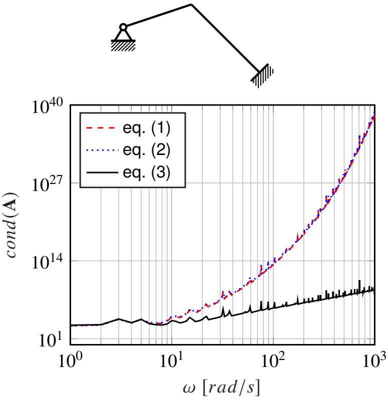
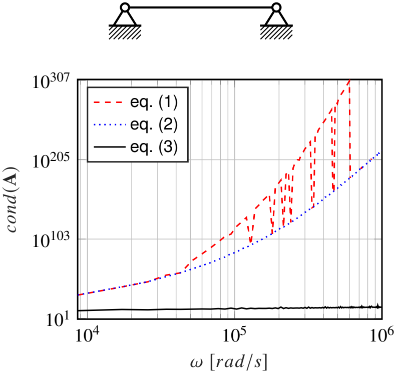
Conditioning number





eq. (1) $W_m(\xi) = P_{m,1}\cos(\lambda_2\xi) + P_{m,2}\sin(\lambda_2\xi) + P_{m,3}\cosh(\lambda_1\xi) + P_{m,4}\sinh(\lambda_1\xi)$

eq. (2) $W_m(\xi) = P_{m,1}\cos(\lambda_2\xi) + P_{m,2}\sin(\lambda_2\xi) + C_{m,3}e^{-\lambda_1\xi} + C_{m,4}e^{\lambda_1\xi}$

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Natural frequencies

- For the arbitrary frame structure
- $\#\omega_{co} = 5988$

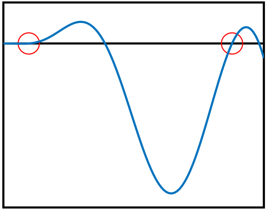
	Numerical Assembly Technique						Finite Element Method		
	double			vpa			Elements per Beam		
	(1)	(2)	(3)	(1)	(2)	(3)	16	32	64
max $\#\omega$	16	203	ω_{co}	52	273	ω_{co}	350 (25[1])	702 (67[1])	1406 (131[1])

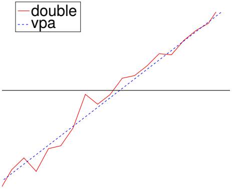
[1] $\#\omega \,|\, \frac{|\omega - \omega_A|}{\omega_A} < \frac{1}{10}$

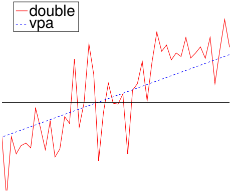
Finding eigenvalues

- Zoomed-in pictures of the frequency function $f(\omega)$ at the natural frequencies ω_i
- $f(x_i)$ is plotted with $x_i \in [\omega_i - \epsilon,\ \omega_i + \epsilon] \mid \epsilon = 10^{-12}$

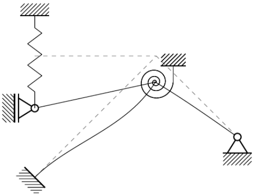
Frequency function

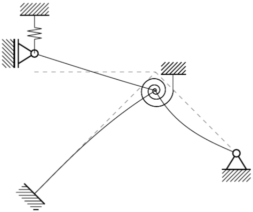


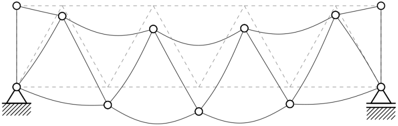


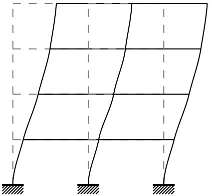


Mode shapes









Conclusion

- ⊕ "Exact" values of the natural frequencies
- ⊕ Stable computations for high frequency ranges
- ⊕ Discretization at nodes

- ⊖ $f(\omega) = \det(\mathbf{A}) \rightarrow$ expensive
- ⊖ Finding zeros of the frequency function in numerical matter

- ➔ Expanding to 3-D including torsional vibrations
- ➔ Implementing arches
- ➔ Considering structural damping

Further application

Automatic generation of a large number of homework assignments for basic courses in Statics and Strength of Materials

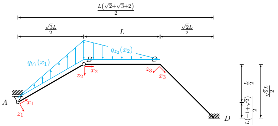
BAUMECHANIK 2 — TU GRAZ
SOMMERSEMESTER 2022 — INSTITUT FÜR BAUMECHANIK

HAUSÜBUNG SYSTEMBERECHNUNG

Berechnen Sie für das dargestellte mechanische System

a) die Biegelinie $w_2(x_2)$, sowie

b) den Verschiebungsverlauf $u_1(x_1)$.



Die Belastungsfunktionen lauten:

$$q_{V_1}(x_1) = \frac{q x_1}{L}$$
$$q_{z_2}(x_2) = \frac{q\,(L - x_2)}{L}$$

Die Steifigkeiten der Einzelstäbe betragen:

$$EI_1 \to \infty$$
$$EA_1 = \frac{EI}{L^2}$$
$$EI_2 = EI$$
$$EA_2 \to \infty$$
$$EI_3 \to \infty$$
$$EA_3 \to \infty$$

The Numerical Assembly Technique for arbitrary planar beam systems based on an improved homogeneous solution

Thomas Kramer, Michael Helmut Gfrerer

GAMM Annual Meeting
Aachen, 15. August 2022